# Extremal curves of the total curvature in homogeneous 3 -spaces 

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## 1 Introduction

A classical and well known result by Fenchel says that the total curvature of any simple closed curve in the Euclidean 3-space satisfies that

$$
\int_{\gamma} \kappa(s) d s \geq 2 \pi
$$

and equality holds if and only if $\gamma$ is a convex plane curve. Therefore, the minimum of the total curvature action over simple closed curves in the Euclidean space is $2 \pi$ and it is reached just on the convex plane curves.

Then it seems natural to consider the total curvature functional acting on a suitable space of curves of a certain surface, or more generally in a Riemannian space, and then to study the associated variational problem.

That was planned by J. Arroyo, M. Barros and O. J. Garay in

- Some examples of critical points for the total mean curvature functional, Proc. Edinburgh Math. Soc. 43 (2000), 587-603,
- Holography and total charge, J. Geom. Phys. 41 (2002), 65-72.

Also M. S. Plyushchay, in

- Masess particle with rigidity as a model for the description of bosons and fermions, Phys. Lett. B 243 (1990), 383-388,
proposed that model to study the dynamics of massless particles.

Arroyo, Barros and Garay's results are summarized as follows:

- As for surfaces, the extremals of the total curvature are achieved by curves of parabolic points.
- As for constant curvature spaces $M^{n}(k)$
- The dynamics is reduced to dimension $n \leq 3$ and curvature $k \geq 0$.
- When $k=0$, the dynamics happens on plane curves in $\mathbb{R}^{3}$.
- When $k>0$, the extremals on a 3 -sphere are achieved by horizontal lifts (also known as Legendre curves), via the Hopf map, of curves on a 2 -sphere.
- Some other partial results providing examples of critical curves in Berger spheres as well as in the complex projective space.
- We see that the round 3 -sphere is, essentially, the only constant curvature Riemannian space where the variational problem makes sense and we know that it is also a homogeneous space.

Then it seems natural to consider the variational problem associated to the total curvature functional for curves in homogeneous 3-spaces whose isometry group is 4-dimensional.

A space $M$ whose isometry group acts transitively on itself is called a homogeneous space.

Let $M(c)$ be a 3-dimensional simply connected space of constant curvature $c$, for any $c \in \mathbb{R}$. It is known that $M(c)$ is homogeneous and its isometry group is 6 -dimensional.

By relaxing the curvature condition, we find a bi-parametric family $\mathrm{E}(c, r)$, where $c, r \in \mathbb{R}$ and $c \neq 4 r^{2}$, of homogeneous spaces with 4-dimensional isometry group (see next table).

| 6 | 4 | 3 |
| :--- | :--- | :--- |
| $\mathbb{H}^{3}, \mathbb{R}^{3}, \mathbb{S}^{3}$ | $\mathbb{H}^{2} \times \mathbb{R}, \mathbb{S}^{2} \times \mathbb{R}$, <br>  <br> PSL <br> Berger spheres | $\mathrm{Sol}_{3}$ |

On the other hand, any homogeneous 3 -space $\mathrm{E}(c, r)$ with rigidity of order four can be viewed as a bundle over a surface with constant curvature. This fibration provides a Riemannian submersion

$$
\mathrm{p}: \mathrm{E}(c, r) \rightarrow \mathrm{B}^{2}(c),
$$

with geodesic fibers (vanishing first O'Neill invariant). In addition, the vertical flow is generated by a unit Killing vector field $\xi$ which allows one to give the following expression for the second O'Neill invariant, $A$,

$$
A_{X} \xi=\bar{\nabla}_{X} \xi=r(X \times \xi)
$$

where $X$ is a horizontal vector field and $r$, the bundle curvature (B. Daniel [4]), is a constant. Therefore, $\mathrm{p}: \mathrm{E}(c, r) \rightarrow \mathrm{B}^{2}(c)$ is a Killing submersion in the sense of Manzano, Espinar and de Oliveira ([5, 6]).

Both constants, $c$ (curvature of the base) and $r$ (bundle curvature), classify the homogeneous 3 -spaces, up to isometries and topology. In other words, each pair of real numbers, ( $c, r$ ), determines, up to topology, a congruence class $\mathrm{E}(c, r)$ of homogeneous 3spaces whose isometry group has either dimension 4 , if $c \neq 4 r^{2}$, or dimension 6 (constant curvature), if $c=4 r^{2}$.

|  | $c<0$ | $c=0$ | $c>0$ |
| :---: | :---: | :---: | :---: |
| $r=0$ | $\mathbb{H}^{2} \times \mathbb{R}$ | - | $\mathbb{S}^{2} \times \mathbb{R}$ |
| $r \neq 0$ | $\widehat{\operatorname{PSL}}(2, \mathbb{R})$ | $\mathrm{Nil}_{3}$ | Berger spheres |

## 2 The total curvature functional

$$
\begin{gathered}
(M,\langle,\rangle) \text { Riemannian manifold; } p_{1}, p_{2} \in M, x_{i} \in T_{p_{i}} M, \\
\Omega=\left\{\gamma:\left[a_{1}, a_{2}\right] \rightarrow M, \quad \gamma\left(a_{i}\right)=p_{i}, \gamma^{\prime}\left(a_{i}\right)=x_{i}, \quad 1 \leq i \leq 2\right\} \\
\mathcal{F}: \Omega \rightarrow \mathbb{R}, \quad \mathcal{F}(\gamma)=\int_{\gamma} \kappa(s) d s .
\end{gathered}
$$

We wish to study the variational problem associated to this functional and find the extremal curves.

Given $\gamma \in \Omega$, the tangent space $T_{\gamma} \Omega$ is identified with the space of fields along $\gamma$ vanishing at the ending points. If $W \in T_{\gamma} \Omega$, we define a curve in $\Omega$ passing through $\gamma$ in the direction of $W$ as

$$
\Gamma:(-\varepsilon, \varepsilon) \rightarrow \Omega, \quad \Gamma(t)=\exp _{\gamma(s)} t W(s)
$$

The first variation was computed Arroyo, Barros and Garay in [1]:

$$
D \mathcal{F}(\gamma)[W]=\left.\frac{\partial}{\partial t}(\mathcal{F}(\Gamma(t)))\right|_{t=0}=\int_{\gamma}\langle\mathcal{E}(\gamma), W\rangle d s+\mathcal{B}(\gamma, W)
$$

where $\mathcal{E}(\gamma)$ is the Euler-Lagrange operator and $\mathcal{B}(\gamma, W)$ is the boundary operator.
The Frenet equations yield

$$
\begin{aligned}
\mathcal{E}(\gamma) & =\tau^{2} N-\tau_{s} B-\tau \eta-R(N, T) T, \\
\mathcal{B}(\gamma, W) & =\sum_{i=1}^{m}\left\langle\nabla_{T} W\left(s_{i}\right), N\left(s_{i}\right)\right\rangle+\sum_{i=1}^{m}\left\langle W\left(s_{i}\right), \nabla_{T} N\left(s_{i}\right)\right\rangle,
\end{aligned}
$$

where $\eta$ belongs to the subbundle normal to that spanned by $\{T, N, B\}(s)$ along $\gamma(s)$.

Then, $\gamma$ is a critical point of $\mathcal{F}$ in $\Omega$ if and only if (see [1]) the following Euler-Lagrange equation is satisfied

$$
\begin{equation*}
R(N, T) T=\tau^{2} N-\tau_{s} B-\tau \eta \tag{1}
\end{equation*}
$$

The solution, widely discussed by Arroyo, Barros and Garay in [1] and [2], is strongly governed by the curvature of the ambient space.

## 3 Lancret helices in homogeneous 3-spaces

We wish to work in $\mathbb{S}^{3}(1)$ where we know that there are a very special curves, Lancret helices, which were characterized by Barros in General helices and a theorem of Lancret, Proc. A.M.S., 125 (1997), 1503-1509.

Namely, in a Riemannian space equipped with a unit Killing field $\xi$, the curves making a constant angle with $\xi$ are called Lancret helices with axis $\xi$.

After Barros' characterization, we knew that the critical curves of $\mathcal{F}$ in $\mathbb{S}^{3}(1)$ are Lancret helices with axis the Hopf field and slope $\pi / 2$.

As every homogeneous 3 -space admits a unit Killing vector field, it seems natural to study the following:

Problem 1: Are there Lancret helices being extremal of the total curvature action in homogeneous 3 -spaces with 4-dimensional isometry group?

Then we undertook the task of getting a geometrical characterization of those helices in $M^{3}=\mathrm{E}(c, r)$.

Let $M^{3}=\mathrm{E}(c, r)$ be a homogeneous 3 -space, $\mathrm{p}: M^{3} \rightarrow \mathrm{~B}^{2}(c)$ the corresponding Riemannian submersion, and let $\xi$ be the associated unit vertical Killing vector field.

- $\xi$ will be the axis of the helices we are looking for.
- Given a curve $\beta(s)$ in $\mathrm{B}^{2}(c)$, let $\bar{\beta}(s)$ be a horizontal lift. Both can be taken arc length parameterized.
- Consider the surface $S_{\beta}=\mathrm{p}^{-1}(\beta)$ in $M^{3}$, which can be parameterized by

$$
X(s, t)=\varphi_{t}(\bar{\beta}(s))
$$

where $\left\{\varphi_{t}: t \in \mathbb{R}\right\}$ is the one-parameter group generated by $\xi$.

- $S_{\beta}$ is a flat surface called Hopf cylinder over $\beta(s)$.
- Its shape operator was computed by Espinar and Oliveira in [5]:

$$
\left(\begin{array}{cc}
\kappa_{g} & -r \\
-r & 0
\end{array}\right)
$$

where $\kappa_{g}$ is the geodesic curvature of $\beta(s)$ in $\mathrm{B}^{2}(c)$.

- Let $\gamma(s)$ be a geodesic of $S_{\beta}$. Then

$$
\frac{d}{d s}\left\langle\gamma^{\prime}(s), \xi(\gamma(s))\right\rangle=\left\langle\bar{\nabla}_{\gamma^{\prime}} \gamma^{\prime}(s), \xi(\gamma(s))\right\rangle+r\left\langle\gamma^{\prime}(s), \gamma^{\prime}(s) \times \xi(\gamma(s))\right\rangle=0
$$

which shows that the angle between $\gamma(s)$ and $\xi$ is constant.
Therefore
The geodesics of Hopf cylinders are Lancret helices.
The converse also holds, namely,
Proposition 3.1 The Lancret helices of the homogeneous 3-spaces with 4 or 6-dimensional isometry group are geodesics of Hopf cylinders.

Some useful formulas:

$$
\begin{gather*}
\kappa=\frac{\kappa_{g}-2 r m}{m^{2}+1}  \tag{2}\\
\tau=\frac{\kappa_{g} m-r\left(m^{2}-1\right)}{m^{2}+1}  \tag{3}\\
\tau=m \kappa+r \tag{4}
\end{gather*}
$$

where $m=\cot \varphi$.

The classical theorem of Lancret ensures that formula (4) with $r=0$ provides a simple characterization of Lancret helices in the Euclidean space. Barros proved (see [3]) that this result also holds for $\mathbb{S}^{3}$ and $\mathbb{H}^{3}$ and observed that the only Lancret helices in $\mathbb{H}^{3}$ are circular ones, i. e., those with curvature an torsion being both constant.

Thus, it seems natural to ask for a Lancret-type theorem in homogeneous 3 -spaces with rigidity of order four. Said otherwise

Problem 2: Does equation (4) characterize Lancret helices in such homogeneous 3-spaces?

We give a negative answer:

Proposition 3.2 Let $M^{3}=\mathrm{E}(c, r)$ be an homogeneous 3 -space with $c \neq 4 r^{2}$, and let $\gamma(s)$ be a Lancret helix in $M^{3}$ with curvature $\kappa(s)$ and torsion $\tau(s)$. Then there exists a curve $\alpha(s)$ in $M^{3}$ with curvature function $\kappa_{\alpha}=\kappa$ and torsion $\tau_{\alpha}=\tau$ which is not congruent to any Lancret curve in $M^{3}$.

## 4 Extremals of the total curvature in $\mathrm{E}(c, r)$

The Riemannian curvature operator of a homogeneous 3-space $M^{3}=\mathrm{E}(c, r)$ was computed by Daniel in [4]:

$$
\begin{aligned}
R(X, Y) Z & =\left(c-3 r^{2}\right)(\langle Y, Z\rangle X-\langle X, Z\rangle Y) \\
& +\left(-c+4 r^{2}\right)[\langle Y, \xi\rangle\langle Z, \xi\rangle X-\langle X, \xi\rangle\langle Z, \xi\rangle Y+\langle X, \xi\rangle\langle Y, Z\rangle \xi-\langle Y, \xi\rangle\langle X, Z\rangle \xi]
\end{aligned}
$$

In particular, along a curve $\gamma(s)$ with Frenet frame $\{T, N, B\}(s)$ :

$$
R(N, T) T=\left(c-3 r^{2}\right) N+\left(-c+4 r^{2}\right)\left[\langle T, \xi\rangle^{2} N-\langle T, \xi\rangle\langle N, \xi\rangle T+\langle N, \xi\rangle \xi\right]
$$

which can be combined with (1) to obtain:

$$
\tau^{2}=r^{2}+\left(c-4 r^{2}\right)\langle B, \xi\rangle^{2}, \quad \tau_{s}=\left(c-4 r^{2}\right)\langle N, \xi\rangle\langle B, \xi\rangle
$$

We can manipulate these equations to find

$$
\begin{equation*}
\tau^{2}=r^{2}+\left(c-4 r^{2}\right)\langle B, \xi\rangle^{2} \quad\left(c-4 r^{2}\right)(r-2 \tau)\langle N, \xi\rangle\langle B, \xi\rangle=0 \tag{5}
\end{equation*}
$$

$$
\tau^{2}=r^{2}+\left(c-4 r^{2}\right)\langle B, \xi\rangle^{2} \quad\left(c-4 r^{2}\right)(r-2 \tau)\langle N, \xi\rangle\langle B, \xi\rangle=0
$$

According to the second equation, we only need to study the following two cases:
(i) $c-4 r^{2}=0$ Then the homogeneous space $\mathrm{E}(c, r)$ has constant curvature $r^{2}$ and then, up to topology, it must be either $\mathbb{R}^{3}($ if $r=0)$ or $\mathbb{S}^{3}$ (if $\left.r \neq 0\right)$. In the former case the extremals are plane curves while in the later the extremals are the Legendrian curves, that is, the horizontal lifts, via the Hopf map, of curves in the two sphere. Anyway, in both cases $\xi$ lies in the normal plane along each extremal.
(ii) $r-2 \tau=0$ Then the first equation of (4) shows that

$$
\langle B, \xi\rangle^{2}=-\frac{3 \tau^{2}}{c-4 r^{2}}
$$

and so $\langle B, \xi\rangle$ is a constant. Furthermore, $\frac{d}{d s}\langle B, \xi\rangle=(r-\tau)\langle N, \xi\rangle$ and so the extremals in this class have either horizontal normal or torsion $\tau=r$. However in the last case $\tau=0$ and so the binormal is horizontal.

Summarizing, we have that

Lemma 4.1 Let $\gamma(s)$ be an extremal of the total curvature functional in $\mathrm{E}(c, r)$, then either
(E1) $\xi$ lies in its normal plane along $\gamma(s)$, or
(E2) $\xi$ lies in its rectifying plane along $\gamma(s)$, or
(E3) $\xi$ lies in its osculating plane along $\gamma(s)$.

Then we have to consider extremals with:

| Horizontal normal | Horizontal binormal |
| :---: | :---: |
| $\langle N, \xi\rangle=0$ | $\langle B, \xi\rangle=0$ |

- 1st case. $\langle N, \xi\rangle=0$

Then $\frac{d}{d s}\langle T, \xi\rangle=\kappa\langle N, \xi\rangle+r\langle T, T \times \xi\rangle=0$, which shows that the curve is a Lancret helix. Then

Proposition 4.2 A Lancret helix in $\mathrm{E}(c, r)$ is an extremal of the total curvature if and only if it has constant torsion (as well as constant curvature) satisfying that

$$
\begin{equation*}
\tau^{2}=c \sin ^{2} \varphi+r^{2}\left(4 \cos ^{2} \varphi-3\right)=\frac{c+r^{2}\left(m^{2}-3\right)}{m^{2}+1} \tag{6}
\end{equation*}
$$

Remark 4.3 It should be noted that, from to the above proposition, the only obstruction to the existence of extremal Lancret helices in $\mathrm{E}(c, r)$ comes from $c+r^{2}\left(m^{2}-3\right)>0$, and so we have the following

Corollary 4.4 Every homogeneous 3 -space $\mathrm{E}(c, r)$, except $\mathbb{H}^{2} \times \mathbb{R}$, admits a real oneparameter class of extremal Lancret helices.

## Examples of extremals with horizontal normal:

1. Riemannian product homogeneous 3 -spaces. Then $r=0$.
$-\mathbb{H}^{2}(c) \times \mathbb{R}$ : no extremal Lancret helices.

- $\mathbb{S}^{2}(c) \times \mathbb{R}$ : a one-parameter class of extremal Lancret helices.

In fact, for any $m \in \mathbb{R}-\{0\}$, we choose in $\mathbb{S}^{2}(c)$ the circle $\beta_{m}$ with geodesic curvature

$$
\begin{equation*}
\kappa_{g}=\frac{\sqrt{c\left(m^{2}+1\right)}}{m} \tag{7}
\end{equation*}
$$

and then take $\gamma_{m}$ as the geodesic with slope $m$ in the Hopf cylinder shaped on $\beta_{m}$. Then $\gamma_{m}$ is an extremal of the total curvature in $\mathbb{S}^{2}(c) \times \mathbb{R}$.
2. Heisenberg group. Now $c=0$. Then, (6) yields $m>\sqrt{3}$, so that for any $\varphi \in(0, \pi / 6)$ there exists an extremal Lancret helix. These are geodesics in Hopf cylinders built over circles in the Euclidean plane with curvature

$$
\kappa_{g}=\frac{r}{m}\left(\sqrt{m^{4}-2 m^{2}-3}+m^{2}-1\right), \quad m=\cot \varphi
$$

- 2nd case. $\langle B, \xi\rangle=0$

Then $\frac{d}{d s}\langle B, \xi\rangle=(r-\tau)\langle N, \xi\rangle=0$ and so $\tau=r$.
What kind of curves are we talking about? This is a hard question.
Let $\psi(s)$ be the angle that $T(s)=\gamma^{\prime}(s)$ makes with $\xi$. Then we have the following characterization of the horizontality of the binormal:

Proposition 4.5 Given a curve $\gamma(s)$ in $\mathrm{E}(c, r)$, the following statements are equivalent:
(1) It has horizontal binormal;
(2) Its curvature function $\kappa(s)$, satisfies that $\kappa(s)+\psi^{\prime}(s)=0$;
(3) It is an extremal of the total curvature with $\tau=r$.

Remark 4.6 (1). It should be observed that, as a consequence of the above proposition, curves in $\mathrm{E}(c, r)$ with curvature function satisfying $\kappa(s)=-\psi^{\prime}(s)$, automatically have constant torsion $\tau=r$.
(2). Note also the existence of curves in $\mathrm{E}(c, r)$ with $\tau=r$ whose binormal is not horizontal. For example, the horizontal lifts of any curve in $\mathrm{B}^{2}(c)$.

Algorithm to construct the class of extremals with horizontal binormal:
(1) First, choose an arbitrary positive function $h(s)$, which will play the role of the curvature function of the extremal, and define the function

$$
\psi(s)=-\int_{0}^{s} h(v) d v
$$

(2) Now take the curve $\beta(s)$ in $\mathrm{B}^{2}(c)$ (unique up to rigid motions) which is determined by its arc-length function $u(s)$ and its curvature function $\kappa(s)$ given, respectively, by

$$
u(s)=\int_{0}^{s} \sin \psi(v) d v, \quad \kappa_{g}(s)=-2 r \cot \psi(s)
$$

(3) Finally, use $\beta(s)$ as a profile curve to construct its Hopf cylinder $S_{\beta}$ in $\mathrm{E}(c, r)$ and then choose in this flat surface the curve, $\gamma(s)$, with slope function (measured with respect to $\xi) \psi(s)$. We conclude that $\gamma(s)$ is an extremal of the total curvature functional on $\mathrm{E}(c, r)$ with horizontal binormal (consequently $\tau=r$ ) and curvature function $h(s)$. Moreover, all of extremals with horizontal binormal are obtained in this way.

Remark 4.7 It should be noted that this argument can be viewed as the solution of the so called "solving natural equations" for curves with horizontal binormal in $\mathrm{E}(c, r)$. In fact, we obtain explicitly the curve from its curvature function through quadratures. Certainly, it is a surprising result having in mind that the spaces $\mathrm{E}(c, r)$ have not the highest rigidity. Moreover, the next theorem measures the size of this class of curves.

We can exploit the above argument to exhibit a one-to-one correspondence between the class of convex plane curves and the family of extremals with horizontal binormal in $\mathrm{E}(c, r)$. It can be obtained from the following result

Theorem 4.8 For every convex curve $\alpha$ in the Euclidean plane, there exists a curve $\gamma$ in $\mathrm{E}(c, r)$ with the same curvature function that $\alpha$ and torsion $\tau=r$, which is an extremal, with horizontal binormal, of the total curvature action on $\mathrm{E}(c, r)$.

A relevant special case appears when $r=0$. In this case the homogeneous 3 -space is a Riemannian product $\mathrm{E}(c, 0)=\mathrm{B}^{2}(c) \times \mathbb{R}$, where (up to topology) $\mathrm{B}^{2}(c)$ is either $\mathbb{S}^{2}(c)$, if $c>0$, or $\mathbb{H}^{2}(c)$, when $c<0$. Then, as a consequence of the above argument we have the following characterization for extremals with horizontal binormal

Corollary 4.9 Any extremal $\gamma(s)$ with horizontal binormal in $\mathrm{B}^{2}(c) \times \mathbb{R}$ lies in a Hopf cylinder built on a geodesic of $\mathrm{B}^{2}(c)$. These curves can be explicitly parameterized as follows

$$
\gamma(s)=\left(\beta(u(s)), \int_{0}^{s} \cos \psi(v) d v\right)
$$

where $\beta(u)$ is a unit speed geodesic in $\mathrm{B}^{2}(c)$ with arc-length function given by

$$
u(s)=\int_{0}^{s} \sin \psi(v) d v \quad \text { and } \quad \psi(s)=-\int_{0}^{s} \kappa(v) d v
$$

$\kappa(s)$ being an arbitrary function making the role of the curvature function of $\gamma(s)$.

Observe that if $\beta$ is a geodesic in $\mathrm{B}^{2}(c)$, then $S_{\beta}$ is a totally geodesic surface in $\mathrm{E}(c, 0)=$ $\mathrm{B}^{2}(c) \times \mathbb{R}$.

On the other hand, extremals with horizontal binormal have torsion zero. Therefore, the first claim of the above corollary can be viewed as a codimension reduction result, which is the classical behavior in spaces with constant curvature. In contrast, extremals in $\mathbb{S}^{2}(c) \times \mathbb{R}$ with horizontal normal do not lie in any totally geodesic surface (see Example with label 7).

## 5 New examples

## A. Extremals in Riemannian products

In $\mathbb{H}^{2}(c) \times \mathbb{R}$ all extremals have horizontal binormal (see Corollary 4.9).
In $\mathbb{S}^{2}(c) \times \mathbb{R}$ we have two kind of extremals:
(i) Extremals with horizontal binormal (see Corollary 4.9);
(ii) Extremals with horizontal normal (see Example with label 7).

The class (ii) consists of Lancret helices having both constant curvature and torsion given, respectively, by

$$
\tau=\sqrt{\frac{c}{m^{2}+1}}, \quad \kappa=\frac{1}{m} \sqrt{\frac{c}{m^{2}+1}} .
$$

As for explicit examples, let us write $\mathbb{S}^{2}=\left\{(x, y, z) \in \mathbb{R}^{3}: x^{2}+y^{2}+z^{2}=1\right\}$ and $\mathbb{H}^{2}=\left\{(x, y, z) \in \mathbb{L}^{3}: x^{2}+y^{2}-z^{2}=-1, z>0\right\}$.

Take the geodesics $\beta(u)=(\cos u, 0, \sin u) \subset \mathbb{S}^{2}$ and $\beta(u)=(\cosh u, 0, \sinh u) \subset \mathbb{H}^{2}$.
Then apply Corollary 4.9 to find extremals.
To do that we build curves in the surface $\beta \times \mathbb{R}$ starting from a prescribed function that makes the role of curvature function.
(A.1) Circles viewed as extremals. Consider a constant function $\kappa(s)=-a$, then we have $\psi(s)=a s$ and therefore

- $\gamma(s)=\left(\cos \left(-\frac{1}{a} \cos (a s)\right), 0, \sin \left(-\frac{1}{a} \cos (a s)\right), \frac{1}{a} \sin (a s)\right):$ extremal in $\mathbb{S}^{2} \times \mathbb{R}$,
- $\gamma(s)=\left(\sinh \left(-\frac{1}{a} \cos (a s)\right), 0, \cosh \left(-\frac{1}{a} \cos (a s)\right), \frac{1}{a} \sin (a s)\right)$ : extremal in $\mathbb{H}^{2} \times \mathbb{R}$.
(A.2) Clothoids regarded as extremals. Pick up the function $\kappa(s)=-s$. We can solve the natural equations to obtain the clothoid in terms of the Fresnel integrals

$$
\alpha(s)=(u(s), t(s))=\left(\int_{0}^{s} \sin \left(v^{2} / 2\right) d v, \int_{0}^{s} \cos \left(v^{2} / 2\right) d v\right) .
$$

Since $\psi(s)=\frac{s^{2}}{2}$, we have that

- $\gamma(s)=\left(\cos \left(\int_{0}^{s} \sin \left(v^{2} / 2\right) d v\right), 0, \sin \left(\int_{0}^{s} \sin \left(v^{2} / 2\right) d v\right), \int_{0}^{s} \cos \left(v^{2} / 2\right) d v\right)$ : extremal in $\mathbb{S}^{2} \times \mathbb{R} \subset \mathbb{R}^{4}$,
- $\gamma(s)=\left(\sinh \left(\int_{0}^{s} \sin \left(v^{2} / 2\right) d v\right), 0, \cosh \left(\int_{0}^{s} \sin \left(v^{2} / 2\right) d v\right), \int_{0}^{s} \cos \left(v^{2} / 2\right) d v\right)$ : extremal in $\mathbb{H}^{2} \times \mathbb{R} \subset \mathbb{L}^{3} \times \mathbb{R}$.
(A.3) Catenaries as extremals. Now, take the function $\kappa(s)=\frac{1}{1+s^{2}}$. The unit speed curve having this curvature function is the catenary

$$
\alpha(s)=(u(s), t(s))=\left(\sqrt{1+s^{2}}, \lg \left(s+\sqrt{1+s^{2}}\right) .\right.
$$

In this case, $\psi(s)=\arctan s$, and, consequently, we obtain that the following curves

- $\gamma(s)=\left(\cos \sqrt{1+s^{2}}, 0, \sin \sqrt{1+s^{2}}, \lg \left(s+\sqrt{1+s^{2}}\right):\right.$ extremal in $\mathbb{S}^{2} \times \mathbb{R}$
- $\gamma(s)=\left(\sinh \sqrt{1+s^{2}}, 0, \cosh \sqrt{1+s^{2}}, \lg \left(s+\sqrt{1+s^{2}}\right):\right.$ extremal in $\mathbb{H}^{2} \times \mathbb{R}$.


## B. Extremals in nilmanifolds

Nilmanifolds are geometries identified with $\mathrm{E}(0, r)$ and, consequently, appear as Riemannian submersions over a Euclidean plane.

The 3 -dimensional Heisenberg group $\mathrm{N}_{r}^{3}$ is $\mathbb{R}^{3}$ endowed with the metric $d s^{2}=d x^{2}+$ $d y^{2}+\theta^{2}$, where $\theta$ is the one-form defined by

$$
\theta=d z+r(y d x-x d y)
$$

It is known that $\xi=\partial_{z}$ is an infinitesimal translation whose associated one-parameter subgroup is

$$
\left\{\varphi_{t}: \mathrm{N}_{r}^{3} \rightarrow \mathrm{~N}_{r}^{3}: t \in \mathbb{R}\right\}, \quad \varphi_{t}(x, y, z)=(x, y, z+t)
$$

The extremals are curves lying in Hopf cylinders. This key fact will be used to give explicit extremals in the Heisenberg group.

Hopf cylinders $S_{\beta}$, shaped on a plane curve $\beta(s)$, are flat surfaces which can be obtained as follows.

Choose an arc-length plane curve, $\beta(s)=(x(s), y(s))$.
Their horizontal lifts $\bar{\beta}(s)$ project down over $\beta(s)$, and then $\bar{\beta}(s)=(x(s), y(s), z(s))$, where horizontality implies that the third coordinate must satisfy $\theta\left(\bar{\beta}^{\prime}(s)\right)=0$.

Hence, horizontal lifts turn out to be determined by the solutions of the differential equation

$$
\begin{equation*}
\frac{d z}{d s}=r\left(x \frac{d y}{d s}-y \frac{d x}{d s}\right) \tag{8}
\end{equation*}
$$

Then $S_{\beta}$ is parameterized by

$$
X(s, t)=\varphi_{t}(\bar{\beta}(s))=(x(s), y(s), z(s)+t)
$$

## B. 1 Extremals with horizontal normal.

They are Lancret helices. Then we apply the corresponding algorithm, given in Section 4, to obtain

Corollary 5.1 For any $\varphi \in(0, \pi / 6)$, there exists an extremal Lancret helix in $N_{r}^{3}$, with slope $m=\cot \varphi$, in a suitable Hopf cylinder.

Lancret helices in $\mathrm{N}_{r}^{3}$ are curves, $\gamma(s)$, forming a constant angle with $\xi$ and they are precisely the set of geodesics of the Hopf cylinders of $\mathrm{N}_{r}^{3}$ (Propositon 3.1). The Lancret helices providing extremals must be geodesics of Hopf cylinders shaped on circles of the Euclidean plane. Let $\beta(s)=\left(R \cos \frac{s}{R}, R \sin \frac{s}{R}\right)$ be the unit speed circle with radius $R$ and centered at the origin. The equation (8) says that horizontal lifts of the above circles are just circular helices

$$
\bar{\beta}(s)=\left(R \cos \frac{s}{R}, R \sin \frac{s}{R}, r R s+c\right), \quad \forall c \in \mathbb{R}
$$

The corresponding Hopf cylinders can be parameterized by

$$
X: \mathbb{R}^{2} \rightarrow \mathrm{~N}^{3}, \quad X(s, t)=\left(R \cos \frac{s}{R}, R \sin \frac{s}{R}, r R s+t\right)
$$

Now, we use Proposition 4.2 to see that extremal Lancret helices in the Heisenberg group are those whose torsion satisfies

$$
\tau^{2}=r^{2}\left(4 \cos ^{2} \varphi-3\right)=\frac{r^{2}\left(m^{2}-3\right)}{m^{2}+1}
$$

and, according to (3), they are geodesics in Hopf cylinders over circles with radius

$$
R=\frac{1}{\kappa_{g}}=\frac{m}{r\left(\sqrt{\left(m^{2}+1\right)\left(m^{2}-3\right)}+m^{2}-1\right)}, \quad m=\cot \varphi .
$$

Then, for any $\varphi \in(0, \pi / 6)$ we obtain the extremal Lancret helices parameterized by
$\gamma_{m}(s)=X\left(\frac{s}{\sqrt{1+m^{2}}}, \frac{m s}{\sqrt{1+m^{2}}}\right)=\left(R \cos \frac{s}{R \sqrt{1+m^{2}}}, R \sin \frac{s}{R \sqrt{1+m^{2}}}, \frac{(r R+m) s}{\sqrt{1+m^{2}}}\right)$.
B. 2 Extremals with horizontal binormal. We use the following algorithm derived from the results in Section 4. It works on any $\mathrm{E}(c, r)$ and, in particular, on the Heisenberg group:
(i) Consider any unit speed curve $\beta(u)$ in $\mathrm{B}^{2}(c)$ with curvature function $\kappa_{g}(u)$.
(ii) Define functions $\psi(u)=-\operatorname{arcot} \frac{\kappa_{g}(u)}{2 r}$ and $s(u)=\int_{0}^{u} \csc \psi(v) d v$.
(iii) Now, in the Hopf cylinder $S_{\beta}=\pi^{-1}(\beta(u(s)))$, choose the curve $\gamma(s)$ with slope $\psi(u(s))$ to obtain an extremal with horizontal binormal in $\mathrm{E}(c, r)$.

Following this procedure we give a few more examples of critical curves belonging to the second class.
(B.2.1) An extremal built on a catenary. The curvature function

$$
\kappa_{g}(u)=\frac{1}{1+u^{2}}
$$

yields the catenary

$$
\beta(u)=(x(u), y(u))=\left(\operatorname{arcsinh}(u), \sqrt{1+u^{2}}\right) .
$$

From here and (8) one has

$$
z(u)=r\left(\sqrt{1+u^{2}} \operatorname{arcsinh}(u)-u\right) .
$$

Then a horizontal lift, $\bar{\beta}$, of $\beta$ is given by

$$
\bar{\beta}(u)=\left(\operatorname{arcsinh}(u), \sqrt{1+u^{2}}, r\left(\sqrt{1+u^{2}} \operatorname{arcsinh}(u)-u\right)\right) .
$$

Then the Hopf cylinder shaped on $\beta, S_{\beta}=\pi^{-1}(\beta)$ is parameterized by

$$
X(u, t)=\left(\operatorname{arcsinh}(u), \sqrt{1+u^{2}}, r\left(\sqrt{1+u^{2}} \operatorname{arcsinh}(u)-u\right)+t\right)
$$

Now in the flat surface $S_{\beta}$ choose the curve $\gamma(s)$ whose slope, measured with respect to the vertical axis, is given by

$$
\psi(u)=-\operatorname{arcot} \frac{1}{2 r\left(1+u^{2}\right)} .
$$

A parameterization of $\gamma(s)$ is

$$
\gamma(s)=X(u(s), t(s))
$$

where

$$
u(s)=-\int_{0}^{s} \frac{2 r\left(1+v^{2}\right)}{\sqrt{1+4 r^{2}\left(1+v^{2}\right)^{2}}} d v, \quad t(s)=\int_{0}^{s} \frac{1}{\sqrt{1+4 r^{2}\left(1+v^{2}\right)^{2}}} d v
$$

which can be expressed in terms of Elliptic functions. Then, $\gamma(s)$ provides an extremal for the total curvature action (with horizontal binormal) on the Heisenberg group (Fig. $1)$.

(a) $\beta$ is a catenary.

Figure 1. A critical curve $\gamma$ (curve in red) for the total curvature in Nil ${ }_{3}$ with $r=1$. Here, $\gamma$ lies on a Hopf cylinder shaped on a planar curve $\beta$ (curve in green), whose horizontal lift $\bar{\beta}$ is shown in black.
(B.2.2) An extremal built on a logarithmic spiral. The curvature function

$$
\kappa_{g}(u)=\frac{1}{u}
$$

yields the curve $\beta(u)=(x(u), y(u))$ where

$$
\begin{equation*}
x(u)=\frac{1}{2} u(\cos (\log (u))+\sin (\log (u))), \quad y(u)=\frac{1}{2} u(\sin (\log (u))-\cos (\log (u))) . \tag{9}
\end{equation*}
$$

Again from (8) we obtain

$$
\begin{equation*}
z(u)=\frac{r u^{2}}{4} \tag{10}
\end{equation*}
$$

Then a horizontal lift of $\beta$ is given by $\bar{\beta}(s)=(x(s), y(s), z(s))$ and the corresponding Hopf cylinder by $X(u, t)=(x(u), y(u), z(u)+t)$.

Now in $S_{\beta}$ we take the curve $\gamma(s)$ whose slope, measured with respect to the vertical axis, is given by

$$
\psi(u)=-\operatorname{arcot} \frac{1}{u} .
$$

A parameterization of $\gamma(s)$ is

$$
\gamma(s)=X(u(s), t(s)),
$$

where

$$
u(s)=-\int_{0}^{s} \frac{2 r v}{\sqrt{1+4 r^{2} v^{2}}} d v=\sqrt{1+s^{2}}, \quad t(s)=\int_{0}^{s} \frac{1}{\sqrt{1+4 r^{2} v^{2}}} d v=\operatorname{arcsinh}(s)
$$

Then, $\gamma(s)$ is also an extremal for the total curvature action (with horizontal binormal) on the Heisenberg group (Fig. 2).

(b) $\beta$ is a logarithmic spiral.

Figure 2. A critical curve $\gamma$ (curve in red) for the total curvature in $N i l_{3}$ with $r=1$. Here, $\gamma$ lies on a Hopf cylinder shaped on a planar curve $\beta$ (curve in green), whose horizontal lift $\bar{\beta}$ is shown in black.
(B.2.3) An extremal constructed out of an involute of a circle. The curvature function

$$
\kappa_{g}(u)=\frac{1}{\sqrt{u}}
$$

yieds the curve $\beta(u)=(x(u), y(u))$, where

$$
\begin{equation*}
x(u)=\frac{1}{2}(\cos (2 \sqrt{u})+\sqrt{u} \sin (2 \sqrt{u})), \quad y(u)=\frac{1}{2}(-2 \sqrt{u} \cos (2 \sqrt{u})+\sin (2 \sqrt{u})) . \tag{11}
\end{equation*}
$$

From (8) we have

$$
\begin{equation*}
z(u)=r \sqrt{u} . \tag{12}
\end{equation*}
$$

Then we find a horizontal lift of $\beta$ and the Hopf cylinder shaped on $\beta$

$$
\bar{\beta}(s)=(x(s), y(s), z(s)), \quad X(u, t)=(x(u), y(u), z(u)+t)
$$

Then, the curve $\gamma(s)$ parameterized by $\gamma(s)=X(u(s), t(s))$, with

$$
\begin{aligned}
& u(s)=-\int_{0}^{s} \frac{2 r \sqrt{v}}{\sqrt{1+4 r^{2} v}} d v=-\frac{1}{4}(2 \sqrt{s(1+4 s)}-\operatorname{arcsinh}(2 \sqrt{s})) \\
& t(s)=\int_{0}^{s} \frac{1}{\sqrt{1+4 r^{2} v}} d v=\frac{1}{2} \sqrt{1+4 s}
\end{aligned}
$$

lies in $S_{\beta}$ forming an angle $\psi$ with respect to the vertical axis given by

$$
\psi(u)=-\operatorname{arcot} \frac{1}{2 r \sqrt{u}}
$$

Therefore, $\gamma(s)$ is an extremal, with horizontal binormal, on the Heisenberg group (Fig. $3)$.


Figure 3. A critical curve $\gamma$ (curve in red) for the total curvature in $N i l_{3}$ with $r=1$. Here, $\gamma$ lies on a Hopf cylinder shaped on an involute of a circle $\beta$ (curve in green), whose horizontal lift $\bar{\beta}$ is shown in black.

## 6 Summary

The current status of this variational approach can be now described as follows:

1. In the hyperbolic space $\mathbb{H}^{3}$, there is no extremal.
2. If $c=4 r^{2}$, then $\mathrm{E}(c, r)$ has constant curvature, $r^{2}$, and, up to topology, it should be either
$2.1 \mathbb{R}^{3}$, when $r=0$, and extremals are plane curves; or
$2.2 \mathbb{S}^{3}\left(r^{2}\right)$, the round sphere with curvature $r^{2}$, when $r \neq 0$. Now, the dynamics works through curves with torsion $\pm r$, i. e., horizontal lifts via the Hopf map of curves in the corresponding round two-sphere.
3. If $c \neq 4 r^{2}$ and $r=0$, then the homogeneous 3 -space is a Riemannian product and there are two possibilities:
3.1 If $c>0$, then $\mathrm{E}(c, 0)=\mathbb{S}^{2}(c) \times \mathbb{R}$ and there are two families of extremals for the total curvature action, namely,
3.1.1 The class of curves with horizontal normal, which is a one-parameter class $\left\{\gamma_{m}, m \in \mathbb{R}-\{0\}\right\}$ of Lancret helices. This family can be described as follows: for any $m \in \mathbb{R}-\{0\}$, choose in $\mathbb{S}^{2}(c)$ the circle $\beta_{m}$ with geodesic curvature

$$
\kappa_{g}=\frac{\sqrt{c\left(m^{2}+1\right)}}{m},
$$

and then take $\gamma_{m}$ as the geodesic with slope $m$ in the Hopf cylinder shaped on $\beta_{m}$.
3.1.2 The class of curves with horizontal binormal, which automatically have torsion zero. This family consists of curves lying in Hopf cylinders built over great circles in $\mathbb{S}^{2}(c)$ (see Corollary 4.9).
3.2 If $c<0$, then $\mathrm{E}(c, 0)=\mathbb{H}^{2}(c) \times \mathbb{R}$. Now the space of extremals is made up of curves with horizontal binormal. Up to rigid motions, they are the curves lying in Hopf cylinders built over geodesics in the hyperbolic plane $\mathbb{H}^{2}(c)$ (see Corollary 4.9).
4. If $c \neq 4 r^{2}$ and $r \neq 0$, the space of extremals consists of two families of curves:
4.1 A real one-parameter class of Lancret helices (see Proposition 4.2).
4.2 The class of curves with horizontal binormal, which automatically has constant torsion $\tau= \pm r$. This class can be parameterized by the space of differentiable real functions.

| $\mathbf{c} \sim \mathbf{r}$ | r | c | Space | Extremals |
| :---: | :---: | :---: | :---: | :---: |
| $c=4 r^{2}$ | $r<0$ |  | $\mathbb{H}^{3}$ | no |
|  | $r=0$ |  | $\mathbb{R}^{3}$ | plane curves |
|  | $r>0$ |  | $\mathbb{S}^{3}\left(r^{2}\right)$ | horizontal liftings with $\tau= \pm r$ |
| $c \neq 4 r^{2}$ | $r=0$ | $c>0$ | $\mathbb{S}^{2}(c) \times \mathbb{R}$ | curves with horizontal normal; 1-parameter family of Lancret helices |
|  |  | $c>0$ | $\mathbb{S}^{2}(c) \times \mathbb{R}$ | curves with horizontal binormal and $\tau=0$ (Cor. 4.9) |
|  |  | $c<0$ | $\mathbb{H}^{2}(c) \times \mathbb{R}$ | curves with horizontal binormal and $\tau=0$ (Cor. 4.9) |
| $c \neq 4 r^{2}$ | $r \neq 0$ |  |  | curves with horizontal normal; 1-parameter family of Lancret helices (Prop. 4.2) |
|  |  |  |  | curves with horizontal binormal and $\tau=r$ (Prop. 4.5) |

Acknowledgements. MB has been partially supported by MTM2013-47828-C2-1P (MINECO-FEDER) and J. Andalucía Regional Grant P09-FQM-4496. AF has been partially supported by MINECO (Ministerio de Economía y Competitividad) and FEDER (Fondo Europeo de Desarrollo Regional) project MTM2012-34037. OJG has been partially supported by a MICINN grant MTM2010-20567 and UPV/EHU GIU13/08-UFI11/52.

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