Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1209

Differential Geometry Peñíscola 1985

Proceedings

Edited by A. M. Naveira, A. Ferrández and F. Mascaró

Springer-Verlag

HERMITIAN NATURAL DIFFERENTIAL OPERATORS

F. J. Carreras, A. Ferrández and V. Miquel
Departamento de Geometría y Topología
Universidad de Valencia
Burjasot (Valencia) SPAIN

§1.Introduction

In [Gi 1] P. Gilkey studies the invariants of Riemannian manifolds with values in forms and the same is done in [A-B-P] using a more elegant approach. Epstein [E] introduces and elucidates the concept of natural tensor as a generalization of invariant with values in forms.

In later papers, [Gi 2], [Do 1], [G-H], the concept of hermitian invariant is introduced and, following [E], the hermitian natural tensors are studied in [F-M].

In a similar way as in [E], natural differential operators are defined in [S]. In this paper we give the general notion of hermitian natural differential operator (briefly, HNDO) on almost hermitian manifolds.

The main tool (Theorem 2.10) to classify HNDO's is the same as in the Riemannian case, with the only modifications introduced by the fact that there are many hermitian natural connections (see [F-M]) and then the expression of a HNDO is not unique. The essential contribution of this paper is providing a list of examples of HNDO's and showing that there are some relations between the almost hermitian geometry and the spectrum of some of them.

In §2 we recall the necessary background and state the classification theorem for HNDO's.

In §3 and §4 we give some examples of HNDO's of type $D: \Gamma(\Lambda^p M) \longrightarrow \Gamma(\Lambda^{p+1} M)$ and $D: \Gamma(\Lambda^p M) \longrightarrow \Gamma(\Lambda^{p-1} M)$ of order one and obtain all those which are homogeneous of maximal weight when p = 0, 1.

In §5 some examples of HNDO's $D: \Gamma(\Lambda^p M) \longrightarrow \Gamma(\Lambda^p M)$ are given and we get all the homogeneous of maximal weight when p = 0. There is a HNDO, for each p, that will play a

----

Work partially supported by C.A.I.C.Y.T. 1985-87, Nº 120.

prominent role in this paper; namely, given a homogeneous hermitian regular connection D on the tangent bundle, we can define the associated D-laplacian $\Delta^D = d^D d^{D*} + d^{D*} d^D$ (d^{D*} is the adjoint of d^D (see remark 5.6)). This operator is used in §6, where we apply the techniques of [Gi 3] and [Gi 4] in order to determine the first two terms in the asymptotic expansion of Δ^D acting on 1-forms. This shows that the spectrum of Δ^D on functions and 1-forms allows us to know when an almost Kaehler or a nearly Kaehler manifold is Kaehler. In [Do 2] and [Gi 3] the Kaehler condition is found out from the spectrum of different operators acting on (p,q)-forms on a hermitian manifold. As far as we know, our results can be considered as a starting point for the study of the spectrum on almost hermitian manifolds which are not complex; and, on the other hand, as an attempt of getting at the Kaehler condition from the spectrum of real operators. For the geometry of nearly Kaehler manifolds see [GR 1,2] and interesting examples of almost Kaehler manifolds are in [C-F-G].

In a forthcoming paper we shall deal with the complex laplacian as the restriction to Hermitian manifolds of a HNDO on almost hermitian manifolds, working on the complexified tangent space.

After the completion of this paper we became aware of the recent work of Donnelly ([Do 3]), where he obtains the formula of Theorem 6.4 by using different methods.

§2. Hermitian natural tensors and hermitian natural differential operators.

2.1. Let E be a functor from the category of hermitian vector spaces (V,g,J) into itself (see [E-K] or [S]) satisfying

(i) $E(V) \subset \otimes^r V$, for any (V,g,J); and

(ii) E(V) is invariant under the action of J induced on $\otimes^r V$.

We suppose also given

(iii) an ordered basis $E(v_i)$ of E(V), for each ordered basis (v_i) of a vector space V; and

(iv) $E(fv_i) = (Ef)(E(v_i))$, for vector spaces V, W and any isomorphism $f \in Hom(V,W)$.

We denote the dual vector space $(EV)^* \subset \otimes^r V^*$ by E^*V and we consider on E^*V the restriction E^*g of the metric induced on $\otimes^r V^*$ and the restriction E^*J of the endomorphism induced by J on $\otimes^r V$.

If (v_i) is an ordered basis of V and $E(v_i) = w_i$, we define the ordered basis $E(v^i)$ of E^*V to be the ordered basis (w^k) so that $w^k(w_j) = \delta^k{}_j$

A functor E satisfying (i) will be called a functor of rank r.

2.2. Given an almost hermitian manifold (M,g,J), a functor E as in 2.1 induces riemannian bundles (EM,Eg) and (E^*M, E^*g) over M, which are riemannian subbundles of ($\otimes^r TM$,g') and ($\otimes^r T^*M$, g'), respectively, where g' is the riemannian structure induced by g. On these bundles we have the endomorphisms of fibre bundles EJ: EM $\longrightarrow$ EM and E^*J: E^*M $\longrightarrow$ E^*M which are the restrictions to EM and E^*M of the endomorphisms J': $\otimes^r TM \longrightarrow \otimes^r TM$ and J': $\otimes^r T^*M \longrightarrow \otimes^r T^*M$, respectively, induced by J. They verify $(EJ)^2 = (E^*J)^2 = (-1)^r id$, and $Eg(EJ\,\bullet, EJ\,\bullet) = Eg(\bullet,\bullet)$ and $E^*g(E^*J\,\bullet, E^*J\,\bullet) = E^*g(\bullet,\bullet)$. Furthermore, it follows from 2.1(iii) that a local coordinate system x determines unique local bases of sections $E(\partial/\partial x^i)$, $E(dx^i)$ for EM, E^*M, respectively.

2.3.DEFINITION: Let E, F be functors as in 2.1. A hermitian natural tensor field t of type (E,F) assigns to each almost hermitian manifold (M,g,J) a tensor field $t_{(M,g,J)} \in \Gamma(EM \otimes F^*M)$ such that if f: (M,g,J) $\longrightarrow$ (M',g',J') is a holomorphic ($J' \circ f_* = f_* \circ J$) isometry of M onto an open subset of M', then $f_* t_{(M,g,J)} = t_{(M',g',J')}\,|_{f(M)}$. t is said to be homogeneous of weight w if $t_{(M,c^2 g,J)} = c^w\, t_{(M,g,J)}$, c being a non-zero real number.

As it is pointed out by Epstein [E], the problem of classifying all natural tensor fields becomes very complicated; however, there is a natural concept of regularity for such tensor fields, which was introduced in [Gi 1] and [A-B-P]. In order to settle the same concept for almost hermitian manifolds, we first need the following:

2.4.DEFINITION: Let (M,g,J) be an almost hermitian manifold of real dimension 2n and let p be a point of M. A coordinate system x centered at p will be called a J-coordinate system if $(\partial/\partial x^{n+i})(p) = J(\partial/\partial x^i)(p)$, $i = 1,\dots,n$.

Then , we have

2.5.DEFINITION: A hermitian natural tensor field t of type (E,F) is said to be regular if for each almost hermitian manifold (M,g,J) and each J-coordinate system x on an open subset U of M, the coefficients of $t_{(M,g,J)}$ with respect to the local basis $E \otimes F\,(\partial/\partial x^i \otimes dx^j)$ are given by universal polynomials in

$$g_{ij},\quad \Omega_{rs},\quad g^{kl},\quad \partial^{|\alpha|}(g_{ij})/\partial x^{\alpha},\quad \partial^{|\beta|}(\Omega_{rs})/\partial x^{\beta}.$$

where α, β are multiindices and Ω is the Kaehler form defined as usual by $\Omega(X,Y) = g(JX,Y)$.

Next theorem summarizes some results of [F-M] in order to apply them for computing hermitian natural differential operators.

2.6.THEOREM: Let t be a regular hermitian natural tensor of type (E,F). Then $t_{(M,g,J)} \in \Gamma(EM \otimes F^*M)$ is the restriction of an element of $\Gamma(\otimes^r TM \otimes \otimes^s T^*M)$, ($r$ = rank E, s = rank F), which is a linear combination of the elementary monomials

$$m(\Omega,R) = \Sigma\, g_{i_1 i_2} \cdots g_{i_{2k-2} i_{2k}}\, g^{j_1 j_2} \cdots g^{j_{2l-1} j_{2l}}\, \Omega_{\alpha_1} \cdots \Omega_{\alpha_p}\, R_{\beta_1} \cdots R_{\beta_q}$$

where each α_i (resp. β_j) is a multi-index $\alpha_i = (u_1,\dots,u_{n_{\alpha_i}})$, $(\beta_j = (v_1,\dots,v_{n_{\beta_j}}))$, $\Omega_{\alpha_i} = \nabla^{n_{\alpha_i}}_{u_3 \dots u_{n_{\alpha_i}}}(\Omega)_{u_1 u_2}$, $R_{\beta_j} = \nabla^{n_{\beta_j}}_{v_5 \dots v_{n_{\beta_j}}}(R)_{v_1 \dots v_4}$, and, if $N = 2k + 2p + 4q + \Sigma_{i=1}^{p} \varepsilon_i + \Sigma_{j=1}^{q} \eta_j$ ($\varepsilon_i = n_{\alpha_i} - 2$ = number of covariant derivatives in Ω_{α_i}, $\eta_j = n_{\beta_j} - 4$ = number of covariant derivatives in R_{β_j}), we have N-s contractions of upper and lower indices (and possible alternations or symmetrizations in the upper or lower indices non-contracted). Notice that $r = 2l - N + s$. Furthermore, the weight of such a monomial is $w(m(\Omega,R)) = s - r - \Sigma\, \varepsilon_i - \Sigma\, \eta_j - 2q$.

Similarly to hermitian natural tensors we can define hermitian natural differential operators as follows:

2.7.DEFINITION: Let E, F, G, H be functors as in 2.1. A hermitian natural differential operator $\mathcal{D}$ of type (E,F,G,H) assigns to each almost hermitian manifold (M,g,J) a differential operator $\mathcal{D}_{(M,g,J)} : \Gamma(EM \otimes F^*M) \longrightarrow \Gamma(GM \otimes H^*M)$ such that if $f: (M,g,J) \longrightarrow (M',g',J')$ is a holomorphic isometry of M onto an open set of M', then $\mathcal{D}_{(M,g,J)} = f^* \mathcal{D}_{(M',g',J')}$.

Now, we are going to express the regularity condition for HNDO's. Let (M,g,J) be an almost hermitian manifold and x a local J-coordinate system on an open subset U of M. Then x determines local bases of sections $(e_\alpha)_{\alpha \in A}$, $(f^\beta)_{\beta \in B}$, $(g_\gamma)_{\gamma \in C}$ and $(h^\delta)_{\delta \in D}$ for EM, F*M, GM and H*M, respectively. Let $\mathcal{D}: \Gamma(EM \otimes F^*M) \longrightarrow \Gamma(GM \otimes H^*M)$ be a differential operator of order k. Then, locally, we can write

$$\mathcal{D}(s^\alpha{}_\beta\, e_\alpha \otimes f^\beta) = \Sigma_{r=0}^{k}\, a^{\beta\gamma i_1 \dots i_r}_{\alpha\delta}\, (\partial^r s^\alpha{}_\beta / \partial x^{i_1} \dots \partial x^{i_r})\, g_\gamma \otimes h^\delta ,$$

where the functions $a^{\beta\gamma i_1 \dots i_r}_{\alpha\delta}$ are symmetric in $i_1,\dots,i_r$.

2.8.DEFINITION: A HNDO $\mathcal{D}$ is said to be regular if the coefficients $a^{\beta\gamma i_1 \dots i_r}_{\alpha\delta}$ of $\mathcal{D}_{(M,g,J)}$, in any local J-coordinate system, are given by universal polynomials in g_{ij}, g^{kl}, Ω_{rs}, $\partial^{|\alpha|}(g_{ij})/\partial x^{\alpha}$, $\partial^{|\beta|}(\Omega_{rs})/\partial x^{\beta}$.

The weight of a HNDO is defined as in the case of hermitian natural tensors.

In order to get a general expression of a HNDO, we need the concept of hermitian natural connection, which we take from [F-M].

2.9.DEFINITION: A hermitian natural connection is a map which assigns to each almost hermitian manifold (M,g,J) a linear connection $D^{(M,g,J)}$ on TM such that if $f:(M,g,J) \longrightarrow (M',g',J')$ is a holomorphic isometry of M onto an open set of M', then

$D^{(M,g,J)}{}_X Y = D^{(M',g',J')}{}_{f_*X} f_*Y$ for every vector fields X,Y on M.

We shall say that a hermitian natural connection D is regular if, for every local J-coordinate system, the Christoffel symbols of D are universal polynomials in the components of the metric tensor, the Kaehler form, their derivatives and the components of the metric induced on T^*M.

In [F-M] a list of all the homogeneous hermitian natural connections is given.

Let E, F be functors as in 2.1. Let (M,g,J) be an almost hermitian manifold and let D be a homogeneous (of weight zero) regular hermitian natural connection on TM. Then, D induces another connection D on $EM \otimes F^*M$ in a natural way. We write $D^k = D \circ \overset{k}{\cdots} \circ D$, and define differential operators D_k making commutative the diagrams

$$\begin{tikzcd}
\Gamma(EM \otimes F^*M) \arrow[rr, "D_k"] \arrow[dr, "D^k"'] & & \Gamma(S^k(T^*M) \otimes EM \otimes F^*M) \\
& \Gamma(\otimes^k T^*M \otimes EM \otimes F^*M) \arrow[ur, "S^k \otimes 1"'] &
\end{tikzcd}$$

where $S^k : \otimes^k T^*M \longrightarrow S^k(T^*M)$ is defined by $S^k(v_1 \otimes \dots \otimes v_k) = (1/k!) \sum_{\sigma \in S} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}$.

Then, it is easy to see that the symbol $\gamma_k(D_k) \in \mathrm{Hom}(S^k(T^*M) \otimes EM \otimes F^*M,\ S^k(T^*M) \otimes EM \otimes F^*M)$ of D_k is the identity map . (For the definition of the symbol $\gamma_r(\mathcal{D})$ of an operator $\mathcal{D}$ of order r see [S]). Furthermore, D_k is a homogeneous regular HNDO of order k and weight zero.

Therefore, the proof of theorem 3.7 in [S] works also here to show the following

2.10.THEOREM: Let $\mathcal{D}$ be a HNDO of type (E,F,G,H) and order k. Then, for each homogeneous regular hermitian connection D, there exist k+1 unique hermitian natural bundle maps $t_r: \Gamma(S^r(T^*M) \otimes EM \otimes F^*M) \longrightarrow \Gamma(GM \otimes H^*M)$, $0 \leq r \leq k$, such that

$$\mathcal{D} = \Sigma_{r=0}{}^{k}\, t_r \circ D_r,$$

and the t_r are regular if and only if $\mathcal{D}$ is. Furthermore,

$$t_r = \gamma_r(\mathcal{D} - \Sigma_{l=r+1}{}^{k}\, t_l \circ D_l).$$

2.11.REMARK: A bundle map t_r as that given in 2.10 can be identified , in a natural way, with a tensor field $t_r \in \Gamma(S^r(TM) \otimes E^*M \otimes FM \otimes GM \otimes H^*M)$ and so, saying that t_r is a hermitian natural bundle map means that it is hermitian natural when considered as a tensor field.

Given $\mathcal{D} = \Sigma_{r=0}{}^{k}\, t_r \circ D_r$ as in 2.10, $\mathcal{D}$ is homogeneous of weight w if and only if each t_r is homogeneous of weight w. From 2.6, the maximal weight of t_k is a+d-b-c-k (a=rankE, b=rank F, c=rank G, d=rank H). Then we shall say that $\mathcal{D}$ has maximal weight if it is homogeneous of weight a+d-b-c-k.

§3.The set of HNDO's of type ($\mathbb{R}$, Λ^pTM, $\mathbb{R}$, Λ^{p+1}TM) and order one.

3.1. In this section we shall deal with homogeneous regular HNDO's $\mathcal{D}: \Gamma(\Lambda^p T^*M) \longrightarrow \Gamma(\Lambda^{p+1}T^*M)$ of order one and maximal weight. By 2.10 such operators have the form

$$\mathcal{D} = t_0 + t_1 \circ D, \tag{3.1}$$

where t_0 and t_1 are homogeneous regular natural tensors of weight w = p+1-p-1 =0, $t_0 \in \Gamma(\Lambda^p TM \otimes \Lambda^{p+1}T^*M)$ and $t_1 \in \Gamma(TM \otimes \Lambda^p TM \otimes \Lambda^{p+1}T^*M)$. Then, the classification of the operators $\mathcal{D}$ reduces to that of the tensors t_0 and t_1.

First of all, we study the space of tensors t_1.

3.2.PROPOSITION: The space of tensors t_1 appearing in (3.1) is spanned by the tensors t_1 whose action on $\omega \in \Gamma(T^*M \otimes \Lambda^p T^*M)$, with components $\omega_{k_1 \dots k_{p+1}}$ in an orthonormal local J-frame $\{e_i\}$, is given by

$$(t_1\omega)_{j_1\dots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \mathrm{sgn}\,(\sigma)\Big(\sum_{i_0\dots i_r} \omega_{i_0 i_0 i_1 i_1^* \dots i_r i_r^* j^*_{\sigma(1)} \dots j^*_{\sigma(s)} j_{\sigma(s+1)} \dots j_{\sigma(s+u)}}\Big)$$

$$\times\, \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \dots \Omega_{j_{\sigma(p)} j_{\sigma(p+1)}},$$

where $0 \leq r \leq (p-1)/2$, $0 \leq s,u \leq p-1$, or by

$$(t_1\omega)_{j_1\dots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \mathrm{sgn}(\sigma)\Big(\sum_{i_1\dots i_r} \omega_{j^*_{\sigma(1)} \dots j^*_{\sigma(s)} j_{\sigma(s+1)} \dots j_{\sigma(s+u)} i_1 i_1^* \dots i_r i_r^*}\Big) \times$$

$$\times\, \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \dots \Omega_{j_{\sigma(p)} j_{\sigma(p+1)}},$$

where $0 \leq r \leq p/2$, $1 \leq s \leq p+1$, $0 \leq u \leq p$, or by

$$(t_1\omega)_{j_1\dots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \mathrm{sgn}\,(\sigma)\Big(\sum_{i_1\dots i_r} \omega_{j_{\sigma(1)} \dots j_{\sigma(s)} j^*_{\sigma(s+1)} \dots j^*_{\sigma(s+u)} i_1 i_1^* \dots i_r i_r^*}\Big)$$

$$\times\, \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \dots \Omega_{j_{\sigma(p)} j_{\sigma(p+1)}},$$

where $0 \leq r \leq p/2$, $1 \leq s \leq p+1$, $0 \leq u \leq p$.

<u>Proof:</u> As we know, $w(t_1) = 0$ and then, from 2.6, $\varepsilon_i = \eta_j = q = 0$ and the space of tensors t_1 will be spanned by elementary monomials of the form

$$\Sigma^{*p}_{p+1}\; g_{i_1 i_2} \dots g_{i_{2k-1} i_{2k}}\, g^{j_1 j_2} \dots g^{j_{2l-1} j_{2l}}\, \Omega_{u_1 u_2} \dots \Omega_{u_{2b-1} u_{2b}}, \tag{3.2}$$

where $2k+2b = 2l$ and $\Sigma^*{}_{p+1}{}^{p}$ means that $2l-(p+1)$ upper indices are contracted with $2l-(p+1)$ lower ones and p upper and $p+1$ lower indices are skewsymmetrized.

The possible contractions using $g^{\cdot\cdot}$, $g_{\cdot\cdot}$ and $\Omega_{\cdot\cdot}$ give us elements of the form $g^{\cdot\cdot}$, $g_{\cdot\cdot}$, $\Omega_{\cdot\cdot}$, $\Omega^{\cdot\cdot}$, $J^{\cdot}_{\cdot}$, $\delta^{\cdot}_{\cdot}$, and, thus, the monomials (3.2) can be written as

$$\mathrm{Alt}^{p}_{p+1}\{g^{i_1 i_2} \dots g^{i_{2r-1} i_{2r}}\; \Omega^{i_{2r+1} i_{2r+2}} \dots \Omega^{i_{2s-1} i_{2s}}\, J^{i_{2s+1}}_{j_1} \dots J^{i_{2s+u}}_{j_u} \times$$

$$\times\; \delta^{i_{2s+u+1}}_{j_{u+1}} \dots \delta^{i_{p+1}}_{j_{u+v}}\, g_{j_{u+v+1} j_{u+v+2}} \dots g_{j_{u+v+2w-1} j_{u+v+2w}} \times$$

$$\times\; \Omega_{j_{u+v+2w+1} j_{u+v+2w+2}} \dots \Omega_{j_p j_{p+1}}\},$$

where $\mathrm{Alt}_{p+1}{}^{p}$ means that p upper indices and p+1 lower indices are skewsymmetrized Therefore, the elementary monomials with some $g_{\cdot\cdot}$ or more than one $g^{\cdot\cdot}$ vanish and so the monomials (3.2) can be finally written as

$$\mathrm{Alt}_{p+1}^{p}\{g^{i_0 i_1}\Omega^{i_2 i_3}\ldots\Omega^{i_{2r}i_{2r+1}}J_{j_1}^{i_{2r+2}}\ldots J_{j_s}^{i_{2r+s+1}}\delta_{j_{s+1}}^{i_{2r+s+2}}\ldots\delta_{j_{s+u}}^{i_p}\,\Omega_{j_{s+u+1}j_{s+u+2}}\ldots\Omega_{j_p j_{p+1}}\}$$

where $0 \le r \le (p-1)/2$, $0 \le s,u \le p-1$, and it is not necessary that $g^{i_0 i_1}$ appears in (3.3). Notice also that only one index of $g^{\cdot\cdot}$ can be skew-symmetrized if the monomial is not zero. Also, the indices of all the $\Omega^{\cdot\cdot}$ must be skewsymmetrized, because, if not, (3.3) represents a zero map or a map defined only on $\Lambda^{p+1}T^*M$ and not on all $T^*M \otimes \Lambda^p T^*M$.

Let t_1 be the tensor given by (3.3) (with $g^{\cdot\cdot}$ appearing in its expression) and let $\omega \in \Gamma(T^*M \otimes \Lambda^p T^*M)$. Then

$$\begin{aligned}(t_1\omega)_{j_1\ldots j_{p+1}} = \sum_{i_0\ldots i_p}\sum_{\sigma\in S_{p+1}}\mathrm{sgn}(\sigma)\sum_{\tau\in S_p}\mathrm{sgn}(\tau)\, g^{i_0 i_{\tau(1)}}\Omega^{i_{\tau(2)}i_{\tau(3)}}\ldots\Omega^{i_{\tau(2r)}i_{\tau(2r+1)}}\times\\ \times J_{j_{\sigma(1)}}^{i_{\tau(2r+2)}}\ldots J_{j_{\sigma(s)}}^{i_{\tau(2r+s+1)}}\,\delta_{j_{\sigma(s+1)}}^{i_{\tau(2r+s+2)}}\ldots\delta_{j_{\sigma(s+u)}}^{i_{\tau(p)}}\times\\ \times\Omega_{j_{\sigma(s+u+1)}j_{\sigma(s+u+2)}}\ldots\Omega_{j_{\sigma(p)}j_{\sigma(p+1)}}\,\omega_{i_0 i_1\ldots i_p}\,,\end{aligned}$$

where $0 \le r \le (p-1)/2$, $0 \le s,u \le p-1$.

If in (3.3) the $g^{\cdot\cdot}$ does not appear, we have the following two possible expressions for $t_1\omega$:

$$\begin{aligned}(t_1\omega)_{j_1\ldots j_{p+1}} = \sum_{i_0\ldots i_p}\sum_{\sigma\in S_{p+1}}\mathrm{sgn}(\sigma)\sum_{\tau\in S_p}\mathrm{sgn}(\tau)\,\Omega^{i_{\tau(1)}i_{\tau(2)}}\ldots\Omega^{i_{\tau(2r-1)}i_{\tau(2r)}}\times\\ \times J_{j_{\sigma(1)}}^{i_0}\,J_{j_{\sigma(2)}}^{i_{\tau(2r+1)}}\ldots J_{j_{\sigma(s)}}^{i_{\tau(2r+s-1)}}\delta_{j_{\sigma(s+1)}}^{i_{\tau(2r+s)}}\ldots\delta_{j_{\sigma(s+u)}}^{i_{\tau(p)}}\times\\ \times\Omega_{j_{\sigma(s+u+1)}j_{\sigma(s+u+2)}}\ldots\Omega_{j_{\sigma(p)}j_{\sigma(p+1)}}\,\omega_{i_0 i_1\ldots i_{p+1}}\,,\end{aligned}$$

or

$$(t_1\omega)_{j_1\cdots j_{p+1}} = \sum_{i_0\cdots i_p}\sum_{\sigma\in S_{p+1}} \mathrm{sgn}(\sigma)\sum_{\tau\in S_p} \mathrm{sgn}(\tau)\, \Omega^{i_{\tau(1)}i_{\tau(2)}}\cdots\Omega^{i_{\tau(2r-1)}i_{\tau(2r)}} \times$$

$$\times\; J_{j_{\sigma(1)}}^{i_{\tau(2r+1)}}\cdots J_{j_{\sigma(s)}}^{i_{\tau(2r+s)}}\,\delta_{j_{\sigma(s+1)}}^{i_0}\,\delta_{j_{\sigma(s+2)}}^{i_{\tau(2r+s+1)}}\cdots\delta_{j_{\sigma(s+u)}}^{i_{\tau(p)}} \times$$

$$\times\; \Omega_{j_{\sigma(s+u+1)}j_{\sigma(s+u+2)}}\cdots\Omega_{j_{\sigma(p)}j_{\sigma(p+1)}}\;\omega_{i_0 i_1\cdots i_p}$$

where $0 \leq r \leq p/2$, $1\leq s \leq p+1$, $0\leq u \leq p$, which, when we take a J-orthonormal frame, are the required formulas.

3.3. The tensors t_1 given in Proposition 3.2 are listed in Table I, where we have used the following notations:

$(J\omega)(X_1, \ldots, X_{p+1}) = (-1)^{p+1}\,\omega(JX_1, \ldots, JX_{p+1})$,

$(c_{11}\omega)(X_1, \ldots, X_{p-1}) = \Sigma_{i=1}^{2n}\,\omega(e_i, e_i, X_1, \ldots, X_{p-1})$,

$(c_J{}^p\,\omega)(X,Y) = \Sigma_{i_1\cdots i_{(p-1)/2}}\,\omega(X, Y, e_{i_1}, Je_{i_1}, \ldots, e_{i_{(p-1)/2}}, Je_{i_{(p-1)/2}})$ when $p+1$ is even,

$(J^s\omega)(X_1, \ldots, X_{p+1}) = (-1)^s\,\omega(X_1, JX_2, \ldots, JX_{s+1}, X_{s+2}, \ldots, X_{p+1})$,

$(J_1{}^s\omega)(X_1, \ldots, X_{p+1}) = (-1)^s\,\omega(JX_1, \ldots, JX_s, X_{s+1}, \ldots, X_{p+1})$,

$(c_J{}^r\omega)(X_1, \ldots, X_{p+1-r}) = \Sigma_{i_1\cdots i_{r/2}}\,\omega(X_1, \ldots, X_{p+1-r}, e_{i_1}, Je_{i_1}, \ldots, e_{i_{r/2}}, Je_{i_{r/2}})$.

3.4. In Table II we have listed the expressions of some of the operators $t_1{\circ}D$, when they have a simple form. We have used the following notations: if D is a linear connection in TM, the operators $d^D : \Gamma(\Lambda^pT^*M) \longrightarrow \Gamma(\Lambda^{p+1}T^*M)$ and $\delta^D : \Gamma(\Lambda^pT^*M) \longrightarrow \Gamma(\Lambda^{p-1}T^*M)$ are defined by

$$d^D\omega = \mathrm{Alt}(D\omega) \qquad\qquad \delta^D\omega = -\,c_{11}(D\omega).$$

They satisfy $(d^D)^2 = 0 = (\delta^D)^2$ if and only if D is symmetric (and then $d^D = d = d^\nabla$, and $\delta^D = \delta = \delta^\nabla$, where ∇ is the Levi-Civita connection). Moreover, d^D is a skew-derivation.

Now we study the space of tensors t_0.

3.5.PROPOSITION: The space of tensors t_0 in (3.1) is spanned by those tensors t_0 whose action on an $\omega\in\Gamma(\Lambda^pT^*M)$, with components $\omega_{k_1\cdots k_p}$ with respect to an orthonormal J-frame $\{e_i\}$ is given by one of the following expresions:

$$(t_0\omega)_{j_1\cdots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \operatorname{sgn}(\sigma) \sum_{i_1\cdots i_r} \omega_{i_1 i_1^* \cdots i_r i_r^* j^*_{\sigma(1)}\cdots j^*_{\sigma(s)} j_{\sigma(s+1)}\cdots j_{\sigma(s+u)}} \times$$

$$\times\ \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p-3)} j_{\sigma(p-2)}} \nabla_{j_{\sigma(p-1)}} \Omega_{j_{\sigma(p)} j_{\sigma(p+1)}},$$

where $2r+s+u = p$;

$$(t_0\omega)_{j_1\cdots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \operatorname{sgn}(\sigma) \sum_{i_1\cdots i_r i_p} \omega_{i_1 i_1^* \cdots i_r i_r^* j^*_{\sigma(1)}\cdots j^*_{\sigma(s)} j_{\sigma(s+1)}\cdots j_{\sigma(s+u)} i_p} \times$$

$$\times\ \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p-2)} j_{\sigma(p-1)}} \nabla_{i_p} \Omega_{j_{\sigma(p)} j_{\sigma(p+1)}},$$

where $2r+s+u = p-1$;

$$(t_0\omega)_{j_1\cdots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \operatorname{sgn}(\sigma) \sum_{i_1\cdots i_r i_p} \omega_{i_1 i_1^* \cdots i_r i_r^* j^*_{\sigma(1)}\cdots j^*_{\sigma(s)} j_{\sigma(s+1)}\cdots j_{\sigma(s+u)} i_p} \times$$

$$\times\ \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p-2)} j_{\sigma(p-1)}} \nabla_{j_{\sigma(p)}} \Omega_{i_p j_{\sigma(p+1)}},$$

$$(t_0\omega)_{j_1\cdots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \operatorname{sgn}(\sigma) \sum_{i_1\cdots i_r i_{p-1} i_p} \omega_{i_1 i_1^* \cdots i_r i_r^* j^*_{\sigma(1)}\cdots j^*_{\sigma(s)} j_{\sigma(s+1)}\cdots j_{\sigma(s+u)} i_{p-1} i_p} \times$$

$$\times\ \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p-1)} j_{\sigma(p)}} \nabla_{i_{p-1}} \Omega_{i_p j_{\sigma(p+1)}},$$

$$(t_0\omega)_{j_1\cdots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \operatorname{sgn}(\sigma) \sum_{i_1\cdots i_r i_{p-1} i_p} \omega_{i_1 i_1^* \cdots i_r i_r^* j^*_{\sigma(1)}\cdots j^*_{\sigma(s)} j_{\sigma(s+1)}\cdots j_{\sigma(s+u)} i_{p-1} i_p} \times$$

$$\times\ \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p-1)} j_{\sigma(p)}} \nabla_{j_{\sigma(p+1)}} \Omega_{i_{p-1} i_p},$$

$$(t_0\omega)_{j_1\cdots j_{p+1}} =$$

$$\sum_{\sigma\in S_{p+1}} \operatorname{sgn}(\sigma) \sum_{i_1\cdots i_r i_{p-2} i_{p-1} i_p} \omega_{i_1 i_1^* \cdots i_r i_r^* j^*_{\sigma(1)}\cdots j^*_{\sigma(s)} j_{\sigma(s+1)}\cdots j_{\sigma(s+u)} i_{p-2} i_{p-1} i_p} \times$$

$$\times\ \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p)} j_{\sigma(p+1)}} \nabla_{i_{p-2}} \Omega_{i_{p-1} i_p},$$

$$(t_0\omega)_{j_1\cdots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \mathrm{sgn}(\sigma) \sum_{i_1\cdots i_r} \omega_{i_1 i_1^*\cdots i_r i_r^* j^*_{\sigma(1)}\cdots j^*_{\sigma(s)} j_{\sigma(s+1)}\cdots j_{\sigma(s+u)}} \times$$
$$\times\, \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p-1)} j_{\sigma(p)}} \delta\Omega_{j_{\sigma(p+1)}},$$

where $2r+s+u = p$;

$$(t_0\omega)_{j_1\cdots j_{p+1}} = \sum_{\sigma\in S_{p+1}} \mathrm{sgn}(\sigma) \sum_{i_1\cdots i_r i_p} \omega_{i_1 i_1^*\cdots i_r i_r^* j^*_{\sigma(1)}\cdots j^*_{\sigma(s)} j_{\sigma(s+1)}\cdots j_{\sigma(s+u)} i_p} \times$$
$$\times\, \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p)} j_{\sigma(p+1)}} \delta\Omega_{i_p}$$

and the expressions obtained from the above list by adding a * in one, two or three of the indices in $\nabla\Omega$ or $\delta\Omega$.

<u>Proof:</u> From 3.1 we have that $w(t_0) = 0$, and, then, from 2.6, $q = 0 = \eta_j$, and there exists one index i such that $\varepsilon_i = 1$ and $\varepsilon_k = 0$ for any $k \neq i$. Therefore, the space of tensors t_0 is spanned by elementary monomials of the form

$$m(\Omega,\nabla\Omega) = \Sigma^{*p}_{p+1}\, g_{i_1 i_2}\cdots g_{i_{2k-1} i_{2k}}\, g^{j_1 j_2}\cdots g^{j_{2l-1} j_{2l}}\, \Omega_{u_1 u_2}\cdots \Omega_{u_{2b-1} u_{2b}}\, \nabla_v \Omega_{rs},$$

where $2k + 2b + 3 = 2l + 1$ and $\Sigma^*{}_{p+1}{}^p$ means that p upper indices and p+1 lower indices are skew-symmetrized, and 2 l-p upper indices are contracted with 2 l-p lower ones. The possible contractions using $g^{\cdot\cdot}$, $g_{\cdot\cdot}$, $\Omega_{\cdot\cdot}$ and $\nabla_\cdot\Omega_{\cdot\cdot}$ yield (up to sign) elements of the form

$g^{\cdot\cdot}$, $g_{\cdot\cdot}$, $\Omega^{\cdot\cdot}$, $\Omega_{\cdot\cdot}$, $J^{\cdot}_{\cdot}$, $\delta^{\cdot}_{\cdot}$, $\qquad \nabla_\cdot\Omega_{\cdot\cdot}$, $\nabla^\cdot\Omega_{\cdot\cdot}$, $\nabla_\cdot J^{\cdot}_{\cdot}$, $\nabla^\cdot J^{\cdot}_{\cdot}$, $\nabla_\cdot\Omega^{\cdot\cdot}$, $\nabla^\cdot\Omega^{\cdot\cdot}$,

$\nabla_{\cdot *}\Omega_{\cdot\cdot}$, $\nabla_\cdot\Omega_{\cdot *\cdot}$, $\nabla_{\cdot *}\Omega_{\cdot *\cdot}$, $\nabla^{\cdot *}\Omega_{\cdot\cdot}$, $\nabla^\cdot\Omega_{\cdot *\cdot}$, $\nabla^{\cdot *}\Omega_{\cdot *\cdot}$,

$\nabla_{\cdot *}J^{\cdot}_{\cdot}$, $\nabla_\cdot J^{\cdot}_{\cdot *}$, $\nabla_{\cdot *}J^{\cdot}_{\cdot *}$, $\nabla^{\cdot *}J^{\cdot}_{\cdot}$, $\nabla^\cdot J^{\cdot}_{\cdot *}$, $\nabla^{\cdot *}J^{\cdot}_{\cdot *}$,

$\nabla_{\cdot *}\Omega^{\cdot\cdot}$, $\nabla_\cdot\Omega^{\cdot *\cdot}$, $\nabla_{\cdot *}\Omega^{\cdot *\cdot}$, $\nabla^{\cdot *}\Omega^{\cdot\cdot}$, $\nabla^\cdot\Omega^{\cdot *\cdot}$, $\nabla^{\cdot *}\Omega^{\cdot *\cdot}$, $\qquad \delta\Omega_\cdot$, $\delta J^\cdot$, $\delta\Omega_{\cdot *}$, $\delta J^{\cdot *}$,

where $\bullet^* = J\bullet$. The contractions not listed above can be reduced to these by the symmetries of $\nabla\Omega$; that is, $\nabla_i\Omega_{jk} = -\nabla_i\Omega_{kj}$ and $\nabla_i\Omega_{j*k*} = -\nabla_i\Omega_{jk}$ (see [G-H]). Then, since $g^{\cdot\cdot}$ and $g_{\cdot\cdot}$ cannot appear in a nonzero monomial (because their skew-symmetrizations are zero), the only non-vanishing elementary monomials are (up to sign):

$$\mathrm{Alt}^{p}_{p+1}\{V^{i_1\cdots i_p}_{j_1\cdots j_{p-2}}\nabla_{j_{p-1}}\Omega_{j_p j_{p+1}}\},\qquad \mathrm{Alt}^{p}_{p+1}\{V^{i_1\cdots i_{p-1}}_{j_1\cdots j_{p-1}}\nabla^{i_p}\Omega_{j_p j_{p+1}}\},$$

$$\mathrm{Alt}^{p}_{p+1}\{V^{i_1\cdots i_{p-1}}_{j_1\cdots j_{p-1}}\nabla_{j_p}J^{i_p}_{j_{p+1}}\},\qquad \mathrm{Alt}^{p}_{p+1}\{V^{i_1\cdots i_{p-2}}_{j_1\cdots j_p}\nabla^{i_{p-1}}J^{i_p}_{j_{p+1}}\},$$

$$\mathrm{Alt}^{p}_{p+1}\{V^{i_1\cdots i_{p-2}}_{j_1\cdots j_p}\nabla_{j_{p+1}}\Omega^{i_{p-1}i_p}\},\qquad \mathrm{Alt}^{p}_{p+1}\{V^{i_1\cdots i_{p-3}}_{j_1\cdots j_{p+1}}\nabla^{i_{p-2}}\Omega^{i_{p-1}i_p}\},$$

$$\mathrm{Alt}^{p}_{p+1}\{V^{i_1\cdots i_p}_{j_1\cdots j_p}\,\delta\Omega_{j_{p+1}}\},\qquad \mathrm{Alt}^{p}_{p+1}\{V^{i_1\cdots i_{p-1}}_{j_1\cdots j_{p+1}}\,\delta J^{i_p}\},$$

and the monomials obtained from this list when we add * to one, two or three of the indices in $\nabla\Omega$, ∇J, $\delta\Omega$ or δJ, where the tensor V is given by

$$V^{i_1\cdots i_k}_{j_1\cdots j_l} = \Omega^{i_1 i_2}\cdots\Omega^{i_{2r-1}i_{2r}}J^{i_{2r+1}}_{j_1}\cdots J^{i_{2r+s}}_{j_s}\delta^{i_{2r+s+1}}_{j_{s+1}}\delta^{i_k}_{j_{s+u}}\Omega_{j_{s+u+1}j_{s+u+2}}\cdots\Omega_{j_{l-1}j_l}.$$

The proposition follows by taking an orthonormal J-frame.

<u>3.6.</u> We list in Table III the tensors t_0 given in Proposition 3.5, when p = 0, 1. If $\omega \in \Gamma(T^*M)$, ω^* will denote the image of ω by the canonical isomorphism between T*M and TM given by the Riemannian metric g.

§<u>4. The set of HNDO's of type ($\mathbf{R}$, $\Lambda^p TM$, $\mathbf{R}$, $\Lambda^{p-1}TM$) and order one.</u>

<u>4.1</u> In this section we shall deal with homogeneous regular HNDO's $\mathcal{D}: \Gamma(\Lambda^p T^*M) \longrightarrow \Gamma(\Lambda^{p-1}T^*M)$ of order one and maximal weight. As we know, such a $\mathcal{D}$ has a general form

$$\mathcal{D} = t_0 + t_1 \cdot D, \tag{4.1}$$

where t_0 and t_1 are homogeneous regular hermitian natural tensors of weight $w=p-1-(p+1) = -2$, $t_0 \in \Gamma(\Lambda^p TM \otimes \Lambda^{p-1}T^*M)$ and $t_1 \in \Gamma(TM \otimes \Lambda^p TM \otimes \Lambda^{p-1}T^*M)$.

Similarly to §3, we have

<u>4.2.PROPOSITION:</u> The space of tensors t_1 in (4.1) is spanned by those tensors whose action on an $\omega \in \Gamma(T^*M \otimes \Lambda^p T^*M)$ (with components $\omega_{k_1\cdots k_{p+1}}$ with respect to an orthonormal local J-frame (e_i)) is given by

$$(t_1\omega)_{j_1\cdots j_{p-1}} = \sum_{\sigma\in S_{p-1}}\mathrm{sgn}(\sigma)\sum_{i_0\cdots i_r}\omega_{i_0 i_0 i_1 i_1^*\cdots i_r i_r^* j^*_{\sigma(1)}\cdots j^*_{\sigma(s)} j_{\sigma(s+1)}\cdots j_{\sigma(s+u)}}\times$$
$$\times\ \Omega_{j_{\sigma(s+u+1)}j_{\sigma(s+u+2)}}\cdots\Omega_{j_{\sigma(p-2)}j_{\sigma(p-1)}},$$

where $0 \leq r \leq (p-1)/2$; $0 \leq s,u \leq p-1$; or by the expressions like the other two given in 3.2 but changing S_{p+1} by S_{p-1} and with $r \geq 1$.

4.3.PROPOSITION: The space of tensors t_0 in (4.1) is spanned by those tensors whose action on an $\omega \in \Gamma(\Lambda^p T^*M)$ is given by the same formulas as in 3.5, but changing p+1 by p-1.

Observe that the change of p+1 by p-1 can also be applied to 3.2 to get 4.2. Therefore, Table I can be considered as a list of generators of tensors t_1 in 4.2, changing p+1 by p-1 (for which, in addition, formulas with no Ω are to be deleted, and in formulas with some Ω, one of these should be deleted). The generators of the space of tensors t_0, in 4.3, when $p \leq 2$, are given in Table IV.

§5. The set of HNDO's of type ($\mathbb{R}$, $\Lambda^p TM$, $\mathbb{R}$, $\Lambda^p TM$) and order two.

5.1. In this section we will consider homogeneous regular HNDO's $D : \Gamma(\Lambda^p T^*M) \longrightarrow \Gamma(\Lambda^p T^*M)$ of order two and maximal weight. Again, by *2.10*, these operators can be written as

$$(5.1) \qquad D = t_0 + t_1 \cdot D + t_2 \cdot D_2 ,$$

where t_0, t_1 and t_2 are homogeneous regular hermitian natural tensors of weight $w = p-p-2 = -2$.

As in §§3 and 4, we have the following results:

5.2.PROPOSITION: The space of tensors $t_2 \in \Gamma(S^2T^*M \otimes \Lambda^p TM \otimes \Lambda^p T^*M)$, appearing in (5.1), is spanned by those tensors whose action on an $\omega \in \Gamma(S^2T^*M \otimes \Lambda^p T^*M)$ (with components $\omega_{k_1 \dots k_{p+2}}$ with respect to an orthonormal local J-frame $\{e_i\}$) is given by one of the following expressions:

$$(t_2 \omega)_{j_1 \dots j_p} = \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) \sum_{i_0 \dots i_r} \omega_{i_0 i_0 i_1 i_1^* \dots i_r i_r^* j_{\sigma(1)}^* \dots j_{\sigma(s)}^* j_{\sigma(s+1)} \dots j_{\sigma(s+u)}} \times$$
$$\times \quad \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \dots \Omega_{j_{\sigma(p-1)} j_{\sigma(p)}} ,$$

$$(t_2 \omega)_{j_1 \dots j_p} = \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) \sum_{i_0 \dots i_r} \omega_{i_0 i_1 i_0 i_1^* i_2 i_2^* \dots i_r i_r^* j_{\sigma(1)}^* \dots j_{\sigma(s)}^* j_{\sigma(s+1)} \dots j_{\sigma(s+u)}} \times$$
$$\times \quad \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \dots \Omega_{j_{\sigma(p-1)} j_{\sigma(p)}} ,$$

$$(t_2\omega)_{j_1\cdots j_p} = \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) \sum_{i_0\cdots i_r} \omega_{j^*_{\sigma(1)} i_0 i_1 i^*_1 \cdots i_r i^*_r i_0 j^*_{\sigma(2)} \cdots j^*_{\sigma(s)} j_{\sigma(s+1)} \cdots j_{\sigma(s+u)}} \times$$
$$\times \quad \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p-1)} j_{\sigma(p)}} ,$$

$$(t_2\omega)_{j_1\cdots j_p} = \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) \sum_{i_0\cdots i_r} \omega_{j_{\sigma(s+1)} i_0 i_1 i^*_1 \cdots i_r i^*_r j^*_{\sigma(1)} \cdots j^*_{\sigma(s)} i_0 j_{\sigma(s+2)} \cdots j_{\sigma(s+u)}} \times$$
$$\times \quad \Omega_{j_{\sigma(s+u+1)} j_{\sigma(s+u+2)}} \cdots \Omega_{j_{\sigma(p-1)} j_{\sigma(p)}} .$$

<u>5.3.PROPOSITION:</u> The space of tensors $t_1 \in \Gamma(TM \otimes \Lambda^p TM \otimes \Lambda^p T^*M)$, appearing in (5.1) is spanned by those tensors whose action on an $\omega \in \Lambda(T^*M \otimes \Lambda^p T^*M)$ is given by one of the expressions in Proposition 3.5, with the following slight modifications:

(a) $\sigma \in S_p$, and

(b) we can also take the contraction in the first two indices of ω in all the expressions.

In general, the expressions of the generators of the space of tensors $t_0 \in \Gamma(\Lambda^p TM \otimes \Lambda^p T^*M)$ are very complicated, and we shall not write them here. We shall consider only the case p = 0.

<u>5.4.PROPOSITION:</u> The space of tensors $t_0 \in \Gamma(\Lambda^0 TM \otimes \Lambda^0 T^*M)$ in (5.1) is spanned by the hermitian natural functions

$$\tau,\ \tau^*,\ \|\nabla\Omega\|^2,\ \|d\Omega\|^2,\ \|\delta\Omega\|^2,\ \|N\|^2,$$

where N is the Nijenhuis tensor, τ is the scalar curvature and τ^* is, as usually, defined by $\tau^* = (1/2)\, \Sigma_{i,j=1}{}^{2n}\, R_{ii^*jj^*}$.

<u>Proof:</u> Since $w(t_0) = -2$, we have , from 2.6, $q = 0 = \eta_j$ and then, either there are indices i,k such that $\varepsilon_i = \varepsilon_k = 1$ and $\varepsilon_l = 0$ for $l \neq i,k$, or there is an index i such that $\varepsilon_i = 2$ and $\varepsilon_l = 0$ for every $l \neq i$; then the result follows from [G-H, Theorem 7.1].

From 5.1 and 5.4 we have

<u>5.5.COROLLARY:</u> Every homogeneous regular HNDO $\mathcal{D}$ of maximal weight acting on functions is of the form

$$\mathcal{D}f = a\,\Delta f + b\,\delta J(f) + c\,(J\,\delta J)(f) + (d_1\,\tau + d_2\,\tau^* + d_3\,\|\nabla\Omega\|^2 + d_4\,\|d\Omega\|^2 + d_5\,\|\delta\Omega\|^2 + d_6\,\|N\|^2)\,f,$$

where a, b, c, d_i are real numbers and Δ is the ordinary laplacian.

<u>Proof:</u> When $p = 0$ tensors t_2 in 5.2 reduce to $t_2(\omega) = c_{11}\omega$, and the tensors t_1 in 5.3 reduce to $t_1(\omega) = \omega(\delta J)$ or $t'_1(\omega) = \omega(J\,\delta J)$. Then, taking $D = \nabla$, we get $(t_2 \cdot D_2)(f) = c_{11}(\nabla^2 f) = g^{ij}(\nabla^2_{ij} f) = \Delta f$, and $t_1(\nabla f) = df(\delta J)$ and $t'_1(\nabla f) = df(J\,\delta J)$, and the result follows from here and proposition 5.4.

<u>5.6.</u> For compact manifolds we consider the scalar product on $\Gamma(\Lambda T^*M)$ given by

$$(\alpha,\beta) = \int_M \alpha \wedge * \beta \,.$$

For this scalar product we have that if D is a HNDO as in 5.5 then D is selfadjoint if and only if $b = c = 0$ (see [McK-S], p.46).

An interesting homogeneous regular HNDO acting on p-forms is the D-laplacian, defined by

$$\Delta^D = d^D d^{D*} + d^{D*} d^D \,,$$

where D is a metric homogeneous regular hermitian natural connection on TM, and d^{D*} is the adjoint of d^D with respect to the above scalar product. It is clearly elliptic and selfadjoint, and we shall study its spectral asymptotic expansion in §6.

<u>§6. The asymptotic expansion of Δ^D acting on 1-forms.</u>

Within this section M will be a compact almost-hermitian manifold of real dimension $m = 2n$. First, we recall some well known facts.

Let E be a vector bundle over M, and $L : \Gamma(E) \longrightarrow \Gamma(E)$ a second order differential operator with symbol given by the metric tensor. Let E_x be the fibre of E over a point $x \in M$. Let us choose a smoth fibre metric $\langle\ ,\ \rangle$ on E, and let $L^2(E)$ be the completion of $\Gamma(E)$ with respect to the global integrated inner product $(\ ,\ )$. For $t > 0$, $\exp(-tL)$: $L^2(E) \longrightarrow L^2(E)$ is an infinitely smoothing operator of trace class. Let $K(t,x,y,L) : E_y \longrightarrow E_x$ be the kernel of $\exp(-tL)$. If $x=y$, K has an asymptotic expansion as $t \to 0^+$, of the form

$$K(t,x,x,L) \sim (4\pi t)^{-m/2} \Sigma_{k=0}^{\infty} t_k H_k(x,L),$$

where the $H_k(x,L)$ are endomorphisms of E_x.

If L is selfadjoint, let $\{\lambda_i, \theta_i\}_{i \in \mathbb{Z}^+}$ be a spectral resolution of L into a complete orthonormal basis of eigenvalues λ_i and eigensections θ_i. Then,

$$\mathrm{tr}\, K(t,x,x,\mathcal{L}) = \Sigma_i \exp(-t\,\lambda_i) \langle\theta_i,\theta_i\rangle_x \sim (4\pi\, t)^{-m/2} \Sigma_{k=0}\, a_k(x,\mathcal{L})t^k,$$

where $a_k(x,\mathcal{L}) = \mathrm{tr}\, H_k(x,\mathcal{L})$. Now, if we integrate on M, we have

$$\Sigma_i \exp(-t\,\lambda_i) \sim (4\pi\, t)^{-m/2} \Sigma_{k=0}^{\infty} a_k(\mathcal{L})\, t^k, \qquad \text{with } a_k(\mathcal{L}) = \int_M a_k(x,\mathcal{L}).$$

<u>6.1.THEOREM ([Gi 3,4]):</u> Let $\underline{D}$ be a connection on E, and ∇ the Levi-Civita connection, and denote also by $\underline{D}$ the connection induced by $\underline{D}$ and ∇ on $T^*M \otimes E$. Let $\mathcal{L}_{\underline{D}}$ be the reduced Laplacian defined by $\mathcal{L}_{\underline{D}}s = -g^{ij}\,\underline{D}^2_{ij}s$ for every $s \in \Gamma(E)$. If $\underline{D}$ is the unique connection on E such that $\mathcal{E} = \mathcal{L}_{\underline{D}} - \mathcal{L} : \Gamma(E) \longrightarrow \Gamma(E)$ is a 0^{th} order operator, then we have

(a) $H_0 = I$

(b) $H_1 = (1/6)\,(-\tau\, I + 6\,\mathcal{E})$.

Now, let E, F be functors as in 2.1, and let $\mathcal{L}$ be a homogeneous regular HNDO of type (E,F,E,F) of order two with symbol given by the metric tensor, then it has maximal weight -2, and, according with theorem 2.10, it can be written in the form

$$\mathcal{L} = -g^{ij}\,\nabla^2_{ij} + t_1 \cdot \nabla + t_0\,,$$

where ∇ is the connection induced on $EM \otimes F^*M$ by the Levi-Civita connection. Next, we compute $\mathcal{E}$ in terms of t_0 and the tensor $\underline{B} = \underline{D} - \nabla : TM \otimes EM \otimes F^*M \longrightarrow EM \otimes F^*M$.

<u>6.2.PROPOSITION:</u> For every $s \in \Gamma(EM \otimes F^*M)$,

$$\mathcal{E}\, s = -g^{ij}\,(\nabla_i(\underline{B})_j s + \underline{B}_i\underline{B}_j s) - t_0 s \tag{6.2}$$

and the connection $\underline{D}$ on $EM \otimes F^*M$ such that $\mathcal{E}$ is a 0^{th} order operator is given by the linear fibre bundle map $\underline{B} = \underline{D} - \nabla$, defined by

$$\underline{B}_X s = -(1/2)\, t_1\,(X^b \otimes s) \tag{6.3}$$

for every $X \in \Gamma(TM)$ and $s \in \Gamma(EM \otimes F^*M)$, where ^{b}: $TM \longrightarrow T^*M$ is the canonical isomorphism induced by the metric.

<u>Proof:</u> From the definition of $\mathcal{L}_{\underline{D}}$ we have

$$\mathcal{L}_{\underline{D}}\, s = -g^{ij}\,(\underline{D}_i\underline{D}_j s - \underline{D}_{\nabla_i j} s) = -g^{ij}\,(\nabla^2_{ij}s) - 2\,g^{ij}\,\underline{B}_i(\nabla_j s) - g^{ij}\,(\nabla_i(\underline{B})_j s + \underline{B}_i\underline{B}_j s).$$

Since, in its standard form, $\mathcal{L}_{\underline{D}}$ can be written as

$$\mathcal{L}_{\underline{D}} = -g^{ij}\,\nabla^2_{ij} + t^{\underline{D}}_1 \cdot \nabla + t^{\underline{D}}_0,$$

we have that

$$t^{\underline{D}}_1 (\alpha \otimes s) = -2\, g^{ij}\, \underline{B}_i(\alpha_j\, s) = -2\, \underline{B}_{\alpha^*} s$$

for $s \in \Gamma(EM \otimes F^*M)$ and $\alpha \in \Gamma(T^*M)$. On the other hand, $\gamma_2(L) = \gamma_2(L_{\underline{D}})$, so that the condition that E be a 0^{th} order operator is equivalent (by Theorem 2.10) to $t^{\underline{D}}_1 = t_1$; whence (6.3) follows. Then, we have $E = t^{\underline{D}}_0 - t_0 = -g^{ij}(\nabla_i(\underline{B})_j s + \underline{B}_i\underline{B}_j s) - t_0(s)$.

Next we study the spectral asymptotic expansion of the operator Δ^D, defined in 5.6. First we determine the operator d^{D*}.

<u>6.3.PROPOSITION:</u> The adjoint operator of d^D, with respect to the inner product given in 5.6, is $d^{D*} = \delta^D - \iota_{\bar{B}}$, where $\bar{B} = \Sigma_{i=1}{}^{2n} B_i i$.

<u>Proof:</u> From the fact that d^D is a skew-derivation and that $\delta^D = (-1)^{np+n+1} * d^D *$ on p-forms, it follows that, if $\alpha \in \Gamma(\Lambda^p(T^*M))$, then

$$\int_M d^D\alpha \wedge *\beta = \int_M d^D (\alpha \wedge *\beta) + \int_M \alpha \wedge *\delta^D\beta.$$

On the other hand, we have

$$d^D (\alpha \wedge *\beta) = d (\alpha \wedge *\beta) + \bar{B}^b \wedge (\alpha \wedge *\beta),$$

and, for $X \in \chi(M)$ and $\mu \in \Gamma(\Lambda^r(T^*M))$,

$$* \iota_X\mu = (-1)^{2n-1} (*\mu) \wedge X^b = (-1)^{r+1} X^b \wedge *\mu,$$

whence,

$$\bar{B}^b \wedge \alpha \wedge *\beta = (-1)^p \alpha \wedge \bar{B}^b \wedge *\beta = \alpha \wedge (\iota_{\bar{B}}\beta).$$

Then,

$$d^D (\alpha \wedge *\beta) = d (\alpha \wedge *\beta) - \alpha \wedge *\iota_{\bar{B}}\beta,$$

and, if we integrate,

$$\int_M d^D\alpha \wedge *\beta = -\int_M \alpha \wedge \iota_{\bar{B}}\beta + \int_M \alpha \wedge *\delta^D\beta = \int_M \alpha \wedge *(\delta^D - \iota_{\bar{B}})\beta.$$

Then, $d^{D*} = \delta^D - \iota_{\bar{B}}$.

Let $\Delta^D{}_p$ the D-laplacian Δ^D acting on p-forms (in particular, $\Delta^D{}_0 = \Delta$, the ordinary real laplacian). Then,

<u>6.4.THEOREM:</u> Let M be a compact almost hermitian manifold of real dimension 2n. Then,

$$a_1(\Delta^D{}_1) = \int_M (-((n+3)/3)\,\tau + A),$$

where $A = -(1/2)\ \Sigma\, T_{ijk}\, T_{ikj}$, T being the torsion tensor of D.

Proof: It follows in the same way as for the ordinary laplacian (see, for instance, [P]), that the action of Δ^D on a 1-form μ is given by

$$(6.4) \qquad (\Delta^D\mu)(v) = -(g^{ij} D^2{}_{ij}\,\mu)(v) - g^{ij}(R^D{}_{iv}\,\mu)_j - g^{ij}(D_{T(i,v)}\,\mu)_j -$$
$$- (D_{\bar{B}}\,\mu)(v) + (D_v\,\mu)(\bar{B}) - (\nabla_v\,\mu)(\bar{B}) + \mu(\nabla_v\bar{B}),$$

where v is a vector field on M. On the other hand, if we write $D_X\,\mu = \nabla_X\,\mu + \tilde{B}_X\,\mu$, then, $\tilde{B}$ is related with B by

$$(6.5) \qquad \tilde{B}_X\,\mu = -\mu\,(B(X,\bullet)).$$

(Notice that $\tilde{B}$ is analogous to the tensor $\underline{B}$ defined before 6.2, however we change the notation because we use $\tilde{B}$ for the connection D on T*M induced by the connectio n D on TM, and it is different of the connection $\underline{D}$ given in 6.1).

Now, for each point $x \in M$ we can consider a local frame $\{e_i\}$ around x which is orthonormal at x and radially parallel from x, so that $(\nabla_i(e_j))_x = 0$. Then it follows from (6.4) that, at x,

$$(6.6) \qquad \Delta^D\,\mu = -\nabla^2{}_{ii}\,\mu - (\nabla_i\tilde{B})_i\,\mu - 2\,\tilde{B}_i(\nabla_i\,\mu) - \tilde{B}_i(\tilde{B}_i\,\mu) + \mu(R^D{}_{i\bullet}i)$$
$$- (\nabla_{T(i,\bullet)}\mu)i - (\tilde{B}_{T(i,\bullet)}\,\mu)i - \mu(\nabla_\bullet\bar{B}) + (\tilde{B}_\bullet\,\mu)(\bar{B}).$$

If we write Δ^D in its canonical form (given by Theorem 2.10),

$$\Delta^D = -\nabla^2{}_{ii} + t_1 \bullet \nabla + t_0,$$

we have

$$(6.7) \qquad t_1(\omega\otimes\mu) = -2\,\tilde{B}_i(\omega_i\mu) - \omega_{T(i,\bullet)}\,\mu_i$$

for every 1-form ω, and

$$(6.8) \qquad t_0\,\mu = -(\nabla_i\tilde{B})_i\,\mu - \tilde{B}_i(\tilde{B}_i\,\mu) + \mu(R^D{}_{i\bullet}i) - (\tilde{B}_{T(i,\bullet)}\,\mu)_i - \mu(\nabla_\bullet\bar{B}) + (\tilde{B}_\bullet\mu)\bar{B}.$$

Then, from (6.3),

$$(6.9) \qquad \underline{B}_X\,\mu = -(1/2)\,t_1\,(X^b\otimes\mu) = -\mu(B_X\bullet) + (1/2)\,T_{\mu^\#\bullet X}$$

Since our frame is parallel,

$$(6.10) \qquad -(\nabla_i \underline{B}_i)\,\mu \;=\; \mu((\nabla_i B)_i \bullet) - (1/2)\,\nabla_i(T)_{\mu^\bullet \bullet i},$$

and

$$(6.11) \qquad -\underline{B}_i(\underline{B}_i\,\mu) \;=\; -\mu(B_i(B_i \bullet)) + (1/2)\,T_{(\mu(B(i,\bullet)))^\bullet \bullet i} + (1/2)\,T_{\mu^\bullet B(i,\bullet) i} -$$

$$- (1/4)\,T_{(\langle T(\mu^\bullet,\bullet), i\rangle)^\bullet \bullet i} =$$

$$= -B_{ij\mu^\bullet} B_{i\bullet j} + (1/2)\,B_{ij\mu^\bullet} T_{j\bullet i} + (1/2)\,T_{\mu^\bullet ji} B_{i\bullet j} - (1/4)\,T_{j\bullet i} T_{\mu^\bullet ji}.$$

Now, from (6.10), (6.11) and (6.2) we get the trace of $\mathcal{E}$

$$-\mathrm{tr}\,\mathcal{E} \;=\; -\tau^D + (1/4)\,\|T\|^2 - \|\bar{B}\|^2 - (\nabla_k \bar{B})_k,$$

but from (4.3) in [M],

$$\tau^D = \tau^\nabla + 2\,(\nabla_k \bar{B})_k + B_{ijk} B_{jik} - \|\bar{B}\|^2,$$

and then, denoting τ^∇ by τ,

$$\mathrm{tr}\,\mathcal{E} \;=\; -\tau + \delta\bar{B}^b - B_{ijk} B_{jik} + (1/4)\,\|T\|^2.$$

Let $A = -B_{ijk} B_{jik} + (1/4)\,\|T\|^2$. Then , since $B_{ijk} = (1/2)\,(T_{ijk} + T_{kij} - T_{jki})$, we have $A = -(1/2)\,T_{ijk} T_{ikj}$.

Now, from 6.1(b) and the above expressions, we have

$$a_1(\Delta^D{}_1) \;=\; \int_M (1/6)\,(-2n\,\tau - 6\,\tau + 6\,\delta\bar{B}^b + 6\,A) \;=\; \int_M (-((n+3)/3)\,\tau + A),$$

since $\int_M \delta\bar{B}^b = 0$.

<u>6.5.COROLLARY:</u> Let D be the characteristic connection, and let $Spc^p(M)$ the spectrum of Δ^D acting on p-forms. Let M_1 be a Kaehler manifold, and let M_2 be a nearly Kaehler or an almost Kaehler manifold. If $Spc^p(M_1) = Spc^p(M_2)$ for $p = 0,1$ then M_2 is a Kaehler manifold.

<u>Proof:</u> First, observe that, when D is the characteristic connection, we have (see [M])

$$A \;=\; (1/2)\,\|T_1\|^2 - (1/4)\,\|T_2\|^2,$$

where T_1 and T_2 are tensor fields on M satisfying that T_1 vanishes on almost Kaehler manifolds and T_2 vanishes on nearly Kaehler manifolds. On the other hand, since $\Delta^D{}_0 = \Delta_0$,

$$a_1(\Delta^D{}_0) \;=\; \int_M (1/6)\,(-2n\,\tau) \;=\; -(n/3)\int_M \tau.$$

Then,

$$\int_M A \;=\; a_1(\Delta^D{}_1) - ((n+3)/3)\, a_1(\Delta^D{}_0)\,.$$

Now, if $Spc^p(M_1) = Spc^p(M_2)$ for $p = 0,1$, then, we have

$$a_1(\Delta^D{}_1)(M_1) = a_1(\Delta^D{}_1)(M_2), \qquad\qquad a_1(\Delta^D{}_0)(M_1) = a_1(\Delta^D{}_0)(M_2),$$

but, if M_1 is Kaehler, $\int_{M_1} A = 0$, so that

$$a_1(\Delta^D{}_1)(M_1) - ((n+3)/3)\, a_1(\Delta^D{}_0)(M_1) = 0,$$

and then,

$$\int_{M_2} A = a_1(\Delta^D{}_1)(M_2) - ((n+3)/3)\, a_1(\Delta^D{}_0)(M_2) =$$

$$a_1(\Delta^D{}_1)(M_1) - ((n+3)/3)\, a_1(\Delta^D{}_0)(M_1) = 0.$$

If M_2 is almost Kaehler, then $T_1 = 0$, $A = -\,(1/4)\,\|T_2\|^2$ and $\int_{M_2} A = 0$, whence $\|T_2\|^2 = 0$, which implies that M_2 is Kaehler. Similarly, if M_2 is nearly Kaehler, then $T_2 = 0$, $A = -\,(1/2)\,\|T_1\|^2$ and $\int_{M_2} A = 0$, whence $\|T_1\| = 0$ and M_2 is Kaehler.

TABLE I

	p	t_1
1	$p \geq 0$	$t_1\omega = \mathrm{Alt}(\omega)$
2	$p \geq 0$	$t_1\omega = \mathrm{Alt}(J\omega)$
3	$p \geq 1$	$t_1\omega = (c_J^p c_{11}\omega) \quad \Omega\wedge \overset{(p+1)/2}{\ldots\ldots} \wedge\Omega$
4	$p \geq 2$	$t_1\omega = \mathrm{Alt}(J^m\omega)$
5	$p \geq 2$	$t_1\omega = \mathrm{Alt}(J_1^m\omega)$
6	$p \geq 2$	$t_1\omega = \mathrm{Alt}(c_J^m\omega)\wedge\Omega\wedge\ldots\overset{(p+1-m)/2}{\ldots\ldots}\wedge\Omega$
7	$p \geq 2$	$t_1\omega = \mathrm{Alt}(Jc_J^m\omega)\wedge\ \Omega\wedge\ldots\ \ldots\wedge\Omega$
8	$p \geq 2$	$t_1\omega = \mathrm{Alt}(J^m c_J^n\omega \otimes \Omega\otimes\ldots\ \ldots\otimes\Omega)$
9	$p \geq 2$	$t_1\omega = \mathrm{Alt}(J_1^m c_J^n\omega \otimes \Omega\otimes\ldots\ \ldots\otimes\Omega)$
10	$p \geq 1$	$t_1\omega = (c_{11}\omega)\wedge\Omega$
11	$p \geq 2$	$t_1\omega = (Jc_{11}\omega)\wedge\Omega$
12	$p \geq 3$	$t_1\omega = \mathrm{Alt}(J^m c_{11}\omega\otimes\Omega)$
13	$p \geq 4$	$t_1\omega = (c_J^m c_{11}\omega)\wedge\ \Omega\wedge\overset{(p-m)/2}{\ldots\ldots}\wedge\Omega$
14	$p \geq 4$	$t_1\omega = (Jc_J^m c_{11}\omega)\wedge\ \Omega\wedge\overset{(p-m)/2}{\ldots\ldots}\wedge\Omega$
15	$p \geq 5$	$t_1\omega = \mathrm{Alt}(J^n c_J^m c_{11}\omega\otimes\Omega\otimes\overset{(p-m)/2}{\ldots\ldots}\otimes\Omega$

TABLE II

	$t_1 \circ D$
1'	$(t_1 \circ D)\omega = d^D\omega$
2'	$(t_1 \circ D)\omega = Jd^D\omega$
3'	$(t_1 \circ D)\omega = (c_J^p\delta^D\omega) \quad \Omega^{(p+1)/2}$
6'	$(t_1 \circ D)\omega = \mathrm{Alt}(c_J^m D\omega)\wedge\Omega^{(p+1-m)/2}$
7'	$(t_1 \circ D)\omega = \mathrm{Alt}(Jc_J^m D\omega)\wedge\Omega^{(p+1-m)/2}$
10'	$(t_1 \circ D)\omega = \delta^D\omega\wedge\Omega$
11'	$(t_1 \circ D)\omega = J\delta^D\omega\wedge\Omega$
13'	$(t_1 \circ D)\omega = c_J^m\delta^D\omega\wedge\Omega^{(p-m)/2}$
14'	$(t_1 \circ D)\omega = Jc_J^m\delta^D\omega\wedge\Omega^{(p-m)/2}$

TABLE III

$t_0 \quad (p = 1)$
$t_0\omega = \nabla_{\omega}\#\Omega$
$t_0\omega = \omega(\mathrm{Alt}_2(\nabla J))$
$t_0\omega = \omega \wedge \delta\Omega$
$t_0\omega = J\omega \wedge \delta\Omega$
$t_0\omega = \omega(\delta J)\Omega$
$t_0\omega = \nabla_{J\omega}\#\Omega$
$t_0 = J^1\nabla_{\omega}\#\Omega$

$t_0 \quad (p = -1)$
$t_0\omega = J^1\nabla_{J\omega}\#\Omega$
$t_0\omega = \omega(\mathrm{Alt}(J_1\nabla J))$
$t_0\omega = J\omega(\mathrm{Alt}(\nabla J))$
$t_0\omega = (J\omega)(\mathrm{Alt}(J_1\nabla J))$
$t_0\omega = \omega \wedge J\delta\Omega$
$t_0\omega = J\omega \wedge J\delta\Omega$
$t_0\omega = \omega(J\delta J)\Omega$

$t_0 \quad (p = 0)$
$t_0 f = f\delta\Omega$

$t_0 \quad (p = 0)$
$t_0 f = fJ\delta\Omega$

TABLE IV

$t_0 \quad (p = 2)$
$t_0\omega = \omega(\mathrm{Alt}^2\nabla J)$
$t_0\omega = \omega(\nabla\,\Omega^{\cdot\cdot})$
$t_0\omega = (c_J^1\omega)\delta\Omega$
$t_0 = c_1^1(\omega \otimes \delta J)$
$t_0 = c_1^1(J^1\omega \otimes \delta J)$
$t_0 = J^1\omega(\mathrm{Alt}^2(\nabla J))$
$t_0 = J\omega\,\mathrm{Alt}^2(\nabla J)$

$t_0 \quad (p = 2)$
$t_0\omega = J^1\omega(\nabla\Omega^{\cdot\cdot})$
$t_0\omega = \omega(J^1\nabla\Omega^{\cdot\cdot})$
$t_0\omega = J^1\omega(J^1\nabla\Omega^{\cdot\cdot})$
$t_0\omega = (c_J^1\omega)J\delta\Omega$
$t_0\omega = c_1^1(\omega \otimes J\delta J)$
$t_0\omega = c_1^1(J^1\omega \otimes J\delta J)$

$t_0 \quad (p = 1)$
$t_0 = \omega(\delta J)$

$t_0 \quad (p = 1)$
$t_0 = \omega(J\delta J)$

REFERENCES

[A-B-P] M.F. Atiyah, R. Bott and K. Patodi, "On the heat equation and the index theorem" Invent. Math. 19 (1973), 279-330.

[C-F-G] L.A. Cordero, M. Fernández and A. Gray, "Symplectic manifolds with no Kaehler structure", to appear in Topology.

[D 1] H. Donnelly, "Invariance theory of Hermitian manifolds" Proc. Amer. Math. Soc. 58 (1976), 229-233.

[D 2] H. Donnelly, "A spectral condition determining the Kaehler property", Proc. Amer. Math. Soc. 47 (1975), 187-195.

[D 3] H. Donnelly, "Heat Equation Asymptotics with Torsion", Indiana Univ. Math. J. Vol. 34. (1985), 105-113.

[E] D.B.A. Epstein, "Natural tensors on Riemannian manifolds", J. Diff. Geom. 10 (1975), 631-635.

[E-K] D.B.A. Epstein and M. Kneser, "Functors between categories of vector spaces", Lecture Notes in Math. Vol. 99, (1969), 154-170.

[F-M] A. Ferrández and V. Miquel, "Hermitian natural tensors", Preprint.

[G 1] P. Gilkey, "curvature and the eigenvalues of the Laplacian for elliptic complexes", Advances in Math. 10 (1973), 344-382.

[G 2] P. Gilkey, "Spectral geometry and the Kaehler condition for complex manifolds", Invent. Math. 26 (1974), 231-258, and "Corrections" , Invent. Math. 29 (1975), 81-82.

[G3] P. Gilkey, "The spectral geometry of real and complex manifolds", Proc. of Sympos. in Pure Math. Vol. 27 (1975), 265-280.

[G 4] P. Gilkey, "Recursion relations and the asymptotic behavior of the eigenvalues of the Laplacian", Compositio Math. 38 (1979), 201-240.

[GR 1] A. Gray, "Nearly Kahler manifolds", J. Diff. Geom. 4 (1970), 283-309.

[GR 2] A. Gray, "The structure of Nearly Kaehler manifolds", Math. Ann. 248 (1976), 233-248.

[G-H] A. Gray and L. M. Hervella, "The sixteen classes of almost hermitian manifolds and their linear invariants", Annali di Mat. pura ed applicata IV, vol. XXIII (1980), 35-58.

[McK-S] H. P. McKean, Jr and I. M. Singer, "Curvature and eigenvalues of the Laplacian", J. Diff. Geom. 1 (1967), 43-69.

[M] V. Miquel, "Volumes of certain small geodesic balls and almost hermitian geometry", Geometriae Dedicata 15 (1984), 261-267.

[P] W. A. Poor, "Differential Geometric Structures", Mc Graw-Hill, Inc., 1981.

[S] P. Stredder, "Natural differential operators on Riemannian manifolds and representations of the orthogonal and special orthogonal groups", J. Diff. Geom. 10 (1975), 647-660.

[T] K. Tsukada, "Hopf manifolds and spectral geometry", Trans. Amer. Math. Soc. 270 (1982), 609-621.