Dynamics of charges and solitons

Manuel Barros[1], Angel Ferrández[2] and Oscar J. Garay[3]

[1]Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, 1807 Granada, Spain. E-mail address: `mbarros@ugr.es`

[2]Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain. E-mail address: `aferr@um.es`

[3]Deparment of Mathematics, Faculty of Science and Technology, University of the Basque Country, Bilbao, Spain. E-mail address: `oscarj.garay@ehu.es`

Abstract

We first show that trajectories traced by charges moving in rotational magnetic fields are, precisely, the geodesics of surfaces of revolution with coincident axis. Thus, people living in a surface of revolution are not able to sense the magnetic Hall effect induced by the surrounding magnetic field and perceive charges as influenced, exclusively, by the gravity action on the surface of revolution. Secondly, the extended Hasimoto transformations are introduced and then used to identify trajectories of charges moving through a Killing rotational magnetic fields in terms of non-circular elastic curves. As a consequence, we see that in this case charges evolve along trajectories which are obtained as extended Hasimoto transforms of solitons of the filament equation.

MSC: 53B21, 53B50, 53C42.

Keywords: Lorentz force, Killing magnetic field, elastic curve, extended Hasimoto transformation.

1 Introduction

Extending the standard setting to the Riemannian context, a *magnetic field* on an n-dimensional Riemannian space, (M^n, g), is nothing but a closed 2-form on M^n, F, i.e., $F \in \Lambda_2(M^n)$ and $dF = 0$. Then, the *Lorentz force* experienced by a particle in a magnetic field is the skew-symmetric operator, Φ, defined by

$$g(\Phi(X), Y) = F(X, Y)\,, \tag{1}$$

for vector fields X, Y on M^n. Now, the dynamics of a charge moving in a magnetic field is described by trajectories, $\gamma(s)$, that are solutions of the so called *Lorentz equation*

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma')\,,$$

where ∇ is the Levi-Civita connection associated with the metric g and $'$ stands for the derivative with respect to the curve parameter. For simplicity, we call the solutions of this equation *magnetic trajectories*. As it happens for geodesics, it is quite clear that magnetic trajectories can not be arbitrarily parametrized; in fact if $\gamma(t)$ is a solution of (1), then

$$\frac{d}{dt}\langle\gamma',\gamma'\rangle = 2\langle\Phi(\gamma'),\gamma'\rangle = 0\,,$$

and, consequently, magnetic trajectories are parametrized proportionally to arc-length. For simplicity, unless otherwise stated, from now on we will usually assume that magnetic curves are parametrized by its arc-length. It is traditional to signify this by using s as the variable name when dealing with such paths. Furthermore, dimension three shows several significant peculiarities which make the study of magnetic fields quite special in this case. Firstly, we notice that the existence of a natural identification between two-forms and vector fields provides a one-to-one correspondence between closed two-forms and vector fields with divergence zero. Hence, as in the classical approach, a magnetic field in a Riemannian 3-space can be seen as a divergence free vector field. In particular, every Killing vector field is a magnetic field. Secondly, it turns out that in dimension three the Lorentz force associated with a magnetic field V is determined by the cross product, i.e., $\Phi(X) = V \times X$, and, therefore, the Lorentz equation can be rewritten as

$$\nabla_{\gamma'}\gamma' = V \times \gamma', \tag{2}$$

what makes evident the Hall effect (encoded in this equation). Formula 2 shows clearly also that the dynamics of charges can not be described by trajectories which are integral curves of the magnetic field, V.

On the other hand, according to Daniel Bernoulli's model of an elastic rod in equilibrium, an *elastic curve* in $\mathbb{R}^3$ (or, simply, *elastica*) is a minimizer of the *elastic energy*. For simplicity, we will henceforth use the term elastic curve or elastica for the extremals of the *total squared curvature* among curves of the same length and first order boundary data, in other words, an elastica α will be a critical curve of the following functional

$$\mathcal{F}_\lambda(\alpha) = \int_\alpha (\kappa^2 + \lambda)\, ds\,, \tag{3}$$

acting on a space of curves joining two fixed points satisfying suitable first order boundary conditions, and where κ denotes the curvature of α and $\lambda \in \mathbb{R}$ works as a Lagrange multiplier. The elastica variational problem is closely related to certain geometric evolution problems and their associated integrable evolution equations. In fact, in many different contexts we are often bound to study curves, α, in $\mathbb{R}^3$ evolving over time and tracing out surfaces of the following form

$$\alpha(s,t) = \varphi_t(\alpha(s))\,, \qquad \alpha(s,0) = \alpha(s)\,,$$

where φ_t is a 1-parameter group of diffeomorphisms of $\mathbb{R}^3$. In addition, often we may also think of the surface $\alpha(s,t)$ as obtained when the curve evolves satisfying a certain differential equation, which is usually known as an *evolution equation*. Once we know a solution $\alpha(s,t)$ of an evolution equation, the generating curve, $\alpha(s)$, is usually called the *initial condition* of the solution.

Perhaps one of the best known examples of this situation is provided by the so called *Hasimoto surfaces*. They are the solutions $\alpha(s,t)$ of the following evolution equation

$$\frac{\partial \alpha}{\partial t} = \frac{\partial \alpha}{\partial s} \times \frac{\partial^2 \alpha}{\partial s^2}\,, \tag{4}$$

which plays an important role in fluid dynamics because it describes the motion of an isolated vortex filament in an incompressible fluid. This is why it is called the *filament equation* or the *localized induction equation* (LIE). It was first formulated by L. Da Rios in 1906 [12], and later by R. Betchov [3], so it is also known as the *Betchov-Da Rios equation*.

In this context, it is a simple exercise to see that if a plane curve, $\alpha(s)$, evolves in the direction of its binormal vector sweeping out a right cylinder, then it provides a filament (a solution of LIE), if and only if $\alpha(s)$, is a circle. In other words, circular right cylinders are the only right cylinders which are Hasimoto surfaces. On the other hand, suppose that $\alpha(s) = (x(s), 0, z(s))$ lies in the plane $y = 0$ and that it evolves by rotating around the z-axis, so that it sweeps out a surface of revolution. That is

$$\alpha(s,t) = R_t(\alpha(s)) = (x(s)\cos t, x(s)\sin t, z(s))\,,$$

and then we look for solutions of (4) of this type. It is known that Hasimoto rotation surfaces are swept out by planar elasticae, [8]. This can be seen by noticing that those surfaces of revolution which are Hasimoto surfaces are obtained by rotating around an axis planar curves whose curvature function is a multiple of the distance to some straight line parallel to the axis. By suitable choice of the axis, we have $\kappa(s) = x(s)$, what implies that these curves must be planar elasticae (see the appendix, for more details). Hence, it seems natural to study filaments that evolve under (4) retaining their shapes, in other words, *congruence solutions*, or *solitons*, of LIE. From a geometric point of view, if the initial condition is arc-length parametrized, this is equivalent to consider curves whose binormal flow is generated by a one-parameter subgroup of rigid motions in $\mathbb{R}^3$. In particular, since the restriction to the filaments induces a Killing field along them, it results that the corresponding initial conditions are elasticae in $\mathbb{R}^3$ (see [7], [8] and references therein for more details).

The main purpose of this paper is to study the dynamics of charges in rotational domains moving under the influence of a rotational magnetic field and explore their connection with classical elasticae.

In Sect. 2 we see that charges moving in a rotational magnetic field evolve as geodesics of surfaces of revolution (and conversely). Then, the Lorentz force is expressed in terms of cylindrical coordinates what will be useful in later sections. Sect. 3 is devoted to introduce what we call the 1-parameter family of *extended Hasimoto transformations*, $\mathcal{H}_r(\alpha)(s)$. Also in this section, extended Hasimoto transforms of curves evolving under the filament equation are characterized as solutions of a coupled system containing the Schrödinger equation. This is a simple extension of the well known result due to Hasimoto [6]. In Sect. 4 we specialize to the Killing rotational magnetic fields and show that their trajectories are completely determined by the elastica and the extended Hasimoto transformations. Finally, in the appendix we will show a well known property of planar elasticae which is used along the paper.

2 Rotational magnetic fields

Rotational Killing vector fields in $\mathbb{R}^3$ can be geometrically described as the infinitesimal generators of one-parameter subgroups of rotations around an axis. Without loss of generality, we may choose the rotation axis to coincide with the z-axis of $\mathbb{R}^3$ and, then, we select the rotational Killing vector field $\xi(x, y, z) = (-y, x, 0) = -y\partial_x + x\partial_y$. Its space of principal orbits is $M = \mathbb{R}^3 - (\{z\}\text{-axis})$ and the associated one-parameter group of isometries is given by $G_\xi = \{\phi_\theta : \theta \in \mathbb{R}\}$, where

$$\phi_\theta(p) = p \cdot R_\theta = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Thus, the space of orbits can be identified with the half-plane $P = \{(x, 0, z) : x > 0\}$ and the quotient map, $\pi : M \to P$, is defined by $\pi(\rho\cos\theta, \rho\sin\theta, z) = (\rho, 0, z)$, where we use cylindrical coordinates $(x, y, z) = (\rho\cos\theta, \rho\sin\theta, z)$.

A domain $\Omega \subset M$ will be called *rotational* if it contains the orbit of any of its points, that is, for all $p \in \Omega$, $\phi_\theta(p) \in \Omega$ for all $\theta \in \mathbb{R}$. It is clear that $\Omega = \pi^{-1}(U)$ is a rotational domain when $U \subset P$ is an open subset in the space of orbits. Moreover every rotational domain is obtained in this way. Now, if $\Omega = \pi^{-1}(U)$ is an invariant domain and $h \in C^\infty(\Omega)$ then the divergence of $h\,\xi$ vanishes, if an only if, $\xi(h) = 0$, which is equivalent to $h = f \circ \pi$, for some $f \in C^\infty(U)$. In this case, we will say that $V = h\,\xi = h\,\partial_\theta$ is a *rotational magnetic* field.

Since in this work the dynamics of charges in rotational domains moving under the influence of a rotational magnetic field is analyzed, we will focus on the solutions of the following Lorentz equation

$$\gamma''(s) = \left((f \circ \pi) \cdot \xi \times \gamma'\right)(s)\,, \tag{6}$$

in $\Omega = \pi^{-1}(U)$, with $f \in C^\infty(U)$, and $'$ denoting the derivative of the arc-length parameter of γ.

As we are interested in identifying the magnetic trajectories of a rotational magnetic field as geodesics of some surface, in what follows we will only consider magnetic curves of rotational magnetic fields which do not touch the field's axis. Moreover, we will need to restrict ourselves to curves $\gamma(s) = (x(s), y(s), z(s))$ with non-horizontal tangent, i.e., $z'(s) \neq 0$. Then we have the following interesting result:

Theorem 2.1 *Assume that $\gamma(s)$ is a curve in $\mathbb{R}^3$ with non-horizontal tangent at any point, i.e., $z'(s) \neq 0$, $\forall s$. Then γ is a magnetic trajectory of a rotational magnetic field if and only if it is a geodesic of a surface of revolution with the same axis as the magnetic field.*

Proof. Let $\gamma(s) = (\rho(s)\cos\theta(s), \rho(s)\sin\theta(s), z(s))$ be a geodesic (assumed to be arc-length parametrized) of a surface of revolution S_β, which is obtained by rotating a profile curve $\beta(s) = (\rho(s), 0, z(s))$ around the z-axis. Certainly $\gamma(s) = \beta(s) \cdot R_{\theta(s)}$, with $R_{\theta(s)}$ defined in (5), and so $\gamma'(s) = \beta'(s) \cdot R_{\theta(s)} + \theta'(s)\,\partial_\theta(\gamma(s))$. Now, the tangent plane of S_β along $\gamma(s)$ is spanned by the vector fields $\phi_{\theta(s)}(\beta'(s)) = \beta'(s) \cdot R_{\theta(s)}$ and $\partial_\theta(\gamma(s))$ and, consequently

$$\partial_\theta(\gamma(s)) \times \gamma'(s) = \partial_\theta(\gamma(s)) \times \phi_{\theta(s)}(\beta'(s))$$

is perpendicular to S_β along $\gamma(s)$. However, if $\gamma(s)$ is a geodesic of S_β, then $\gamma''(s)$ is also normal to S_β along $\gamma(s)$ which implies the existence of a certain function $h(\gamma(s))$ satisfying

$$\gamma''(s) = h(\gamma(s))\,\partial_\theta(\gamma(s)) \times \gamma'(s)\,. \tag{7}$$

Let us show that the function h is actually defined along the profile curve $\beta(s)$. To do this, we use the classical *Clairaut relation* for geodesics in surfaces of revolution which ensures the existence of a constant r (from now on, the *Clairaut slope* of the geodesic) satisfying

$$\rho^2(s)\theta'(s) = \rho(s)\cos\varphi(s) = r\,, \tag{8}$$

where $\varphi(s)$, $0 \le \varphi(s) \le \pi/2$, is the angle that $\gamma'(s)$ makes with the parallels of S_β. Now, we can combine (7) with (8) to relate the curvature function, $\kappa(s)$, of $\gamma(s)$ in $\mathbb{R}^3$ with the associated function h

$$h(\gamma(s)) = \frac{\kappa(s)}{\rho(s)\sin\varphi(s)} = \frac{\kappa(s)}{\sqrt{\rho^2(s) - r^2}}\,. \tag{9}$$

Since $\gamma(s)$ is a geodesic in S_β, its normal curvature coincides with $\kappa(s)$ and so we can use the Euler formula to obtain

$$\kappa(s) = \kappa_1(s)\,\cos^2\varphi(s) + \kappa_2(s)\,\sin^2\varphi(s)\,, \tag{10}$$

where $\kappa_1(s)$ and $\kappa_2(s)$ are the principal curvatures of S_β along $\gamma(s)$. Consequently, $h(\gamma(s))$ is constant on each parallel which proves that $h \circ \pi = f(s)$ is a function on β and determines a π-invariant (rotationally invariant) function on the surface of revolution S_β. Now, f can be extended to a neighborhood (for example a tubular neighborhood) say U of the profile curve $\beta(s)$ in the half-plane P. Such extension, which will be also denoted by f, verifies $\partial_\theta(f) = 0$ in the rotational invariant domain $\Omega = \pi^{-1}(U)$ and $\gamma(s)$ becomes a solution of the corresponding Lorentz force equation (6) with potential f.

To show the converse, suppose that $\gamma(s) = (\rho(s)\cos\theta(s), \rho(s)\sin\theta(s), z(s))$ is a solution of (6), for a certain rotational magnetic field. Then, we can write $\gamma(s) = \phi_{\theta(s)}(\beta(s))$ with $\beta(s) = (\rho(s), 0, z(s))$. Now, it is clear that $\gamma(s)$ lies in the surface of revolution S_β. Furthermore, since $\gamma(s)$ is a solution of (6), we have

$$\gamma''(s) = (f \cdot \pi)\,\partial_\theta \times (\phi_{\theta(s)}(\beta'(s)) + \theta'\,\partial_\theta) = (f \cdot \pi)\,\partial_\theta \times \phi_{\theta(s)}(\beta'(s)),$$

which proves that $\gamma''(s)$ is normal to S_β along $\gamma(s)$ and so this is a geodesic of S_β. $\square$

Geodesic parallels of a surface of revolution with axis $\overrightarrow{0z}$ do not satisfy (6) and, therefore, they are not rotational magnetic trajectories. On the other hand, if γ is a geodesic of a surface of revolution with axis $\overrightarrow{0z}$, then Clairaut's relation implies that points where $\gamma'(s_o) = 0$ are isolated, and the previous theorem says that the connected components of the set of points where γ is not tangent to a parallel are magnetic trajectories. Thus, roughly speaking, Theorem 2.1 reduces the study of rotational magnetic flows to the analysis of gravity in surfaces of revolution in the following sense: *Charges moving in rotational magnetic flows evolve as free fall particles in surfaces of revolution*. So people living in a surface of revolution are not able to detect the magnetic Hall effect derived from the magnetic force in the surrounding space and see charges as exclusively subjected to the gravity action of the surface of revolution.

Examples As an illustration of Theorem 2.1, we give a few examples:

(1) Denote by P the half-plane $P = \{(x, 0, z) : x > 0\}$ and choose ψ a constant satisfying $0 < \psi < \frac{\pi}{2}$. Suppose that U is the domain in P determined by $U := \{(x, z) \in P : -\tan\psi < z/x < \tan\psi\}$ and $\Omega = \pi^{-1}(U)$ a rotational domain. Define $f \in C^\infty(U)$ by

$$f(x, z) = \frac{1}{\sqrt{(x^2 + z^2)}\sqrt{(x^2 \sin^2\psi - z^2 \cos^2\psi)}},$$

and represents by $V = f(x, z)\,\partial_\theta$ the associated rotational magnetic field. Now, consider the class of great circles with slope ψ in the family of spheres centered at the origin, and take the set of semi-circles formed by the points of the above great circles belonging to Ω. Then V admits this set of semicircles as magnetic trajectories.

(2) Consider $U = \{(x, 0, z) \in P : (x - A)^2 + z^2 < A^2\}$, where A is a positive constant. It is clear that $\Omega = \pi^{-1}(U)$ is a rotational domain, actually it is an open solid torus. We can consider an obvious foliation by coaxial revolution tori in Ω. These tori, $\{\mathrm{T}_a : a < A\}$, are obtained by rotating around the $\{z\}$-axis the circle with radius $a < A$ which is centered at the point $(A, 0, 0)$. Now, we can choose an admissible Clairaut slope, say r, and look for a rotational magnetic field, in Ω, that admits as magnetic trajectories the geodesics with Clairaut slope r in the above tori. Again, for geodesics γ confined to the outer part of the torus between the v_o and $-v_o$ parallels we omit the points of tangency to these barrier parallels (i.e., points of γ with $z'(s_0) = 0$). These geodesics cross outer equator but not inner equator. As the curvature function in $\mathbb{R}^3$ of these geodesics coincides with the normal curvature function in the corresponding torus, it can be computed (using the Euler formula (10)) obtaining

$$\kappa(s) = \frac{\rho^3(s) - Ar^2}{a\rho^3(s)}\,.$$

Then an easy computation involving (9) shows that the corresponding potential, $f : U \to \mathbb{R}$, must be defined by

$$f(x, 0, z) = \frac{x^3 - Ar^2}{x^3\sqrt{(x^2 - r^2)((x - A)^2 + z^2)}}\,.$$

(3) Let us consider the function $F : P \to \mathbb{R}$ defined by $F(x, 0, z) = \frac{\cosh z}{x}$. Since it is free of critical points, it defines a foliation in P by catenaries $\{\mathrm{C}_a : x = a\cosh z\,/\,a > 0\}$.

For an admissible Clairaut slope, r, we look for a rotational magnetic field admitting as magnetic trajectories the geodesics with Clairaut slope r, in the class of catenoids obtained by rotating the above family of catenaries around the $\{z\}$-axis. As before, for geodesics γ confined to the part of a catenoid limited by a parallel v_o, we omit the points of tangency to the barrier parallel, i .e., points of γ with $z'(s_0) = 0$) (see [11] for an explicit computation of the geodesics of a catenoid). Since catenoids are minimal surfaces in $\mathbb{R}^3$, we first use the *Euler formula* to compute the curvature function in $\mathbb{R}^3$ of their geodesics (which coincides with the corresponding normal curvature in the catenoid) and then we apply (9) to obtain the following potential of this rotational magnetic field

$$f(x,0,z) = \frac{(2r^2 - x^2)\cosh^3 z}{x(x^2 - r^2)^{1/2}(\cosh^2 z + x^2 \sinh^2 z)^{3/2}} \,.$$

Finally, with an eye in later applications, we end this section expressing the Lorentz equation (6) in cylindrical coordinates. Then, a few manipulations will transform it in a ODE system which will be useful in the following.

Setting $(x, y, z) = (\rho\cos\theta, \rho\sin\theta, z)$, it is not difficult to see that the Lorentz force equation (6) can be written as

$$\begin{aligned}
(\rho'' - \rho(\theta')^2 - f\rho z')\cos\theta - (2\rho'\theta' + \rho\theta'')\sin\theta &= 0\,, \\
(2\rho'\theta' + \rho\theta'')\cos\theta + (\rho'' - \rho(\theta')^2 - f\rho z')\sin\theta &= 0\,, \\
z'' &= -f\rho\rho'\,.
\end{aligned}$$

This set of differential equations can be equivalently transformed into the system

$$\begin{aligned}
\rho'' - \rho(\theta')^2 - f\rho z' &= 0\,, \\
2\rho'\theta' + \rho\theta'' &= 0\,, \\
z'' &= -f\rho\rho'\,.
\end{aligned}$$

However, since we know that the solutions are geodesics in surfaces of revolution, the second equation can be combined with the Clairaut constraint (8) along each geodesic and, consequently, the Lorentz force equation can be expressed as

$$\begin{aligned}
\rho'' - \frac{r^2}{\rho^3} - f\rho z' &= 0\,, && (11)\\
\rho^2\,\theta' &= r\,, && (12)\\
z'' &= -f\rho\rho'\,. && (13)
\end{aligned}$$

It is worth pointing out that all solutions, $\gamma(s) = (\rho(s)\cos\theta(s), \rho(s)\sin\theta(s), z(s))$, of the ODE system (11), (12) and (13) are curves with constant speed, that is, they are automatically parametrized proportionally to arclength, because a combination of (11) and (13) gives

$$\rho'' - \frac{r^2}{\rho^3} + \frac{z'z''}{\rho'} = 0\,,$$

which can be written as

$$\left((\rho')^2 + (z')^2 + \frac{r^2}{\rho^2}\right)' = (|\gamma'|^2)' = 0\,.$$

Certainly the complete and explicit integration of the system (11)-(13) for an arbitrary potential $f(\rho, z)$ defined on a certain profile curve is beyond reach. Nevertheless, as we will see in Sect. 4, it can be completely solved for constant f.

3 The extended Hasimoto transformations

The *Hasimoto transformation*, [5, 6], is defined on the space C of connected smooth curves in $\mathbb{R}^3$ without inflection points. It associates to each curve $\alpha \in \mathrm{C}$ a complex valued function with no zeros, the so called *complex curvature*, or *complex wave function*, which is defined by

$$\phi(s) = \kappa(s) \exp\left(i \int_0^s \tau(u)\, du\right), \tag{14}$$

where $\kappa(s) > 0$ and $\tau(s)$ stand, respectively, for the Frenet curvature and torsion of α. It is clear that this transformation is actually defined on the space, $\overline{\mathrm{C}}$, of congruence classes of curves in C, in other words, curves which are invariant under rigid motions in $\mathbb{R}^3$. To recover the curve (or its congruence class) from its complex wave function, we have to consider a suitable quotient space of $\mathrm{F}(\mathbb{R}, \mathbb{C}) \equiv \mathrm{F}^o$, the family of complex valued functions with no zeroes. To proceed with, consider the natural action

$$\mathbb{S}^1 \times \mathrm{F}^o \to \mathrm{F}^o\,, \qquad e^{it} \cdot \phi(s) = \rho(s) \cdot e^{i(\theta(s)+t)}\,, \qquad \text{where} \quad \phi(s) = \rho(s) e^{i\theta(s)}\,.$$

Now, the congruence class of a curve can be recovered from the class $[\phi(s)]$ of its complex wave function. In fact, the curvature κ and the torsion τ can be expressed in terms of $[\phi(s)]$ as follows

$$\kappa(s) = |\phi(s)|\,, \qquad \tau(s) = \frac{d}{ds} \arg[\phi(s)]\,.$$

Consequently, the Hasimoto transformation can be defined as

$$\mathcal{H} : \overline{\mathrm{C}} \to \mathrm{F}^o/\mathbb{S}^1\,, \qquad \mathcal{H}([\alpha(s)]) = [\phi(s)]\,. \tag{15}$$

On the other hand, F^o can be viewed as the space of curves in the punctured plane $\mathbb{R}^2_*$ (we may assume that the origin has been removed) and so $\mathrm{F}^o/\mathbb{S}^1$ is the space of curves in $\mathbb{R}^2_*$ modulo rotations around the origin. From now on, and for simplicity, we will make evident both quotient spaces and, as usual, the Hasimoto transformation (14) will be considered as a one-to-one correspondence between connected smooth curves in $\mathbb{R}^3$ with no inflection points and curves in $\mathbb{R}^2_*$. Hasimoto discovered that LIE (4) is essentially equivalent to a well-known completely integrable PDE, the *focusing cubic nonlinear Schrödinger equation*

$$i\phi_t + \phi_{ss} + \frac{1}{2}(|\phi|^2 + K)\phi = 0\,, \tag{16}$$

with K being a constant. Namely, a curve evolves according to the flow (4) if and only if its complex wave function is a solution of (16).

For our purposes, it will be convenient to introduce the extended Hasimoto transformations. To this end, we consider $\mathbb{R}^3$ with the z-axis removed, which is identified with $(\mathbb{C} - \{0\}) \times \mathbb{R}$ by writing $(x, y, z) = \big(\rho\, e^{i\theta}, z\big)$. If $\mathrm{C}^{\,o}$ denotes the space of curves in $(\mathbb{C} - \{0\}) \times \mathbb{R}$, then a natural projection can be defined as follows

$$\tilde{\pi} : \mathrm{C}^{\,o} \to \mathrm{F}^o, \qquad \tilde{\pi}(\gamma)(s) = \phi(s)\,,$$

where $\gamma(s) = (\phi(s), z(s)) = \big(\rho(s)\, e^{i\theta(s)}, z(s)\big)$. The idea is to use this map to lift the Hasimoto transformation and then obtain the extended Hasimoto transformations which are valued in $\mathrm{C}^{\,o}$. Thus, for $r \in \mathbb{R}$, we define the map

$$\mathcal{H}_r : \mathrm{C} \to \mathrm{C}^{\,o}, \qquad \mathcal{H}_r(\alpha)(s) = (\phi(s), z(s)), \tag{17}$$

where

$$\begin{aligned} \phi(s) &= \rho(s)\, e^{i\theta(s)} = \kappa(s) \exp\left(i \int_0^s \tau(u)\, du \right), \\ z(s) &= \int_0^s \left(-\frac{1}{2} |\phi(u)|^2 + r \right) du\,. \end{aligned} \tag{18}$$

It is obvious that these maps, as well as the original Hasimoto transformation, are actually defined in $\overline{\mathrm{C}}$. Moreover, they are liftings of the Hasimoto map in the sense that

$$\tilde{\pi} \circ \mathcal{H}_r = \mathcal{H}, \qquad r \in \mathbb{R}.$$

From now on, $\mathcal{H}_r$, with $r \in \mathbb{R}$, will be called the *extended Hasimoto transformations* of *level* r. Of course, each extended Hasimoto transformation is injective when considered acting on $\overline{\mathrm{C}}$. To obtain a one-to-one correspondence, we set a relation of equivalence in $\mathrm{C}^{\,o}$ by declaring to curves equivalent if they have the same projection on $z = 0$. In other words, $\gamma_1, \gamma_2 \in \mathrm{C}^{\,o}$ are related, if and only if, $\tilde{\pi}(\gamma_1) = \tilde{\pi}(\gamma_2)$. The quotient space is denoted by $\overline{\mathrm{C}}^{\,o}$ while $\{\gamma\}$ stands for the equivalence class of any curve $\gamma \in \mathrm{C}^{\,o}$. It should be noted that all the images of a curve, $\alpha \in \mathrm{C}$, under the extended Hasimoto transformations, lie in the same equivalence class, that is, $\{\mathcal{H}_{r_1}(\alpha)\} = \{\mathcal{H}_{r_2}(\alpha)\}$ for any $r_1, r_2 \in \mathbb{R}$.

Not only that, within any equivalence class one can find the images under $\mathcal{H}_r$ of a curve in $\mathrm{C}^{\,o}$. In fact, given any curve $\gamma(s) = (\phi(s), z(s))$ in $\mathrm{C}^{\,o}$ we have a curve, $\alpha(s)$, in C which is determined, up to rigid motions, by the complex wave function $\phi(s)$. Now the family of curves

$$\gamma^r(s) = \left(\phi(s), \int_0^s \left(-\frac{1}{2} |\phi(s)|^2 + r \right) ds \right), \quad r \in \mathbb{R}\,,$$

satisfy

$$\gamma^r(s) = \mathcal{H}_r(\alpha)\,, \qquad \{\gamma\} = \{\gamma^r\}\,.$$

Consequently, at any level r, the extended Hasimoto transformation $\mathcal{H}_r$ becomes a bijective mapping when defined on the quotient space, that is, the map

$$\mathcal{H}_r : \overline{\mathrm{C}} \to \overline{\mathrm{C}}^{\,o}\,, \qquad \mathcal{H}_r(\alpha) = \{\gamma^r\}\,, \qquad r \in \mathbb{R}, \tag{19}$$

is a one-to-one correspondence.

Recall that the Hasimoto transformation (14) provides a bridge between LIE (4) and the cubic nonlinear Schrödinger equation (16). Therefore, it seems natural to study the action of the extended Hasimoto transformation on curves evolving by LIE. In this respect, we have the following result:

Theorem 3.1 *A curve $\alpha(s)$ evolves according to the filament flow if and only each extended Hasimoto image $\gamma^r(s) = \mathcal{H}_r(\alpha)(s)$ is a solution of the following completely integrable PDE*

$$\phi_t = i\phi_{ss} + \frac{i}{2}|\phi|^2\phi - iA(t)\phi, \tag{20}$$

$$z_t = |\phi|^2\,(arg[\phi])_s + F(t), \tag{21}$$

for time dependent real valued functions, $A(t)$ and $F(t)$.

Proof. To prove this, we first notice that (20) is a consequence of the previously mentioned Hasimoto's results regarding (16). Thus, we only need to compute z_t, the time variation of the last coordinate. Let us write

$$\phi(s,t) = \kappa(s,t)\, e^{i\int_0^s \tau(u,t)\,du}\,, \tag{22}$$

$$z(s,t) = \int_0^s \left(-\frac{1}{2}|\phi(u,t)|^2 + r\right) du\,. \tag{23}$$

Now, a direct computation involving the cubic nonlinear Schrödinger equation (20) yields

$$\frac{\partial}{\partial t}|\phi(s,t)|^2 = \phi_t\bar{\phi} + \phi\,\bar{\phi}_t = i(\phi_{ss}\bar{\phi} - \bar{\phi}_{ss}\phi)\,,$$

which, using (22) gives

$$\phi_{ss}(s,t) = \left[(\kappa_{ss} - \kappa\tau^2) + i(2\kappa_s\tau + \kappa\tau_s)\right]\, e^{i\int_0^s \tau(u,t)\,du}\,,$$

and consequently

$$\frac{\partial}{\partial t}|\phi(s,t)|^2 = -2\frac{\partial}{\partial s}(\kappa^2\tau)\,. \tag{24}$$

Therefore, we have from (23)

$$z_t(s,t) = -\frac{1}{2}\int_0^s \frac{\partial}{\partial t}|\phi(u,t)|^2\,du = \kappa^2\tau + F(t),$$

which proves (21) . □

Remark 3.2 The function $A(t)$ appearing in the cubic nonlinear Schrödinger (20) can be removed by introducing the new variable

$$\psi(s,t) = \phi(s,t)\, e^{-i\int_0^t A(u)\,du}.$$

Hence, it is not difficult to see that $\psi(s,t)$ is a solution of (20) if and only if $\phi(s,t)$ is a solution of the following cubic nonlinear Schrödinger equation

$$\phi_t = i\phi_{ss} + \frac{i}{2}|\phi|^2\phi.$$

4 Elasticae and trajectories of Killing rotational magnetic fields

Now, the following problem arises naturally: *To determine the curves* $\{\gamma\} \in \overline{\mathrm{C}}^{\,o}$ *which are images, under an extended Hasimoto transformation of an elastica.* Said otherwise, setting $\mathcal{E}_\lambda$ for the space of extremal curves of (3), we are proposing the study of $\{\mathcal{H}_r(\mathcal{E}_\lambda)\}$, for some $r \in \mathbb{R}$. Since Frenet curvatures uniquely determine a curve in $\mathbb{R}^3$ (up to congruence), the solutions of the following Euler-Lagrange equations [9] provide the extremals of the elastic functional (3) acting on spaces of curves in $\mathbb{R}^3$ satisfying given zero and first order boundary data

$$\kappa'' + \frac{1}{2}\kappa^3 - \kappa\tau^2 - \frac{\lambda}{2}\kappa \;=\; 0\,, \tag{25}$$

$$\kappa^2\tau \;=\; c\,. \tag{26}$$

Here κ and τ denote the curvature and torsion functions along γ and $c \in \mathbb{R}$ is a constant. Combining (18), (25) and (26) we see that the curve $\gamma^r(s) = \mathcal{H}_r(\alpha(s))$ must satisfy the following differential equations

$$\rho'' - \rho\,(\theta')^2 + \frac{1}{2}\,\rho^3 - \frac{\lambda}{2}\,\rho \;=\; 0\,, \tag{27}$$

$$\rho^2\,\theta' \;=\; c\,, \tag{28}$$

$$z' \;=\; -\frac{1}{2}\,\rho^2 + r\,. \tag{29}$$

So far, the possible values of the parameter r (the Clairaut slope) have had no influence on our arguments, but now we will see that there is a privileged value of r. In fact, let us compute the speed of $\gamma^r(s) = \mathcal{H}_r(\alpha(s))$. From (28) and (29) we obtain

$$|(\gamma^r(s))'|^2 = (\rho'(s))^2 + (z'(s))^2 + \frac{c^2}{\rho(s)^2}\,,$$

so that

$$\frac{d}{ds}|(\gamma^r(s))'|^2 = 2\rho'\left(\rho'' - \frac{c^2}{\rho^3} + \frac{1}{2}\,\rho^3 - r\rho\right) = 2\rho'\left(\rho'' - \rho(\theta')^2 + \frac{1}{2}\,\rho^3 - r\rho\right).$$

Hence, we can use (27) to see that, if $\alpha(s)$ is not a circle, the curve $\gamma^r(s) = \mathcal{H}_r(\alpha(s))$ is parametrized proportionally to arc-length, that is, $|(\gamma^r(s))'|^2$ is constant. But this happens if and only if $r = \lambda/2$. Moreover, it is not difficult to check that, in this case, the system of differential equations (27), (28) and (29), whose solutions provide the whole space $\mathcal{H}_{\lambda/2}(\mathcal{E}_\lambda)$, is equivalent to the following one

$$\begin{aligned} \rho'' - \rho(\theta')^2 - \rho\, z' &= 0\,, \\ 2\rho'\,\theta' + \rho\,\theta'' &= 0\,, \\ z'' + \rho\,\rho' &= 0\,. \end{aligned}$$

Now, a comparison of this system of differential equations with (11), (12) and (13) shows that it can be written in the following simple form

$$\gamma'' = \partial_\theta \times \gamma'\,, \tag{30}$$

which is just the Lorentz force equation associated with the Killing rotational magnetic field ∂_θ. As a consequence, we have the following statement (notice that, in this case, magnetic trajectories in the statement are not assumed necessarily to be arc-length parametrized)

Theorem 4.1 *Assume that $\alpha(s)$ is an immersed curve in $\mathbb{R}^3$ other than a plane circle. Then, $\alpha(s)$ is an elastica, that is, an extremal of $\mathcal{F}_\lambda$ (acting on spaces of curves satisfying given zero and first order boundary data), if and only if, $\gamma^{\lambda/2}(s) = \mathcal{H}_{\lambda/2}(\alpha(s))$ is a magnetic trajectory of a Killing rotational magnetic field. Therefore, the whole moduli space of magnetic trajectories of ∂_θ is*

$$\bigcup_{\lambda \in \mathbb{R}} \mathcal{H}_{\frac{\lambda}{2}}\left(\mathcal{E}_\lambda\right).$$

Remark 4.2 It should be noticed that the undetermined constant c, appearing in the Euler-Lagrange equation for elasticae (26), can be geometrically interpreted as follows. The constant c is just the Clairaut slope (see (8)) of the $\frac{\lambda}{2}$-Hasimoto image of an elastica, regarded as a geodesic in a surface of revolution (see Theorem 2.1). Figure 1 shows two magnetic trajectories obtained as images of two free elasticae by using the Hasimoto transformation $\mathcal{H}_0$. They correspond to the same Clairaut slope.

Remark 4.3 From (13) one sees that the only magnetic rotational fields admitting a general helix as a magnetic trajectory are the Killing rotational magnetic fields and, in this case, the helix has to be a non-planar circular helix (geodesics of circular cylinders other than circles), $\gamma_{a,b}(t)$ with radius a and pitch $2\pi b$, $a, b \in \mathbb{R}, a > 0$. Moreover, if $b \neq 0$ these circular helices are λ-elasticae for $\lambda = \frac{a^2 - 2b^2}{(a^2+b^2)^2}$ and their Hasimoto r-transforms are again helices for any r, thus, they are magnetic trajectories for the Killing rotational field ∂_θ for any choice of r.

Theorem 4.1 tells us that magnetic trajectories of a Killing rotational magnetic field are obtained as images of λ-elasticae α by the $\mathcal{H}_{\lambda/2}$ Hasimoto transformation: $\gamma^{\lambda/2}(s) = \mathcal{H}_{\lambda/2}(\alpha(s))$. On

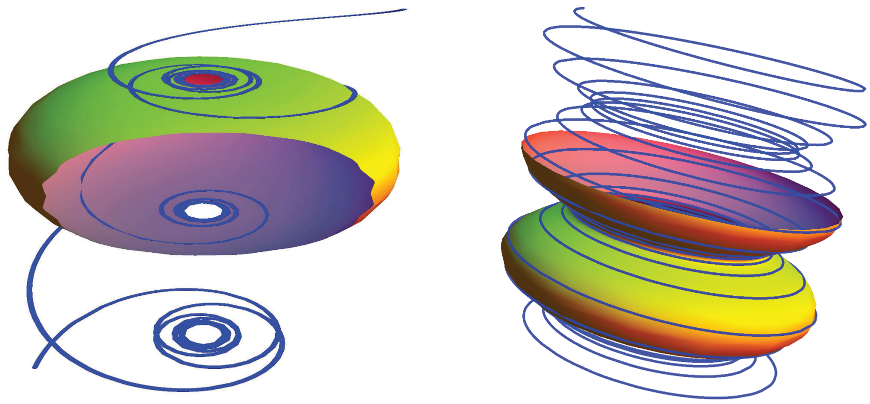

Figure 1: Two magnetic trajectories obtained as images by $\mathcal{H}_0$ of two free elasticae. According to Theorem 1, they lie into two surfaces of revolution (shown in part) on which they have been chosen to have the same Clairaut slope.

the other hand, curvature and torsion of λ-elasticae are known. In fact, the curvature and torsion, κ and τ, respectively are given by [9]

$$\kappa^2(s) = k_o^2\left(1-\frac{p^2}{\omega^2}\,\mathrm{sn}^2\left(\frac{k_o}{2\omega}\,s\,,p\right)\right) = \frac{c}{\tau}\,, \tag{31}$$

where $c \in \mathbb{R}$, $k_o \in \mathbb{R}^+$, $\mathrm{sn}(x,p)$ represents the Jacobi elliptic sine of modulus p, and the other parameters satisfy

$$0 \leq p \leq w \leq 1,\ 2\lambda = \frac{k_o^2}{\omega^2}\left(3\omega^2-p^2-1\right),\, 4c^2 = \frac{k_o^6}{\omega^4}\left(1-\omega^2\right)\left(\omega^2-p^2\right). \tag{32}$$

Now, by using (31) and well known properties of elliptic integrals and Jacobi elliptic functions [1], one gets

$$\int_o^s \tau\, dt = \frac{2\,c\,\omega}{k_o^3}\Pi\left(\frac{\mathrm{p}^2}{\omega^2}, \mathrm{am}\left(\frac{\mathrm{k_o}}{2\omega}\,\mathrm{s},\mathrm{p}\right),\mathrm{p}\right)\,, \tag{33}$$

$$\begin{aligned} z(s) &= \int_o^s \left(\frac{\lambda}{2}-\frac{\kappa^2(t)}{2}\right) dt \\ &= \left(\lambda + k_o^2\left(\frac{1}{\omega^2}-1\right)\right)\frac{s}{2} - \frac{k_o}{\omega}\mathrm{E}\left(\mathrm{am}\left(\frac{\mathrm{k_o}}{\omega}\,\mathrm{s},\mathrm{p}\right),\mathrm{p}\right)\,, \end{aligned} \tag{34}$$

where $\Pi\left(\mathrm{n},\mathrm{s},\mathrm{p}\right)$ denotes the incomplete Elliptic integral of the third kind with characteristic n and modulus p, $\mathrm{E}\left(s,p\right)$ denotes the incomplete Elliptic integral of the second kind with modulus p, and $\mathrm{am}\left(s,p\right)$ is the Jacobi elliptic amplitude of modulus p. Hence, by substitution of (33) and (34) in (18) *we obtain the explicit parametrization of the complete family of magnetic trajectories for a Killing rotational magnetic field (the variation range of the parameters is described right after* (31) *and in* (32)*).* Solving directly the ODE system obtained by expressing the Lorentz equation (6) in cylindrical coordinates, a similar parametrization of magnetic trajectories for a

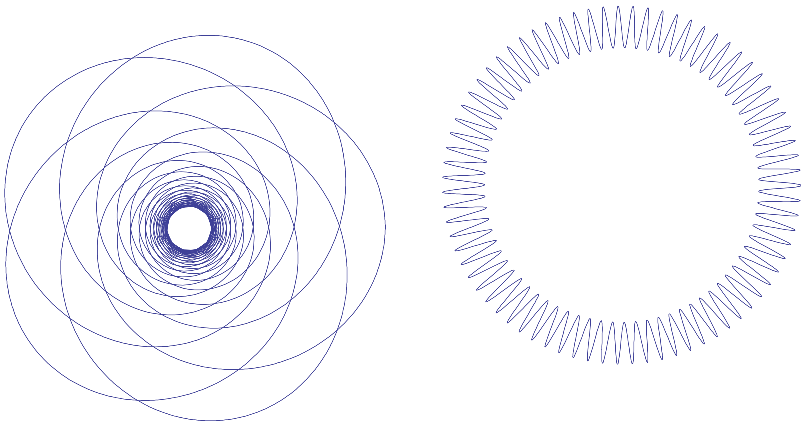

Figure 2: Closed projections of two non-closed magnetic trajectories corresponding to Hasimoto transformations of two λ-elasticae with different λ.

Killing rotational magnetic field was obtained in [4], however, apparently the authors did not realize their connection with elastic curves.

Remark 4.4 Although, in general, rotational magnetic fields admit closed magnetic trajectories, the rotational Killing magnetic field does not. In fact, it is known that every compact surface of revolution has infinite many closed geodesic and, therefore, using Theorem 2.1 these geodesics are closed trajectories for the associated magnetic rotational field. However, from (18) we see that the third component of a magnetic trajectory of the Killing rotational field satisfies $z'(s) = \frac{\lambda}{2} - \frac{\kappa^2(s)}{2}$ which is monotonically decreasing if $\lambda \leq 0$. Moreover, if $\lambda > 0$, then we have from (34) that the coefficient of s of $z(s)$ is not zero. Therefore, $z(s)$ is not periodic in any case, what means that these magnetic trajectories do not close up and, consequently, the surfaces of revolution where they live as geodesics according to Theorem 2.1 can not be compact. On the other hand, let T be the period of $\kappa(s)$ as defined in (31). If there are positive integers m and n such that $mT = n\pi = h$, then h is a period of the first two components of the magnetic trajectories as described in (18). This means that there are choices of the parameters c, k_o, p and w in (33) so that the projections of the corresponding magnetic trajectories on the plane $z = 0$ are closed curves (see Figure 2).

Finally, we identify the magnetic trajectories of the Killing magnetic field ∂_θ as extremals of a variational problem. *Killing vector fields* along curves were introduced in [9, 10]). Given a curve in $\mathbb{R}^3$, say $\alpha(u)$, we consider the three following fundamental geometric invariants: speed, $v(u) = |\alpha'(u)|$, curvature $\kappa(u)$ and torsion $\tau(u)$. Now, a vector field $V(u)$ along $\alpha(u)$ is said to be a Killing field along the curve if

$$V(v) = V(\kappa) = V(\tau) = 0. \tag{35}$$

A straightforward long computation involving the Frenet frame of the curve, $\{T, N, B\}$, provides the following variation formulae (see [10], proposition 1)

$$\begin{aligned} V(v) &= \langle \nabla_T V, T\rangle\, v\,, && (36)\\ V(\kappa) &= \langle \nabla_T^2 V, N\rangle - 2\langle \nabla_T V, T\rangle\, \kappa\,, && (37)\\ V(\tau) &= -T\left(\frac{1}{\kappa}\langle \nabla_T^2 V, B\rangle\right) - \langle \nabla_T V, \tau T + \kappa B\rangle\,, && (38)\end{aligned}$$

and consequently the Killing vector fields along a curve form a six dimensional linear space. Moreover, the restriction to any curve of a Killing vector field in $\mathbb{R}^3$ gives a Killing vector field along that curve. Then every Killing vector field along a curve, α, is the restriction to α of a Killing vector field on $\mathbb{R}^3$, [9].

In particular, taking $V(s) = \kappa(s)\, B(s)$ along a unit speed curve, the equation (36) automatically holds and equations (37) and (38) turn out to be (25) and (26), respectively. Therefore elasticae in $\mathbb{R}^3$ are characterized by the fact that the vector field $V = \kappa\, B$ is Killing along the curve. Since, as it has been mentioned before, a Killing field on a curve extends to a Killing vector field on the whole $\mathbb{R}^3$, we see that elastica provide congruence solutions to LIE (4), [10] (alternatively, they give soliton solutions to the Schrödinger equation (16)). Combining this information with the one-to-one correspondence between non-circular elastic curves and magnetic trajectories of rotational Killing magnetic fields obtained in theorem 4.1 via the extended Hasimoto transformations, we have

Corollary 4.5 *Rotational Killing magnetic flows can be constructed upon those solutions of the cubic nonlinear Schrödinger equation associated to filaments that evolve under LIE (4) without changing shape, only position.*

Moreover, if $\alpha(s)$ is a trajectory of a Killing vector field V, then [2]

$$V(s) = w\, T(s) + \kappa(s)\, B(s)\,, \qquad (39)$$

for some constant $w \in \mathbb{R}$. Thus, either $w = 0$ and V is perpendicular to the trajectory $\alpha(s)$, for any value of s, or $w \neq 0$ and they are never perpendicular. Assume now that $V = \eta\partial_\theta$, $\eta \in \mathbb{R}$. If $w = 0$, then combining (39), the value of $V = \eta\partial_\theta$ along $\alpha(s)$, and the Frenet equations, one may see that $\alpha(s)$ has to be a planar curve satisfying $\alpha_1(s) = \delta\, \kappa(s)$, where $\delta \in \mathbb{R}$ and $\alpha = (\alpha_1, \alpha_2)$ are the coordinates of $\alpha(s)$ with respect to a suitable coordinate system in the plane containing the curve. Therefore, it must be a planar elastica (see the appendix). Thus, if $\alpha(s)$ is a planar elastica, then, $\mathcal{H}_{\lambda/2}(\alpha(s))$ is $\alpha(s)$ itself (up to congruences) and it is a meridian of the rotational Hasimoto surface $S_\alpha(s)$ swept out by $\alpha(s)$. Notice that $S_\alpha(s)$ is foliated by congruent copies of $\alpha(s)$, so that $\mathcal{H}_{\lambda/2}$ sends the whole meridian foliation of $S_\alpha(s)$ to $\alpha(s)$. If $w \neq 0$, then one can use (36)-(38) to show that the trajectory satisfies the Euler-Lagrange equations for the extremals of $\int_\gamma \kappa^2 + w\tau + \lambda$ and then it has to be a non-planar centerline of a Kirchhoff elastic rod [2]. These trajectories correspond to the images by $\mathcal{H}_{\lambda/2}$ of the foliation formed by congruent copies of the initial condition, of the Hasimoto surfaces swept out by a non-planar elasticae.

5 Appendix

For completeness, we include here a proof of a well known property of planar elasticae which has been used in the paper. Assume we are given a unit speed planar curve $\alpha(s)$ whose curvature function $\kappa(s)$ is a multiple of its distance to some straight line L (see also [13]). Without loss of generality, we may assume the line L coincides with the y-axis, so that we have

$$\kappa(s) = \mu\, x(s)\,, \tag{40}$$

where the curve coordinates are $\alpha(s) = (x(s), y(s))$ and $\mu \in \mathbb{R}$. If $\kappa(s)$ is constant, then $\alpha(s)$ is a line and $\kappa(s) = \mu = 0$. So assume, $\kappa(s)$ is not constant. Then, the condition (40) and the definition of the curvature imply

$$x''(s) + \mu\, x(s)\, y'(s) \;=\; 0\,, \tag{41}$$

$$y''(s) - \mu\, x(s)\, x'(s) \;=\; 0\,. \tag{42}$$

Now, the second equation (42) can be integrated to get $y'(s) = \frac{\mu}{2}\, x^2(s) + A$, for some constant A. So, plugging this information into the equation (41), we obtain

$$x''(s) + \frac{\mu^2}{2}\, x^3(s) + A\mu\, x(s) = 0\,,$$

which, using (40) can be written in terms of the curvature giving

$$\kappa''(s) + \frac{1}{2}\, \kappa^3(s) + A\mu\, \kappa(s) = 0\,. \tag{43}$$

Therefore, the curve satisfies (25) and (26) for $\tau = 0$, and $\alpha(s)$ has to be a non-circular elastica in $\mathbb{R}^2$.

Conversely, assume that $\alpha(s)$ is a non-circular planar elastica. Then, since $\tau = 0$, equations (25) and (26) imply that its curvature function $\kappa(s)$ is a solution of (43) for some $A, \mu \in \mathbb{R}$, $\mu \neq 0$. Then, we define a new curve $\beta(s) = (\widetilde{x}(s), \widetilde{y}(s))$ by

$$\widetilde{x}(s) \;=\; \frac{1}{\mu}\, \kappa(s)\,,$$

$$\widetilde{y}(s) \;=\; \int_0^s \left(\frac{\mu}{2}\, \widetilde{x}^2(u) + A\right) du\,.$$

Differentiating the length of the tangent vector $\frac{d\beta}{ds}(s) = \beta'(s)$ and using the equation (43) one sees that

$$\frac{d}{ds}|\beta'(s)|^2 = \frac{d}{ds}\left[(\kappa')^2 + \frac{1}{4}\,\kappa^4 + A\mu\kappa^2 + A^2\mu^2\right] = 2\kappa'\left(\kappa'' + \frac{1}{2}\kappa^3 + A\mu\kappa\right) = 0,$$

which shows that $\beta(s)$ is parametrized proportionally to its arc-length. Furthermore, since $\alpha(s)$ is not a circle, $\widetilde{x}'(s) \neq 0$ and, then, it is not difficult to check that $\beta''(s) = \mu\, \widetilde{x}(s)(-\widetilde{y}'(s), \widetilde{x}'(s))$, which shows that the curvature of β satisfies

$$\kappa_\beta(s) = \mu\, \widetilde{x}(s) = \kappa(s)\,,$$

and, therefore, both curves α and β are congruent. *Hence, planar curves whose curvature function is a multiple of the distance to some straight line are, precisely, the non-circular elastic curves of the plane* (for a related property, see also [13]).

Acknowledgments Research partially supported by the following grants: MB by MINECO-FEDER grant MTM2013-47828-C2-1-P and J. Andalucía Regional Grant P09-FQM-4496; AF by MINECO-FEDER grant MTM2015-65430-P; OJG by MINECO-FEDER grant MTM2014-54804-P and Gobierno Vasco grant IT1094-16. The three authors by Fundación Séneca project 19901/GERM/15, Spain.

References

[1] M. Abramowitz, I. A. Stegun, (Eds), Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, 9th printing, Dover, New York, (1972), pp. 587-607.

[2] M. Barros, A. Romero, Magnetic Vortices, EPL 77 (2007) 34002 p1-5.

[3] R. Betchov, On the curvature and torsion of an isolated vortex filament, J. Fluid Mech. 22 (1965) 471-479.

[4] S.L. Druta-Romaniuc, M.I. Munteanu, Magnetic curves corresponding to Killing magnetic fields in $\mathbb{R}^3$, J. Math. Phys. 52 (2011)113506.

[5] Hasimoto, H.: Motion of a vortex filament and its relation to elastica, J. Phys. Soc. Japan 31, 293-294 (1971).

[6] H. Hasimoto, A soliton on a vortex filament, J. Fluid Mech. 51 (1972) 477-485.

[7] T. Ivey, Helices, Hasimoto surfaces and Bäcklund transformations, Canadian Math. Bull. 43 (2000) 427-439.

[8] S. Kida, A vortex filament moving without change the form, J. Fluid Mech. 112 (1981) 397-409.

[9] J. Langer, D. A. Singer, The total squared curvature of closed curves, J. Diff. Geo. 20 (1984) 1-22.

[10] J. Langer, D. A. Singer, Knotted elastic curves in $\mathbb{R}^3$, J. London Math. Soc. 30 (1984) 512-520.

[11] S. E. Rasor, The Geodesic Lines on the Helicoid, Ann. of Math. 11 (1910) 77-85.

[12] L. D. Rios, On the motion of an unbouded fluid with a vortex filament of any shape, Rend. Circ. Mat. Palermo 22 (1906) 117-132.

[13] D. A. Singer, Curves whose curvature depends on Distance from the origin, Amer. Math. Monthly 106 (1999) 835-841.