Published for SISSA by Springer

Received: *March 9, 2012*
Revised: *April 23, 2012*
Accepted: *April 30, 2012*
Published: *May 15, 2012*

Null scrolls as fluctuating surfaces: a new simple way to construct extrinsic string solutions

Manuel Barros[a] **and Angel Ferrández**[b,1]

[a] *Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, 1807 Granada, Spain*

[b] *Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain*

E-mail: mbarros@ugr.es, aferr@um.es

Abstract: We exhibit a surprising phenomenon that happens in the conformal Lorentz Minkowski three space, which has no counterpart in a Riemannian setting. Whenever a curve, no matter its causal character, propagates transversely through a conformal geodesic null vector field, it is generating the worldsheet of an extrinsic Polyakov string solution. Furthermore, the Polyakov extrinsic energy of these solutions only depend on the world-sheet topology and it can be computed not only intrinsically, but also holographically by measuring the hyperbolic angles in the boundary corners. This geometric approach, to provide extrinsic string solutions, can be considered as an alternative to the Pohlmeyer reduced mechanism. Then, we describe how to translate these solutions to the language of Pohlmeyer theory. We will also show that any curve in the conformal boundary of the anti de Sitter 3-space can be viewed as a piece of the generalized Wilson loops associated with an extrinsic string solution obtained by this geometric mechanism.

Keywords: Conformal Field Models in String Theory, String Field Theory

[1] Corresponding author.

doi:10.1007/JHEP05(2012)068

JHEP05(2012)068

Contents

1 Introduction

As long as we know, it is important to construct classical string solutions in $\mathbb{H}^5_1 \times \mathbb{S}^5$, where we write $\mathbb{H}^5_1$ by $\mathbf{AdS}_5$, because the **AdS/CFT** correspondence. Now, people construct classical string solutions in $\mathbb{H}^3_1 \times \mathbb{S}^3$ and then, after embedding, in the bigger space. Most of these solutions correspond with stationary string surfaces, that is, those with zero mean curvature function (see for example [10, 14, 21] and references therein). The Pohlmeyer reduction provides a powerful and elegant tool in this process, because it makes equivalent the related sigma models in both factors to the sinh-Gordon and sin-Gordon equations, respectively. To apply this construction in the anti de Sitter factor several methods are used. For example, the dressing method is based on the choice of a vacuum solution of the string equation. People usually pick a minimal (spacelike) or stationary (timelike) surface that corresponds with Hopf surfaces obtained, respectively, when lifting geodesics in $\mathbb{H}^2_1$ and $\mathbb{H}^2$ via the corresponding Hopf mappings from $\mathbb{H}^3_1$ to these surfaces. The chosen vacuum solutions play the same role as the Clifford torus in the 3-sphere.

In this paper, we consider the Polyakov extrinsic string action, which is known in differential geometry as the Willmore functional. Its relation with the Nambu-Goto-Polyakov action, that provides the classical string theory, is conceptually discussed. In particular, the latter can be viewed as a sub-theory of the former one. Therefore, each stationary surface, which is a classical string solution, provides in addition an extrinsic string solution.

Nowadays, we know some geometrical methods to construct extrinsic string solutions in both $\mathbb{S}^3$ and $\mathbb{H}^3_1$. The most popular reduces the searching for extrinsic string solutions with a certain degree of symmetry to that of elastic curves, that is, curves that are critical for the total squared curvature, in suitable surfaces (see [3] and references therein, see also [16] as a key reference for the Bernoulli elastica). The action certainly takes place directly in $\mathbb{H}^3_1$, $\mathbb{S}^3$ or in any conformal picture of these backgrounds. As far as we know, the relation of these solutions with the sinh-Gordon and sin-Gordon models, via the Pohlmeyer reduced theory, is unknown.

We describe a surprisingly new geometric method to generate as many extrinsic string solutions as we wish. The core of this method is as simple as the following statement: *Whenever a curve propagates in the* Lorentz-Minkowski three space, $\mathbb{L}^3$, *through a geodesic null vector field, it is generating the worldsheet of an extrinsic string configuration.* We also exhibit an algorithm that allows us to explicitly construct that huge family of solutions. It should be noted that these solutions are not necessarily stationary (zero mean curvature) in $\mathbb{L}^3$. However, its moduli space contains a submoduli, made up of stationary solutions, which, in turn, can be nicely described in terms of three moduli: two functions determining the propagating curve, which can be chosen to be lightlike, and a real number which plays the role of lightlike slope (see [4] for details).

Nevertheless, the surprise, regarding this simple phenomenon, goes further. We will see that every extrinsic string solution, in this new family, carries a topological charge that can be holographically computed. More precisely, we will show that the Polyakov extrinsic energy of these solutions only depend on the worldsheet topology, and, furthermore, it is encoded in the boundary and can be computed just measuring the hyperbolic angles in the boundary corners.

On the other hand, since the Polyakov extrinsic action measures the Willmore energy of worldsheets, the conformal invariance of the model becomes obvious. This allows one to bring the so built extrinsic string solutions, for example using a stereographic projection, to the 3-dimensional anti de Sitter space, $\mathbb{H}^3_1 = \mathbf{AdS}_3$. To this respect, we indicate a method to get those conformal stationary surfaces, that is extrinsic string solutions that provide classical ones in the anti de Sitter context. This is made in the last section, where the natural question, in connection with its relationship with the Pohlmeyer reduced theory, is considered. Though the general treatment seems theoretically clear, it involves formidable computations. Finally, we consider the behavior of these extrinsic string solutions in the conformal boundary of the anti de Sitter three space to study the corresponding generalized Wilson loops.

2 Fluctuating geometry

Many interesting systems in physics and biology are described by fluctuating surfaces. String theories and theory of biological membranes are two enlightening samples in this wide variety of nonlinear phenomena. Though all of these theories describe phenomena of different nature, they are amazingly associated, basically, with the same action. This kind of universality may be strongly related to the fact that very often such actions have an underlying geometric meaning. An important characteristic of these theories is that the fluctuations do not change the topology of the surfaces. Therefore, in agreement with this requirement, we state the following setting for dynamical variables. Let Γ be a set of nonnull piecewise regular curves in $\mathbb{L}^3$ and N_o a spacelike unit normal vector field along Γ. We choose a surface S with boundary ∂S (which could be empty) and denote by $\mathbf{I}_\Gamma(S, \mathbb{L}^3)$

the space of timelike immersions satisfying the following first order boundary conditions

$$\phi(\partial S) = \Gamma, \qquad N_\phi/_\Gamma = N_o,$$

where N_ϕ stands for the Gauss map associated with the immersion ϕ. Roughly speaking, by identifying each immersion $\phi \in \mathbf{I}_\Gamma(S, \mathbb{L}^3)$ with its graph, $\phi(S)$, viewed as a surface with boundary in $\mathbb{L}^3$, then $\mathbf{I}_\Gamma(S, \mathbb{L}^3)$ can be viewed as the space of timelike surfaces in $\mathbb{L}^3$ having the same boundary and being tangent along the common boundary. It should be noted that these first order boundary conditions have already been considered in the compact case (see for example [2, 8] and some references therein).

The discussion of strings historically began with the Nambu-Goto action which is defined, on the space of immersions that fix the boundary but need not be tangent along the common boundary, by

$$\mathcal{NG}(\phi) = c_o \int_S dA_\phi,$$

where c_o is a constant related with the tension of the surface and dA_ϕ is the element of area relative to the induced metric. Now, strings are curves that evolve in the target space generating surfaces that provide extremals of this energy action. This topic, from a geometric point of view, is well understood for a long time and the string solutions correspond with those surfaces with zero mean curvature $H_\phi = 0$. Let us point out a couple of remarks:

(1) The first one is merely formal. Surfaces with zero mean curvature function in a Riemannian setting are called *minimal surfaces*. However, this term is not appropriated for timelike surfaces in a Lorentzian context (for example in the spacetime $\mathbb{L}^3$ or $\mathbf{AdS}_3$). Several authors still use mimetically the term, as well as others use the term *maximal surfaces*. Nevertheless, most of people use the more appropriate one of *stationary surfaces*.

(2) Our second remark concerns to the core of this string theory. It is obvious that the values of the Nambu-Goto energy, in particular *extremal values* (those reached by the string solutions), have an *intrinsic nature*. In other words, those values can be computed inside the worldsheets, so that people living in a string solution are able to measure the Nambu-Goto tension that its world receives from the surrounding spacetime. On the contrary, those people have no idea of the extremal nature of the world where they are living, because the *mean curvature* has an *extrinsic nature*. Thus, though the nature of the Nambu-Goto is intrinsic, it provides string solutions whose critical nature can not be intrinsically valued.

The Nambu-Goto action presents problems if one wishes to quantize the string using a path-integral approach. In this respect, A. M. Polyakov [19] proposed to replace the area action by an *equivalent action* that involves an intrinsic metric besides the induced one from the ambient spacetime metric. Both theories provide the so called *classical string solutions* that correspond with *stationary surfaces* ($H = 0$). It should be noted that the new Polyakov action is still intrinsic from its own origin.

Certainly, from a geometric point of view, this intrinsic-extrinsic disagreement between action and solutions is not satisfactory. If we wish to evolve curves in a target spacetime to generate surfaces being extremals of a certain action, it seems natural to *involve the extrinsic geometry of surfaces in the density of the action.* This idea was materialized in 1986 independently by A. M. Polyakov [20] and H. Kleinert [15]. Both authors introduced the same new string action using different motivations and methods. In fact, Kleinert defined the action trying to imitate the elastic functional for membranes, obviously in a Eucliden context, introduced in 1973 by W. Helfrich (Z. Naturforsch 33a, 305). On the contrary, Polyakov used two kind of arguments. On the one hand, something related with the qualitative properties (critical behavior of the string tension and others) of strings. On the other hand, a fascinating argument showing that it is the only choice (up to divergences) of action that is invariant under similarity transformations. In this way the so called *Polyakov extrinsic action* was born as a string action. More precisely, this action is defined on $\mathbf{I}_\Gamma(S,\mathbb{L}^3)$ and it measures the total extrinsic curvature of the pair $(\phi(S),\phi(\partial S))$ in $\mathbb{L}^3$,

$$\mathcal{PKH}:\mathbf{I}_\Gamma(S,\mathbb{L}^3)\to\mathbb{R},\qquad \mathcal{PKH}(\phi)=\int_S H_\phi^2\,dA_\phi-\int_{\partial S}\kappa_\phi\,ds,$$

where H_ϕ stands for the mean curvature of the immersion $\phi(S)$ and κ_ϕ is the geodesic curvature of $\phi(\partial S)$ in $\phi(S)$.

In this paper, we deal with the dynamics associated with this Polyakov extrinsic string action, which by the way, is the flat version of the so called, in differential geometry, *Willmore functional*. On the space of boundary immersed timelike surfaces, which are tangent along the common boundary, in a generic spacetime, say M, it works as

$$\mathcal{W}(\phi)=\int_S (H_\phi^2+R_\phi)\,dA_\phi-\int_{\partial S}\kappa_\phi\,ds,$$

where R_ϕ stands for the sectional curvature of the target space on the tangent plane of $\phi(S)$, the extrinsic Gaussian curvature of the surface. The systematic study of the variational problem associated with this action was proposed by T. J. Willmore in a meeting at Oberwolfach in 1960. Since then, it has became into a very popular problem of great interest not only in the theory of surfaces, but also in other different contexts of mathematics and physics. This popularity is due in part to the still open Willmore conjecture. However, it is of special interest, because its conformal invariance. More precisely, the above Willmore action is invariant under conformal changes in the target spacetime metric and consequently it is a problem stated in the conformal class of that metric. Obviously, this important property extends that for similarity transformations, which was showed by Polyakov. Moreover, it should be noted that it was known by W. Blaschke and G. Thomsen in 1923, [5, 22], for the conformal class of the Euclidean metric. The so called Blaschke's program was extended to any Riemannian metric and with slight changes to Lorentzian conformal classes (see for example [7] and [23]). As an obvious consequence, *the Polyakov extrinsic string theories in both* $\mathbb{L}^3$ *and* $\mathbf{AdS}_3$ *are equivalent.*

Summarizing, we have two non-equivalent string theories associated with corresponding different string actions:

1. The Nambu-Goto string action, $\mathcal{NGP}$, which is equivalent to the intrinsic Polyakov one, that provides solutions being stationary surfaces, $H = 0$, usually called classical string solutions.

2. The Polyakov extrinsic action that, in contrast to the previous one whose origin is intrinsic, contemplates the extrinsic geometry of the string surfaces in the target spacetime. Similarly, it should be called as the Willmore-Polyakov-Kleinert-Helfrich ($\mathcal{WPKH}$) action, and actually it is defined in the conformal class of the target metric.

The field configurations, or critical points, associated with the $\mathcal{WPKH}$ action are known in differential geometry as Willmore surfaces for the prescribed boundary conditions. In the context of string theories, they are worldsheets of the Polyakov extrinsic string action and so bearing in mind the original extrinsic nature of the action they will be called *extrinsic string solutions*. However, the concept of critical point needs some extra technical considerations. A critical point of such a problem means a critical point of the induced problem on reasonable compact pieces or nonnull polygons. More precisely, a connected, simply connected with nonempty interior, compact domain, $\Omega \subset S$, is said to be a *nonnull polygon* if it has a piecewise smooth boundary, $\partial\Omega$, which is made up of a finite number of nonnull curves. Now, $\phi \in \mathbf{I}_\Gamma(S, \mathbb{L}^3)$ provides a classical string solution if for any nonnull polygon $\Omega \subseteq S$, the restriction $\phi|_\Omega$ is a critical point of the Polyakov extrinsic action on $\mathbf{I}_{\phi(\partial\Omega)}(\Omega, \mathbb{L}^3)$.

The field equation associated with this variational problem, computed in [3], is

$$\Delta_\phi H_\phi + 2H_\phi \left(H_\phi^2 - K_\phi\right) = 0, \tag{2.1}$$

where K_ϕ denotes the Gaussian curvature of $\phi(S)$. In particular, every stationary surface ($H = 0$) is automatically Willmore and consequently the $\mathcal{NGP}$ string theory can be regarded as a sub-theory of the $\mathcal{WPKH}$ string theory. Said otherwise, *in the moduli space of the extrinsic string solutions one can find a sub-moduli space made up of the classical string solutions.*

It should be noted that, in particular, the surface S could be boundary free and in this case no boundary condition is needed. Let us give, as an illustration, a pair of explicit examples (see [6]):

Example 2.1. *A rotational stationary surface: The hyperbolic catenoid.*

Choose $S = \mathbb{R}^+ \times \mathbb{R}$ and define the timelike immersion $\phi \in \mathbf{I}(S, \mathbb{L}^3)$ by

$$\phi(s,t) = (\sinh s \, \sinh t, s, \sinh s \, \cosh t).$$

Then we obtain a stationary surface and consequently it is a classical string solution as well as an extrinsic string solution.

Example 2.2. *A ruled stationary surface: The helicoid of the third kind.*

Take $S = \mathbb{R}^2$ and the timelike immersion $\phi \in \mathbf{I}(S, \mathbb{L}^3)$ given by

$$\phi(s,t) = \left(\frac{t}{\cosh s}, -t \, \tanh s, \sinh s\right).$$

JHEP05(2012)068

It is easy to check that its mean curvature function vanishes identically and so it provides a string solution in both theories.

Besides those classical string solutions that correspond with stationary surfaces, other extrinsic string solutions with non-zero constant mean curvature are known (see [21]). It should be noted that this kind of solutions are examples of the so called Weingarten surfaces and, in some sense, they play the same role as the round sphere in the Euclidean context (see for example [9]). Furthermore, the whole class of extrinsic string solutions admitting a rotational symmetry has been obtained in [3] and it can be briefly described as follows:

1. Rotational worldsheets obtained when rotating a timelike free elastic curve, around a nonnull axis, in a de Sitter plane.

2. Rotational worldsheets with spacelike axis and whose profile string is a free elastic curve in a hyperbolic plane.

3. Rotational worldsheets with lightlike axis and profile string a timelike free elastic curve in an anti de Sitter plane.

3 Null scrolls as extrinsic string solutions

Let $\gamma : I \subset \mathbb{R} \to \mathbb{L}^3$ be a regular curve in $\mathbb{L}^3$ and $B(s)$, $s \in I$, a vector field along the curve which is transversal to $\gamma(s)$ everywhere. Let $\phi : I \times \mathbb{R} \to \mathbb{L}^3$ be the immersion defined by

$$\phi(s,t) = \gamma(s) + t\, B(s).$$

As for the first fundamental form we have $\phi_s = \gamma'(s) + t\, B'(s)$ and $\phi_t = B(s)$, so that $\phi(s,t)$ parametrizes a timelike surface, $S(\gamma, B)$, on the domain $\{(s,t) \in I \times \mathbb{R} : \mathbf{g}(s,t) = \langle \phi_s, \phi_s\rangle\langle \phi_t, \phi_t\rangle - \langle \phi_s, \phi_t\rangle^2 < 0\}$, which we will call ruled surface with base curve $\gamma(s)$ and ruling flow $B(s)$. It is clear that we can consider two classes of timelike ruled surfaces according to the ruling flow is lightlike or not. The ruled surfaces with lightlike ruling flow are called *null scrolls* ([11, 18]). One of the aims of this paper is to show the following:

Theorem 3.1. *Every null scroll is an extrinsic string solution in the Lorentz-Minkowski conformal structure.*

Proof. To prove this result, we first note that given a null scroll $S(\gamma, B)$, parametrized by $\phi(s,t) = \gamma(s) + t\, B(s)$, then we can normalize the ruling flow to have

$$\langle \gamma'(s), B(s)\rangle = -1.$$

Moreover if the base curve is non null, then without loss of generality, we can parametrize it by its arclength so $\langle \gamma'(s), \gamma'(s)\rangle = \epsilon$, where $\epsilon = 1$ if the curve is spacelike, while $\epsilon = -1$ when the base curve was chosen to be timelike. However, we can change, if necessary, the base curve in order to be a lightlike curve (see [1]). In fact, if $\langle \gamma'(s), \gamma'(s)\rangle = \epsilon$, then we look for a curve $\beta(s) = \gamma(s) + t(s)\, B(s)$ satisfying $\langle \beta'(s), \beta'(s)\rangle = 0$ and $\langle \beta'(s), B(s)\rangle \neq 0$.

JHEP05(2012)068

From $\beta'(s) = \gamma'(s) + t'(s)\, B(s) + t(s)\, B'(s)$, we determine the function $t(s)$ as a solution of the following Ricatti differential equation

$$2\, t'(s) = \langle B'(s), B'(s)\rangle\, t^2(s) + 2\langle \gamma'(s), B'(s)\rangle\, t(s) + \epsilon,$$

where it should be noted that $\langle \beta'(s), B(s)\rangle = \langle \gamma'(s), B(s)\rangle = -1$. Consequently, any null scroll $S(\gamma, B)$ can be parametrized by $\phi(s,t) = \gamma(s) + t\, B(s)$ trough a null base curve and

$$\langle \gamma'(s), \gamma'(s)\rangle = \langle B(s), B(s)\rangle = 0 \qquad \text{and} \qquad \langle \gamma'(s), B(s)\rangle = -1.$$

In this setting the induced metric on the null scroll $S(\gamma, B)$ is given by the following matrix

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 2t\,\langle \gamma'(s), B'(s)\rangle + t^2\,\langle B'(s), B'(s)\rangle & -1 \\ -1 & 0 \end{pmatrix}.$$

Now, the Laplacian in this reference is

$$\Delta = -2\frac{\partial^2}{\partial s \partial t} - 2\left(\langle \gamma', B'\rangle + t\,\langle B', B'\rangle\right)\frac{\partial}{\partial t} - \left(2t\langle \gamma', B'\rangle + t^2\,\langle B', B'\rangle\right)\frac{\partial^2}{\partial t^2}.$$

To compute the Gauss map and the shape operator of $S(\gamma, B)$, we define $C(s) = \gamma'(s) \times B(s)$. Then $C(s)$ is a unit spacelike vector field along $\gamma(s)$, which is anywhere orthogonal to $\mathbf{S}(\gamma, B)$ and so it defines the Gauss map of the null scroll along its directrix. The Gauss map is given by

$$N(s,t) = \frac{\phi_s(s,t) \times \phi_t(s,t)}{|g_{11}g_{22} - g_{12}^2|} = C(s) + t\, B'(s) \times B(s).$$

Now, an easy computation allows us to see that

$$B'(s) \times B(s) = -f(s)\, B(s),$$

where $f(s) = \langle \gamma'(s), B'(s) \times B(s)\rangle = \det[\gamma'(s), B'(s), B(s)]$, and then the Gauss map of the null scroll is given by

$$N(s,t) = -t\, f(s)\, B(s) + C(s).$$

Sometimes the function $-f(s)$ is called the *parameter of distribution* of the ruled surface $S(\gamma, B)$.

A straightforward computation yields

$$N_s = -f\, X_s - \left(\langle \gamma', \gamma'' \times B\rangle + t\, f'\right) X_t$$

and

$$N_t = -f\, X_t,$$

so the matrix of the shape operator dN is

$$dN \equiv \begin{pmatrix} f & t\, f' + \langle \gamma', \gamma'' \times B\rangle \\ 0 & f \end{pmatrix}.$$

Therefore, mean and Gauss curvatures functions of the null scroll $\mathbf{S}(\gamma, B)$ are given by

$$H(s,t) = f(s) \qquad \text{and} \qquad K(s,t) = f(s)^2.$$

This gives us an important information, namely

$$\Delta H(s,t) = 0, \quad \text{and} \quad H^2(s,t) - K(s,t) = 0,$$

and the proof finishes. $\square$

The above result can be paraphrased as: *Whenever a curve propagates in* $\mathbb{L}^3$ *transversely through a geodesic null vector field, it is generating the worldsheet of an extrinsic string solution.* However, the case where curves propagate through geodesics, nonnull flows, behaves quite different. To illustrate this situation, we propose the following experiment. Suppose that γ is a curve contained in the Euclidean plane $P = \{(x, y, 0)\}$ in $\mathbb{L}^3$. Let C_γ be the right cylinder over the given curve, which turns out to be a timelike flat surface in $\mathbb{L}^3$. Now, it seems natural to ask *how do we choose the curve* γ *in order to* C_γ *be an extrinsic string solution*? To answer this question, we compute all the ingredients of C_γ appearing in the equation (2.1). Then, we see that C_γ is an extrinsic string solution if and only if the curvature $\kappa(s)$ of $\gamma(s)$ in P satisfies the following second order differential equation

$$2\kappa'' + \kappa^3 = 0, \tag{3.1}$$

which is nothing but the field equation associated with the Bernoulli free elastica in the plane P. This result should be compared with our main result (Theorem 3.1). Now, the general solution of this second order differential equation can be expressed in terms of the Jacobi elliptic cosine

$$\kappa(s) = \sqrt{\kappa_o}\, \mathbf{cn}\left(\sqrt{\frac{\kappa_o}{2}}\, s, \sqrt{\frac{1}{2}}\right), \qquad \kappa_o > 0. \tag{3.2}$$

In general, for curves in P, the *solving natural equation problem* can be done by quadratures. In other words, we can recover, using quadratures, the curve from its curvature function. Indeed, assume that we have the curvature function, say $\kappa(s)$, of a curve $\alpha(s) = (x(s), y(s))$ lying in P. Then, we can see $\kappa(s)$ as the instant variation of the angle $\theta(s)$ that the unit tangent to $\alpha(s)$ makes with the x-axis, that is, $\kappa(s) = \frac{d\theta}{ds}$. From here, one can readily obtain the curve from its curvature, $\alpha(s) = (\int \cos \int \kappa(s) ds, \int \sin \int \kappa(s) ds)$. For a general curvature function, these integrations can not be made explicitly, however, we can do it for elastic curves. In fact, first notice that a first integral of (3.1) is

$$(\kappa'(s))^2 + \frac{1}{4}\kappa^4(s) = c,$$

for an arbitrary positive constant c. Then we can write

$$\begin{aligned} \kappa'(s) &= \sqrt{c}\, \sin\varphi(s), \\ \kappa''(s) &= \sqrt{c}\, \kappa(s)\, \cos\varphi(s). \end{aligned}$$

As a consequence, we obtain $\kappa(s) = \varphi'(s) = \theta'(s)$ and so $\kappa'(s) = \sqrt{c}\, y'(s)$. Therefore, we can choose a coordinate system so that $\gamma(s) = (x(s), y(s)$ with

$$y(s) = \frac{1}{\sqrt{c}}\,\kappa(s) = \frac{\sqrt{\kappa_o}}{\sqrt{c}}\,\mathbf{cn}\left(\sqrt{\frac{\kappa_o}{2}}\,s, \sqrt{\frac{1}{2}}\right), \qquad \kappa_o > 0, \quad c > 0. \tag{3.3}$$

Moreover

$$x'(s) = \cos\varphi(s) = \frac{y''(s)}{\sqrt{c}\, y(s)}, \qquad c > 0,$$

and consequently, we obtain

$$x(s) = \frac{\sqrt{\kappa_o}}{\sqrt{c}}\,\mathbf{E}\left(\mathbf{am}\left(\sqrt{\frac{\kappa_o}{2}}\,s, \sqrt{\frac{1}{2}}\right), \sqrt{\frac{1}{2}}\right), \qquad \kappa_o > 0, \quad c > 0, \tag{3.4}$$

where $\mathbf{E}\,(u, p)$ is the elliptic integral of the second kind of modulus p and $\mathbf{am}\,(t, p)$ is the Jacobi amplitude.

As a consequence of this argument, we obtain that *up to Lorentz transformations, besides Lorentzian planes, the right cylinders built on the two parameter class of plane curves with coordinates given in* (3.4) *and* (3.3) *are the only cylinders with timelike ruling flows that provide extrinsic string solutions.*

4 A holographic principle for scrolls solutions

The geometry of null scrolls is plenty of amazing properties. They certainly provide a basic concept to distinguish between Lorentzian and Euclidean geometries. Let us list some of these geometric important properties:

1. The mean curvature function of a null scroll is, up to the sign, the parameter of distribution.

2. Since $H^2(s,t) = K(s,t)$, we can say that null scrolls are Weingarten surfaces (see [9]). Actually, one can see that null scrolls are the only Lorentzian surfaces of $\mathbb{L}^3$ whose mean and Gaussian curvatures are related as above (see also [9, 12]). This provides an important bridge between intrinsic and extrinsic geometries of null scrolls. Therefore, *people living in a null scroll can see how the null scroll is curved in* $\mathbb{L}^3$.

3. A null scroll is stationary ($H = 0$) if and only if it is flat ($K = 0$).

4. A null scroll is a B-scroll if and only if it has constant mean curvature. This gives an alternative definition of B-scrolls without involving any additional tool such as a Cartan frame (see [11]).

5. The functions on a null scroll which are constant along the ruling flow are harmonic.

Even more, this kind of geometry is able to encode important physical properties too. In fact, we have shown that null scrolls can be viewed as extrinsic string solutions and we would like to compute the topological charges that they could carry. To do that, let us compute the critical values of the Polyakov extrinsic energy of a nonnull polygon, say Ω, in a null scroll $S(\gamma, B)$. The boundary $\partial\Omega$ is a simple closed curve made up of a finite number, say n, of smooth nonnull curves. We denote by θ_j, for $1 \leq j \leq n$, the exterior hyperbolic angles at the corresponding vertices. The hyperbolic angle between two nonnull vectors in a Lorentzian surface was introduced in [3] and the Gauss-Bonnet formula for these nonnull polygons was obtained as follows

$$-\int_\Omega K\,dA + \int_{\partial\Omega} \kappa\,ds + \sum_{j=1}^{n} \theta_j = 0.$$

Then we can compute the Polyakov energy of this polygon as

$$\mathcal{WPKH}(\Omega) = \int_\Omega H^2\,dA - \int_{\partial\Omega} \kappa\,ds = \int_\Omega K\,dA - \int_{\partial\Omega} \kappa\,ds = \sum_{j=1}^{n} \theta_j.$$

As a consequence, if we denote by $\mathcal{P}_n$ the space of nonnull polygons with n corners, then

$$\mathcal{WPKH}(\Omega_n) \in \left(-\frac{n\pi}{4}, \frac{n\pi}{4}\right), \qquad \forall \Omega_n \in \mathcal{P}_n.$$

Therefore, the critical values of the Polyakov extrinsic energy on pieces of null scrolls not only have an intrinsic nature, but also they are topological invariants. These topological charges are encoded in their boundaries, namely, in the corners along the boundaries. This shows a holographic principle for the $\mathcal{WPKH}$ critical values which, obviously, has no reply neither in an Euclidean context nor in the $\mathcal{NGP}$ setting. In other words, every curve, propagating in $\mathbb{L}^3$ through a geodesic lightlike flow, generates an extrinsic string solution which, piecewise, carries topological charges that can be holographically computed by measuring the hyperbolic angles in the boundary corners of pieces.

5 An algorithm to build the big zoo of scroll solutions

We are going to provide a simple method to explicitly construct the scroll solutions for the $\mathcal{WPKH}$ string theory, as well as an algorithm to build as many extrinsic string solutions as we wish. To proceed with, it will be useful to consider the following point of view. First, note that a null scroll, $S(\gamma, B)$, is completely determined, up to Lorentz transformations, when we know $\gamma'(s)$ and $B(s)$ lying in the light cone, and satisfying the "normalization condition"

$$\langle \gamma'(s), B(s) \rangle = -1.$$

Then, choose an orthonormal frame in $\mathbb{L}^3$ where the Lorentz-Minkowski metric is written as $dx^2 + dy^2 - dz^2$, and so the light cone $\mathcal{C}$ by $x^2 + y^2 = z^2$. In this framework any vector $\vec{u} \in \mathcal{C}$ may be written as

$$\vec{u} = (x, y, z) = z\,(\cos\alpha, \sin\alpha, 1).$$

Therefore, (z, α), which we will call *elliptic polar coordinates*, parametrize $\mathcal{C}$. Observe that the corresponding coordinate curves are straight lines and circles, respectively. In this frame, the null scroll data can be written as

$$\gamma'(s) = c(s)\,(\cos\omega(s), \sin\omega(s), 1), \qquad B(s) = r(s)\,(\cos\varphi(s), \sin\varphi(s), 1),$$

JHEP05(2012)068

and the normalization condition yields

$$\cos(\omega-\varphi)=1-\frac{1}{c(s)\,r(s)}.$$

A straightforward computation allows us to obtain the mean curvature of $\mathbf{S}(\gamma,B)$, in terms of this frame, as

$$\begin{aligned}H(s,t)=f(s)&=-\det[\mathbf{x}'(s),B(s),B'(s)]\\&=-c(s)\,r(s)\,r'(s)\det\begin{pmatrix}\cos\omega(s)&\sin\omega(s)&1\\\cos\varphi(s)&\sin\varphi(s)&1\\\cos\varphi(s)&\sin\varphi(s)&1\end{pmatrix}\\&\quad-c(s)\,(r(s))^2\det\begin{pmatrix}\cos\omega(s)&\sin\omega(s)&1\\\cos\varphi(s)&\sin\varphi(s)&1\\-\varphi'(s)\,\sin\varphi(s)&\varphi'(s)\,\cos\varphi(s)&0\end{pmatrix}\end{aligned}$$

and then

$$\begin{aligned}H(s,t)&=-c(s)\,(r(s))^2\left[\varphi'(s)-\varphi'(s)\,\cos(\omega-\varphi)\right]\\&=-c(s)\,(r(s))^2\left[\varphi'(s)-\varphi'(s)+\frac{\varphi'(s)}{c(s)\,r(s)}\right].\end{aligned}$$

Consequently, we find that

$$H(s,t)=f(s)=-r(s)\,\varphi'(s). \tag{5.1}$$

Remark 5.1. *The mean curvature of a null scroll only depends on the lightlike ruling flow. In particular, stationary null scrolls ($H=0$) correspond with parallel lightlike ruling flow, that is, ruling flow with $\varphi(s)$ constant. In this sense, they can be regarded as cylinders with lightlike generatrices. The moduli space of stationary null scrolls has been obtained in [4]. It can be viewed as a kind of circle bundle over the space of congruence classes of lightlike curves in $\mathbb{L}^3$. This result deeply contrast with the case of stationary cylinders with nonnull generatrices, where we only get a Lorentzian plane.*

The algorithm. This new framework to study null scrolls has important consequences. Actually, it allows us to construct explicitly the extrinsic strings propagating through lightlike ruling flows. For example, we can give an algorithm to build explicitly the complete class of extrinsic string solutions with prescribed Polyakov extrinsic density, say a function $h\in\mathcal{C}^\infty(I,R)$, and this fact can be viewed as a kind of *solving natural equations* for scroll extrinsic string solutions. To do it, we first choose any positive function, $r(s)$, defined on the same interval and use (5.1) to compute a third function by

$$\varphi(s)=\int_s^0\frac{h(s)}{r(s)}.$$

Then, we have the following lightlike flow

$$B(s)=r(s)\,(\cos\varphi(s),\sin\varphi(s),1),$$

which can be used as the ruling flow to generate all the extrinsic solutions corresponding to scroll string solutions whose Polyakov extrinsic density is the given function $h(s)$. Then, the profile strings of these solutions have an arbitrary positive time function $c(s)$ and an angular function which must be determined from

$$\omega(s) = \varphi(s) + \arccos\left(1 - \frac{1}{c(s)\, r(s)}\right).$$

Now, use quadratures to obtain the profile strings as

$$\gamma(s) = \int_0^s c(u)(\cos\omega(u), \sin\omega(u), 1)du.$$

In this way, we get that the scroll extrinsic string solution $S(\gamma, B)$ has mean curvature function $h(s)$.

To illustrate this algorithm we give the following

Example 5.2. *Suppose that we wish to obtain all scroll extrinsic string solutions, with constant mean curvature, say $h = 1$, which are generated when propagating, in $\mathbb{L}^3$, the lightlike helix $\gamma(s) = (\sin s, -\cos s, s)$.*

To solve this problem, we need to construct the lightlike ruling flows, that allow one to propagate the string in order to get the solutions. We put

$$B(s) = r(s)\,(\cos\varphi(s), \sin\varphi(s), 1),$$

which must satisfies the following two constraints

$$\begin{aligned} \cos\left(s - \varphi(s)\right) &= 1 - \frac{1}{r(s)} && \text{(normalization condition)},\\ \varphi'(s) &= -\frac{1}{r(s)} && \text{(constant mean curvature condition)}. \end{aligned}$$

Consequently, the angular function $\varphi(s)$ must be a solution of the following differential equation

$$\frac{d\varphi(s)}{ds} = \cos\left[s - \varphi(s)\right] - 1.$$

We use the change $\psi(s) = s - \varphi(s)$ to reduce it to

$$\frac{d\psi(s)}{ds} = 2 - \cos\psi(s),$$

which can be easily solved by separation of variables

$$\frac{d\psi}{2 - \cos\psi} = ds,$$

finding the following general solution

$$\psi(s) = 2\,\arctan\left\{\frac{\sqrt{3}}{3}\tan\left[\frac{\sqrt{3}}{2}\,s + C\right]\right\}, \qquad C \in \mathbb{R},$$

which provides the following parameters for the lightlike ruling flows

$$\varphi(s) = s - 2\,\arctan\left\{\frac{\sqrt{3}}{3}\,\tan\left[\frac{\sqrt{3}}{2}\,s + C\right]\right\},$$
$$r(s) = \frac{1}{1 - \cos\left\{2\,\arctan\left\{\frac{\sqrt{3}}{3}\,\tan\left[\frac{\sqrt{3}}{2}\,s + C\right]\right\}\right\}}.$$

Consequently, there exists just a one-parameter class of lightlike flows that allow us to propagate the lightlike helix $\gamma(s) = (\sin s, -\cos s, s)$ to generate extrinsic string solutions with constant mean curvature $h = 1$. Obviously, this result can be extended to other strings.

6 The solutions viewed in the anti de Sitter world

The anti de Sitter three space, $\mathbf{AdS}_3$, with curvature -1 can be viewed in $\mathbb{C}_1^2$, endowed with the induced metric, as the following quadric

$$\mathbb{H}_1^3 = \{(\xi, \eta) \in \mathbb{C}^2 \,:\, |\xi|^2 - |\eta|^2 = -1\}.$$

As $\mathbb{L}^3$ can be naturally identified with $\mathbb{C} \times (-\mathbb{R})$, let $F : \mathbb{L}^3 \to \mathbb{R}$ be defined by $F(\zeta, k) = |\,\zeta|^2 - k^2$. On the open solid hyperboloid $\Sigma = \{(\,\zeta, k) \in \mathbb{L}^3 \,:\, F(\zeta, k) < 1\}$, let $E : \Sigma \to \mathbb{H}_1^3$ be the stereographic map given by

$$E(\zeta, k) = \left(\frac{2}{1 - F(\zeta, k)}\,\zeta, \frac{2}{1 - F(\zeta, k)}\,k + i\,\frac{1 + F(\zeta, k)}{1 - F(\zeta, k)}\right).$$

Now, the anti de Sitter metric, say g, can be pulled back by E to obtain the following, manifestly conformally flat, metric on Σ

$$\bar{g} = E^*(g) = \frac{4}{(1 - F(\zeta, k))^2}\,(|\,d\zeta|^2 - dk^2). \tag{6.1}$$

The $\mathcal{WPKH}$ action measures the Willmore energy of the string worldsheets, which, as we have explained in section 2, makes obvious its conformal invariance. Therefore, as a consequence of our main theorem, we obtain: *Every curve propagating in the anti de Sitter three space, $(\Sigma, \bar{g})$, through a lightlike straight lines flow, generates an extrinsic string solution.* In other words, the class of null scrolls in $\mathbb{L}^3$, when viewed under the metric $\bar{g}$, still provides extrinsic string solutions in the anti de Sitter 3-space.

It is well known that the classical string theory, that is the $\mathcal{NGP}$ one, in anti de Sitter 3-space is equivalent to sinh-Gordon theory via the Pohlmeyer reduction (see [17]). Therefore, each classical string solution in $(\Sigma, \bar{g})$ can be written, at least theoretically, in terms of a wavefunction (a solution of the sinh-Gordon equation). However, finding explicit solutions via this inverse Pohlmeyer mechanism, in general, involve formidable computations (see [14] and references therein).

The problem of finding the class of scroll extrinsic string solutions in $\mathbb{L}^3$ that provides classical string solutions in $\mathbf{AdS}_3$ is equivalent to the following geometric problem: *find* *those null scrolls which are conformal stationary under the conformal change* (6.1). For simplicity, we denote by $g_o = |\,d\zeta|^2 - dk^2$ the metric in $\mathbb{L}^3$, so the conformal change (6.1) can be written as

$$\bar{g} = \rho^2\, g_o, \qquad \rho(\zeta, k) = \frac{2}{1 - F(\zeta, k)}.$$

Then, one needs to know how the mean curvature functions of a surface are related after making a conformal change in the target metric. In our case, the null scrolls in $\mathbf{L}^3$ giving zero mean curvature (stationary surfaces) in $\mathbf{AdS}_3$ are those whose mean curvature function (in $\mathbb{L}^3$) satisfies

$$H = C(\rho) = g_o(\nabla\rho, C),$$

where we are using the notation introduced in section 3. The chief idea to discuss the last equation is that the mean curvature function of a null scroll is completely codified in the ruling flow and, in addition, it is invariant along that flow. Now, this idea can be combined with the algorithm, to construct extrinsic string solutions with prescribed mean curvature function, that we gave in section 5. Therefore, given any function, say $H(s)$, we consider the class of ruling flows $B(s) = r(s)\,(\cos\varphi(s), \sin\varphi(s), 1)$ with $-r(s)\,\varphi'(s) = H(s)$. Now, the null scrolls associated with that class provide the whole family of extrinsic string solutions whose mean curvature is $H(s)$. However, to compute the conformal stationary solutions, the mean curvature must be

$$H = g_o(\nabla\rho, C) = \det(\gamma'(s), B(s), \nabla\rho).$$

This equation provides all of base curves $\gamma(s) = (x(s), y(s), z(s))$ of extrinsic string solutions $S(\gamma, B)$, having the given ruling flow, which are conformal stationary and so being classical string solutions in $\mathbf{AdS}_3$.

Remark 6.1. *The last argument provides a simple way to construct stationary surfaces, and so classical string solutions, in* $\mathbf{AdS}_3$*. In a forthcoming paper, we will discuss, in detail, the moduli space of classical string solutions obtained in this way. Moreover, we will compare this moduli space of classical string solutions with that obtained directly in* $\mathbf{AdS}_3$ *by considering the idea of null scroll and then characterizing the sub-moduli space of stationary null scrolls.*

Once we have discussed the way to obtain those null scrolls in $\mathbb{L}^3$ which are conformal stationary, and so providing classical string solution in the anti de Sitter three space, let us briefly describe how to translate these solutions to the language of Pohlmeyer reduced theory. To do it, first of all we need to recall the following well known classical statement: *every timelike surface in* $\mathbb{L}^3$*, or in* $\mathbf{AdS}_3$*, can be parametrized by two families of lightlike curves.* This is a chief point that provides the geometric support to the Virasoro constraints. To make it clear, let us use the following better known notation in this context. Suppose we have a parametrization $Y(z, \bar{z})$ of a certain timelike surface. Then, we put

$$z = \frac{1}{2}(u - v), \quad \bar{z} = \frac{1}{2}(u + v), \quad \text{and} \quad \partial = \partial_u - \partial_v, \quad \bar{\partial} = \partial_u + \partial_v,$$

and so $\partial Y = Y_z$ and $\bar{\partial} Y = Y_{\bar{z}}$. Consequently, we get

$$\partial Y \cdot \partial Y = \langle Y_z, Y_z \rangle, \qquad \bar{\partial} Y \cdot \bar{\partial} Y = \langle Y_{\bar{z}}, Y_{\bar{z}} \rangle,$$

which shows that the Virasoro constraints are equivalent to the fact that the original parametrization is made through two families of lightlike curves. As an important consequence, we see that the Virasoro constraint are invariant under conformal changes in the target space, so they are actually established in the conformal class of the target spacetime metric.

For example, consider a null scroll $S(\gamma, B)$ in $\mathbb{L}^3$ naturally parametrizated by

$$\phi(s,t) = \gamma(s) + t\, B(s), \quad B(s) = r(s) \, \left(e^{i\varphi(s)}, 1\right).$$

Then, it is obvious that this parametrization does not encode the Virasoro constraints, because, in general,

$$\langle \phi_s, \phi_s \rangle = 2t \, \langle \gamma'(s), B'(s) \rangle + t^2 \, \langle B'(s), B'(s) \rangle,$$

does not identically vanish. Certainly, one can obtain a parametrization of $S(\gamma, B)$ by two families of lightlike curves, what, in general, could be formidably complicated. For the subclass of stationary surfaces, that is, for those extrinsic string solutions that in addition are classical ones, we can do it easily. In fact, for these solutions the angular function $\varphi(s)$ defining the ruling flow, $B(s)$, is a constant, say φ_o, and so $B(s) = r(s) \, \left(e^{i\varphi_o}, 1\right) = r(s)\, \vec{v}_o$. Said otherwise, the rulings of a classical string solution are parallel. This is the key to define the following parametrization

$$X(s,t) = \gamma(s) + t\, \vec{v}_o,$$

which certainly provides a pair of lightlike parametric curves, and so the Virasoro constraints hold.

Next, we start with a conformal stationary extrinsic string solution, say $S(\gamma, B)$, in $\mathbb{L}^3$, which we parametrize by two families of lightlike curves to include the Virasoro constraints. Then, it provides a classical string solution in the anti de Sitter target which is parametrized in the same way and so the Virasoro constraints hold. Denote by $X(z, \bar{z})$ such a parametrization. In this setting, the induced metric on the null scroll is given by $2\mu(z, \bar{z})\, dz\, d\bar{z}$, where μ is a certain function which can be computed in terms of null scroll data, γ and B. Now, the function defined by

$$\beta(z, \bar{z}) = \frac{4\mu(z, \bar{z})}{1 - F(\zeta(z, \bar{z}), k(z, \bar{z}))},$$

provides a solution of the following generalized sinh-Gordon equation, (see [14] for details)

$$\partial \bar{\partial}\, \beta - e^{\beta} - h\, e^{-\beta} = 0,$$

where h stands for a certain function which can be tediously computed in terms of null scroll data.

JHEP05(2012)068

The stationary surfaces in $\mathbf{AdS}_3$, or equivalently the conformal stationary surfaces in $\mathbb{L}^3$, define, in the boundary of the anti de Sitter space, the so called Wilson loops. However, any extrinsic string solution, even not being stationary, defines certain curves in the boundary which could be called *generalized Wilson loops*. For example, these curves can be easily obtained for stationary surfaces in $\mathbb{L}^3$. In fact, we only need to compute the conformal factor, providing the anti de Sitter space $(\Sigma, \bar{g})$, restricted to the null scroll as

$$F(s,t) = \left| t\, e^{i\varphi_o} + \int c(s)\, e^{i\omega(s)}\, ds \right|^2 - \left(t + \int c(s)\, ds \right)^2 ,$$

and then solve the equation

$$F(s,t) = \left| t\, e^{i\varphi_o} + \int c(s)\, e^{i\omega(s)}\, ds \right|^2 - \left(t + \int c(s)\, ds \right)^2 = 1.$$

Let us exhibit the following example. Consider the lightlike helix given by

$$\gamma(s) = \left(-ie^{is}, s \right), \quad s \in \mathbb{R},$$

so that $\gamma'(s) = \left(e^{is}, 1 \right)$. We now choose the lightlike vector $\vec{v}_o = \left(e^{i\varphi_o}, 1 \right)$ and build the associated stationary null scroll, which we parametrize by two families of lightlike curves to get the Virasoro constraints,

$$X(s,t) = \left(-ie^{is} + te^{i\varphi_o}, s+t \right).$$

For simplicity, we choose $\varphi_o = 0$ to get

$$F(s,t) = F(\zeta(s,t), k(s,t)) = 1 - s^2 + 2t(\sin s - s).$$

Consequently, this extrinsic string solution provides the following generalized Wilson loop in the conformal boundary of the anti de Sitter 3-space

$$w(s) = \gamma(s) + \frac{s^2}{2(\sin s - s)}\, \vec{v}_o = \left(\sin s + \frac{s^2}{2(\sin s - s)}, -\cos s, s + \frac{s^2}{2(\sin s - s)} \right).$$

Finally, we wish to remark that *any curve*, say $\alpha(s)$, no matter its causal character, *in the conformal boundary* $\partial\Sigma$ *of the anti de Sitter 3-space, can be viewed as*, at least, *a piece of the generalized Wilson loops of an extrinsic string solution. Furthermore, if* $\alpha(s)$ *is null-homotopic, the null scrolls materializing the above property determine a dual curve in the opposite size of the conformal boundary.*

To check this fact, we can do the following simple argument. We take any lightlike vector field, say $B(s)$, along the given curve. Then, we construct in $\mathbb{L}^3$ the corresponding null scroll $S(\alpha, B)$. Now, we can obtain (solving a Ricatti equation) a lightlike curve, say $\gamma(s)$, in the open solid hyperboloid Σ which generates the above null scroll, i. e., $S(\alpha, B) = S(\gamma, B)$. This null scroll provides an extrinsic string solution not only in $\mathbb{L}^3$, but also in the anti de Sitter 3-space $(\Sigma, \bar{g})$. Obviously this solution is reflected in the conformal boundary through a set of curves including the previous one.

The study of the role that these generalized Wilson loops would play in the **AdS/CFT** correspondence could be an interesting and however complicated problem that needs further research.

Acknowledgments

The authors wish to thank the referee for his/her constructive comments and suggestions for improvement in the article. MB has been partially supported by Spanish MEC-FEDER Grant MTM2010-18099 and J. Andalucía Regional Grant P09-FQM-4496. AF has been partially supported by MICINN project MTM2009-10418, and Fundación Séneca project 04540/GERM/06, Spain. This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Regional Agency for Science and Technology (Regional Plan for Science and Technology 2007-2010).

References

[1] L. J. Alías, A. Ferrández, P. Lucas and M. A. Meroño, *On the Gauss map of B-scrolls*, *Tsukuba J. Math.* **22** (1998) 371.

[2] M. Barros, *A Geometric algorithm to construct new solitons in the $O(3)$ nonlinear σ-model*, *Phys. Lett.* **B 553** (2003) 325 [hep-th/0211143] [INSPIRE].

[3] M. Barros, M. Caballero and M. Ortega, *Rotational surfaces in $\mathbb{L}^3$ and solitons in the nonlinear σ-model*, *Commun. Math. Phys.* **290** (2009) 437 [arXiv:0705.2206] [INSPIRE].

[4] M. Barros and A. Ferrández *How big is the family of stationary null scrolls?*, in preparation.

[5] W. Blaschke, *Vorlesungen über Differentialgeometrie III*, Springer-Verlag, Germany (1929).

[6] R.M.B. Chaves, M.P. Dusan and M. Magid, *Björling problem for timelike surfaces in the Lorentz-Minkowski space*, *J. Math. Anal. Appl.* **377** (2011) 481.

[7] B.Y. Chen, *Total mean curvature and submanifolds of finite type*, World Scientific, Singapore (1984).

[8] A. Dall'Acqua, K. Deckelnick and H. Christoph Grunau, *Classical solutions to the Dirichlet problem for Willmore surfaces of revolution*, *Adv. Calc. Variat.* **1** (2008) 379.

[9] F. Dillen and W. Kühnel, *Ruled Weingarten surfaces in Minkowski 3-space*, *Manuscr. Math.* **98** (1999) 307.

[10] H. Dorn, N. Drukker, G. Jorjadze and C. Kalousios, *Space-like minimal surfaces in AdS $\times S$*, *JHEP* **04** (2010) 004 [arXiv:0912.3829] [INSPIRE].

[11] L.K. Graves, *Codimension one isometric immersions between Lorentz spaces*, *Trans. Amer. Math. Soc.* **252** (1979) 367.

[12] Z. H. Hou and F. Ji, *On lorentzian surfaces with $H^2 = K$ in Minkowski 3-space*, *J. Math. Anal. Appl.* **334** (2007) 54.

[13] W. Helfrich, *Elastic properties of lipid bilayers. Theory and possible experiments*, *Z. Naturforsch.* **C 28** (1973) 693.

[14] A. Jevicki, K. Jin, C. Kalousios and A. Volovich, *Generating AdS string solutions*, *JHEP* **03** (2008) 032 [arXiv:0712.1193] [INSPIRE].

[15] H. Kleinert, *The membrane properties of condensing strings*, *Phys. Lett.* **B 174** (1986) 335 [INSPIRE].

[16] J. Langer and D.A. Singer, *The total squared curvature of closed curves*, *J. Diff. Geom.* **20** (1984) 1.

[17] A. Larsen and N.G. Sanchez, *Sinh-Gordon, cosh-Gordon and Liouville equations for strings and multistrings in constant curvature space-times*, *Phys. Rev.* **D 54** (1996) 2801 [hep-th/9603049] [INSPIRE].

[18] H. Liu, *Ruled surfaces with lightlike ruling in* 3*-Minkowski space*, *J. Geom. Phys.* **59** (2009) 74.

[19] A.M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B 103** (1981) 207 [INSPIRE].

[20] A.M. Polyakov, *Fine structure of strings*, *Nucl. Phys.* **B 268** (1986) 406 [INSPIRE].

[21] K. Sakai and Y. Satoh, *Constant mean curvature surfaces in AdS_3*, *JHEP* **03** (2010) 077 [arXiv:1001.1553] [INSPIRE].

[22] G. Thomsen, *Über konforme Geometrie I: Grundlagen der konformen Flächentheorie*, *Abh. Math. Sem. Univ. Hamburg* **21** (1923) 56.

[23] J.L. Weiner, *On a problem of Chen, Willmore et al.*, *Indiana Univ. Math. J.* **27** (1978) 19.

JHEP05(2012)068