Extremal curves of the total curvature in homogeneous 3-spaces

Manuel Barros[1], Angel Ferrández[2*] and Óscar J. Garay[3]

[1]Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, 1807 Granada, Spain. E-mail address: mbarros@ugr.es

[2]Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain. E-mail address: aferr@um.es

[3]Department of Mathematics, Faculty of Science and Technology, Univ. of the Basque Country UPV/EHU, Aptdo 644, 48014, Bilbao, Spain. E-mail address: oscarj.garay@ehu.es

Abstract

The space of curves which are extremal for the total curvature energy is well understood in isotropic homogeneous 3-spaces, said otherwise, spaces of constant curvature. In this paper we obtain that space of extremals in homogeneous 3-spaces whose isometry group has dimension four, that is, rotationally symmetric homogeneous 3-spaces. Most of the geometry in these spaces is governed by the existence of a unit Killing vector field, ξ, sometimes called the Reeb vector field, which turns the homogeneous 3-space into the source of a Riemannian submersion whose target space is a surface with constant curvature. Here, we show that a curve is an extremal of the total curvature energy if and only if ξ lies into either the rectifying plane or the osculating plane along that curve. Then, we prove that every rotationally symmetric homogeneous 3-space, except $\mathbb{H}^2 \times \mathbb{R}$, admits a real one-parameter class of extremals with horizontal normal (Lancret helices). The whole family of extremals is completed with a second class made up of those curves with horizontal binormal. In contrast with the first class, it appears in any rotationally symmetric space, with no exception, and it can be modulated in the space of real valued functions. We also work out geometric algorithms to solve the so called solving natural equations for extremals problem, allowing us to determine them explicitly in many cases. In addition, we solve the closed curve problem by showing the existence of two families of closed extremals. Namely, a rational one-parameter class of closed Lancret helices that appears at any rotationally symmetric homogeneous 3-space, except in $\mathbb{H}^2 \times \mathbb{S}^1$, and a second class of extremals with horizontal binormal, which can be identified with the class of convex closed curves in the Euclidean plane. We also present a quantization principle, à la Dirac, for extremal values of the total curvature energy acting on closed curves in any rotationally symmetric homogeneous 3-space.

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(*) Corresponding author A. Ferrández.

1 Introduction

According to a classical result of Whitney and Grauestein, the total curvature of plane curves is, under suitable boundary conditions, an invariant in the homotopy class of the curve. Even more, each homotopy class of curves, satisfying suitable boundary conditions, is completely characterized by the common total curvature. As an obvious consequence, if fluctuations do not change the topology of curves, then a variational approach to the total curvature action of curves in the Euclidean plane turns out to be trivial.

On the other hand, another classical and well known result, due to Fenchel, assures that the total curvature of any simple closed curve in the Euclidean 3-space satisfies

$$\int_\gamma \kappa(s)\, ds \geq 2\pi,$$

with equality if and only if γ is a convex plane curve. Therefore, the minimum of the total curvature action over simple closed curves in the Euclidean space is 2π and it is reached just on the convex plane curves. Hence, an obvious and naive question is to decide whether, besides those minima, there is any other extremal (critical) of the total curvature functional acting on closed curves in the Euclidean space.

More precisely, it seems natural to consider the total curvature functional acting on a suitable space of curves in a certain surface, or more generally in a Riemannian space, and then to study the associated variational problem. Although the study of the total curvature of curves in Riemannian spaces has been intensively considered along the literature (see [8] and references therein), the systematic study of the associated variational approach was initiated in [1, 2]. However, we have to point out that M. S. Plyushchay, [13], proposed it as a model to study massless particles with rigidity.

Some progress has been made in this direction which, as far as we know, can be briefly summarized as follows. The variational approach for the total curvature in surfaces was first considered in [2], where it was shown that extremals of the total curvature are reached by curves consisting of parabolic points. Stability of extremals was also holographically characterized there. Then, the problem in spaces with the highest rigidity (constant curvature) was considered in [1], where it was shown that the dynamics associated with the total curvature action is consistent only in round 3-spheres. More precisely, when studying the total curvature dynamics of curves in an n-dimensional Riemannian space, $M^n(k)$, with constant curvature k, we have:

- The dynamics is reduced to dimension $n \leq 3$.

- The curvature must be non negative, $k \geq 0$, so it does not make sense in, for example, the hyperbolic space.

- For $k = 0$ and up to topology, the dynamics actually occurs over arbitrary plane curves in $\mathbb{R}^3$. This, in particular, gives the answer to the above stated naive question.

- For $k > 0$, up to topology, the extremals over a round 3-sphere are just the so called Legendrian curves, that is, curves which are horizontal lifts, via the Hopf map, of curves in the 2-sphere.

Other partial results, providing examples and families of extremals in Berger spheres and in the complex projective plane, can be found in [1] (see also [4, 6, 7] for extremals in warped product spaces).

In this paper we will focus on this variational approach for curves in homogeneous 3-spaces whose isometry group has dimension four (sometimes called rotationally symmetric homogeneous 3-spaces [11]). In particular, we obtain the complete classification of extremals for the total curvature action in these backgrounds. The main tool to determine the extremals is the existence of an infinitesimal translation, ξ, which allows us to see each homogeneous 3-space with rigidity of order four as a fibration over a surface with constant curvature, which works, in turn, as the space of ξ-orbits. As a consequence, a congruence class of these homogeneous 3-spaces is determined, up to topology, by a pair of constants: the curvature of the base, c, and the so called bundle curvature, r. Then, following [9], we will use the popular notation $\mathrm{E}(c, r)$,with $c \neq 4r^2$, to denote those homogeneous 3-spaces having four dimensional isometry group (if $c = 4r^2$ the space has constant curvature, said otherwise, highest rigidity, and six dimensional isometry group). Furthermore, we provide algorithms to explicitly construct the extremals, which can be regarded as the solution of the so called solving natural equations for extremals problem. The results of this paper provide progress in the knowledge of the dynamics associated with the total curvature action of curves in 3-dimensional homogeneous geometries (including Thurston ones [15, 16]), or, said otherwise using the Plyushchay terminology ([13]), of the dynamics of a massless boson in those backgrounds. The current status of this variational approach can be now described as follows:

1. In the hyperbolic space, $\mathbb{H}^3$, there is no dynamics of curves associated with the total curvature action, because it does not provide any extremal.
2. If $c = 4r^2$, then $\mathrm{E}(c, r)$ has constant curvature, r^2, and, up to topology, it should be either
   - 2.1 $\mathbb{R}^3$, when $r = 0$, and the dynamics works through curves with vanishing torsion, i. e., plane curves; or
   - 2.2 $\mathbb{S}^3(r^2)$, the round sphere with curvature r^2, when $r \neq 0$. Now, the dynamics works through curves with torsion $\pm r$. In other words, horizontal lifts via the Hopf map of curves in the corresponding round two-sphere.
3. If $c \neq 4r^2$ and $r = 0$, then the homogeneous 3-space is a Riemannian product and there are two possibilities:
   - 3.1 If $c > 0$, then $\mathrm{E}(c, 0) = \mathbb{S}^2(c) \times \mathbb{R}$ and there are two families of extremals for the total curvature action, namely,

3.1.1 The class of curves with horizontal Frenet normal. It forms a real one-parameter class $\{\gamma_m, m \in \mathbb{R} - \{0\}\}$ of Lancret helices, that is, curves making a constant angle with ξ. More precisely, this family can be described as follows: for any $m \in \mathbb{R} - \{0\}$, choose in $\mathbb{S}^2(c)$ the circle β_m with geodesic curvature

$$\kappa_g = \frac{\sqrt{c(m^2+1)}}{m},$$

and then take γ_m as the geodesic with slope m in the Hopf cylinder shaped on β_m.

3.1.2 The class of curves with horizontal binormal, which automatically have torsion zero. Up to rigid motions, this family is formed by curves lying in Hopf cylinders built over great circles in $\mathbb{S}^2(c)$ (see Corollary 4.12).

3.2 If $c < 0$, then $\mathrm{E}(c, 0) = \mathbb{H}^2(c) \times \mathbb{R}$. Now the space of extremals is made up of curves with horizontal binormal. Up to rigid motions, they are the curves lying in Hopf cylinders built over geodesics in the hyperbolic plane $\mathbb{H}^2(c)$ (see Corollary 4.12).

4. If $c \neq 4r^2$ and $r \neq 0$, the space of extremals consists of two families of curves:

4.1 A real one-parameter class of Lancret helices (see Proposition 4.3).

4.2 The class of curves with horizontal binormal, which automatically has constant torsion $\tau = \pm r$. This class can be parameterized by the space of differentiable real functions. They are summarized as well as explicitly and geometrically described after Proposition 4.8.

Besides the isotropic and the rotationally symmetric homogeneous spaces, whose isometry groups have, respectively, dimension six and four, a third class completes the list of 3-dimensional homogeneous spaces [14, 15, 16]. Homogeneous spaces in this class have an isometry group of dimension three, the corresponding geometries are called solv-geometries or Bianchi geometries and they are associated with the group Sol_3. Therefore, a natural open problem, which will not be addressed here, is to study the dynamics associated with the total curvature of curves in Sol_3.

Our interest, then, is directed towards the closed curve problem. In other words, we wonder about the existence of closed extremals, that is, critical points of the total curvature energy acting on closed curves in $\mathrm{E}(c, r)$. We completely solve this question showing two big families of closed solutions:

(i) A first class, $\mathbf{F}_N$, found in any $\mathrm{E}(c, r)$ other than $\mathrm{H}^2(c) \times \mathbb{S}^1$, is formed by a rational one-parameter family of closed Lancret helices, equivalently, curves with horizontal normal.

(ii) A second class, $\mathbf{F}_B$, which can be found in any $\mathrm{E}(c, r)$, is formed by closed solutions with horizontal binormal. This family is identified with the set of convex closed curves in the Euclidean plane.

Moreover, we give a quantization principle, in the sense of Dirac, which assures that the critical levels of the total curvature energy acting on closed curves in $\mathrm{E}(c, r)$ are not arbitrary, but rational multiples of a certain fixed quantity of energy.

2 The total curvature action

Troughout this paper we will consider clamped curves, with a finite number of inflection points, in an n-dimensional Riemannian space $(M, \langle\, , \rangle)$. More precisely, for any pair of points, $p_1, p_2 \in M$ and tangent vectors $x_i \in T_{p_i}M$, we consider the space, Ω, of smooth immersed curves with a finite number of points where the curvature vanishes (inflection points) that are clamped to those points, that is,

$$\delta : [a_1, a_2] \to M \qquad \delta(a_i) = p_i, \;\; \delta'(a_i) = x_i, \;\; 1 \leq i \leq 2.$$

Note that this space includes, in particular, the space of closed curves when $p_1 = p_2$ and $x_1 = x_2$. Now, in Ω we consider the action that measures the total curvature of curves, that is,

$$\mathcal{F} : \Omega \to \mathbb{R}, \qquad \mathcal{F}(\delta) = \int_\delta \kappa(s)\, ds,$$

where κ denotes the curvature function of curves in Ω. A natural problem is the study of the variational problem associated with this functional and specially to determine the curves which are extremals of this action.

Given $\delta \in \Omega$, the tangent space, $T_\delta\,\Omega$, of Ω at δ is identified with the space of vector fields along δ which vanish at the end points. If $W \in T_\delta\,\Omega$, then one can define a curve in Ω passing through δ in the direction of W, say for instance

$$\Gamma : (-\varepsilon, \varepsilon) \to \Omega, \qquad \Gamma(t) = \exp_{\delta(s)} tW(s)\,,$$

and, the first variation formula is defined by $D\mathcal{F}(\delta) : \mathbf{T}_\delta\,\Omega \to \mathbb{R}$

$$D\mathcal{F}(\delta)[W] = \frac{\partial}{\partial t}\left(\mathcal{F}(\Gamma(t))\right)\bigg|_{t=0}.$$

It can be computed, using a standard argument which involves some integration by parts, and was set down in [1]

$$D\mathcal{F}(\delta)[W] = \int_\delta < \mathcal{E}(\delta), W > ds + \mathcal{B}(\delta, W),$$

where $\mathcal{E}(\delta)$ and $\mathcal{B}(\delta, W)$ denote the so called Euler-Lagrange and boundary operators, respectively. The Frenet equations yield (see [1] for details)

$$\mathcal{E}(\delta) = \tau^2\, N - \tau_s B - \tau\eta - R(N, T)T,$$

$$\mathcal{B}(\delta, W) = \sum_{i=1}^{m} < \nabla_T W(s_i), N(s_i^+) - N(s_i^-) > + \sum_{i=1}^{m} < W(s_i), \nabla_T\, N(s_i^+) - \nabla_T\, N(s_i^-) >,$$

where ∇ and R are, respectively, the Levi-Civita connection and the Riemannian curvature of the metric $\langle\,,\rangle$ on M. Moreover, T and N stand for the unit tangent and the unit normal (which is well defined if $n = 2$, while perhaps it is not defined in the inflections points $\delta(s_i)$, $1 \leq i \leq m$, when $n \geq 3$). Also B is the unit binormal and in this case $\tau \geq 0$ denotes the torsion and τ_s its derivative with respect to the arc-length parameter. Finally, η belongs to the subbundle normal to the one spanned by $\{T, N, B\}(s)$ along $\delta(s)$, except at most at the inflection points. Now, the last formulas allow us to characterize the extremals for $\mathcal{F}$ in Ω. In fact, δ is a critical point of $\mathcal{F}$ in Ω if and only if the following conditions hold (see [1])

(1) N, B and τ are well defined at the inflection points of δ, and

(2) the following Euler-Lagrange equation is satisfied

$$R(N,T)T = \tau^2\, N - \tau_s B - \tau\eta. \tag{1}$$

It seems clear that the degree of integrability we have, on the above stated field equation, strongly depends on the control we get on the curvature of the target space. Note that the influence of this space on the equation is exerted through the sectional curvature of the osculating plane of the trajectories and along them.

Let us briefly survey what is known on this equation and so on the critical points of the total curvature action. If M is the Euclidean plane, $\mathbb{R}^2$, then equation (1) becomes into an identity and this is so because, if fluctuations do not change the topology of curves, then the action $\mathcal{F}$ is constant on each homotopy class of curves in Ω. For example, if Ω consists of closed curves, the action $\mathcal{F}$ provides an integer multiple (the rotation index) of 2π. In the general case that value must be corrected by the angle between x_1 and x_2.

More generally, if M has dimension two, then (1) can be written as $K(\delta(s)) = 0$, where K stands for the Gaussian curvature of the surface. Therefore, extremal curves are made up of parabolic points. This case, including the stability of extremals, has been amply considered in [2].

On the other hand, the dynamics associated with $\mathcal{F}$ in a space of constant curvature c, say $M^n(c)$, for any dimension n, was completely solved in [1]. Let us briefly summarize it. First of all, note that in this case, the Euler-Lagrange equation (1) turns out to be

$$\tau^2 = c, \qquad \tau_s = 0, \qquad \tau\,\eta = 0,$$

and the third equation implies that the dynamics evolves in backgrounds with dimension $n \leq 3$. Moreover, the trajectories have constant torsion and $c \geq 0$. Next, it is clear that there are, up to topology, two possibilities to get extremals: the Euclidean plane, where the dynamics is trivial, and a round 3-sphere. This reduces the problem to determine those curves in a 3-sphere, say with radius one, whose torsion is, up to orientation, one. These curves were geometrically obtained in [1]: They are congruent to a horizontal lift (also called a Legendrian curve), via the Hopf map, of a curve in the round two sphere with radius 1/2.

Since the round 3-sphere is, essentially, the only constant curvature Riemannian space where the variational problem makes sense and, on the other hand, it is also a homogeneous 3-space, it seems natural to study the existence of extremals for the total curvature of curves in homogeneous 3-spaces M^3. It is known that the isometry group, $I(M^3)$, of M^3 is a Lie group whose dimension is 6, 4 or 3. From now on, we will say that it has rigidity of order 6, 4 or 3, respectively. In the first case, the space has constant curvature and so it is covered by either $\mathbb{S}^3$ (positive curvature), $\mathbb{R}^3$ (vanishing curvature) or $\mathbb{H}^3$ (negative curvature). So a natural breeding ground to study our variational problem is provided by the homogeneous 3-spaces in the second rigidity level, that is, when $\dim(I(M^3)) = 4$. This is certainly the widest and most interesting family of homogeneous 3-spaces, which sometimes are called rotationally symmetric homogeneous 3-spaces. Among its simply connected members, one can find, besides the Riemannian products $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, the Berger spheres, the Heisenberg group, Nil_3, and the universal covering of the special lineal group $\mathrm{SL}(2, \mathbb{R})$. Obviously, the family also includes quotient of these spaces by suitable isometry subgroups, namely,

- Lens spaces, $\mathrm{L}_n = \mathbb{S}^3/\mathbb{Z}_n$, $n \geq 2$, including the projective space, $\mathbb{RP}^3 = \mathrm{L}_2$, with the corresponding induced Berger metrics;

- Heisenberg bundles, including those over flat tori; and

- The projective special linear group $\mathrm{PSL}(2, \mathbb{R}) = \widetilde{\mathrm{PSL}}(2, \mathbb{R})/\mathbb{Z}_2$ and other quotients of $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$.

Any homogeneous 3-space M^3 with rigidity of order four can be viewed as a bundle over a surface with constant curvature. To exploit this approach, we will follow the notation of [9]. This fibration provides a Riemannian submersion, $\mathrm{p} : M^3 \to \mathtt{B}^2(c)$, with geodesic fibers (vanishing first O'Neill invariant) over a constant curvature (say c) surface. In addition, the vertical flow is generated by a unit Killing vector field ξ which allows one to give the following expression for the second O'Neill invariant, A,

$$A_X \xi = \overline{\nabla}_X \, \xi = r \, (X \times \xi),$$

where X is a horizontal vector field and r, the bundle curvature [9], is a constant. Therefore, $\mathrm{p} : M^3 \to \mathtt{B}^2(c)$ is a Killing submersion in the sense of [10, 12]. Throughout this paper, we will denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections on M^3 and $\mathtt{B}^2(c)$, respectively. Both constants, c (curvature of the base) and r (bundle curvature), classify the homogeneous 3-spaces up to isometries and topology. In other words, each pair of real numbers, (c, r), determines, up to topology, a congruence class $\mathrm{E}(c, r)$ of homogeneous 3-spaces whose isometry group has either dimension 4, if $c \neq 4r^2$, or dimension 6 (constant curvature), if $c = 4r^2$. From now on we will focus on homogeneous three spaces, $\mathrm{E}(c, r)$, whose isometry group has dimension 4 and consequently $c \neq 4r^2$. Furthermore, we will consider oriented homogeneous three spaces and, in particular, we will use orientation to define the cross product.

3 Lancret helices in homogeneous 3-spaces

In a Riemannian space endowed with a unit Killing vector field ξ (also called an infinitesimal translation), curves making a constant angle with ξ are known as Lancret helices with axis ξ (see [3] for details on Lancret helices in the 3-sphere). As every homogeneous 3-space admits a unit Killing vector field, it seems natural to study the following:

Problem 1: *Are there Lancret helices being extremal of the total curvature action in rotationally symmetric homogeneous spaces?*

This problem will be answered in the next section (see Proposition 4.2). To this end, it will be useful to characterize, geometrically, Lancret helices in homogeneous 3-spaces. Let $M^3 = \mathrm{E}(c,r)$ be a homogeneous 3-space, $\mathrm{p} : M^3 \to \mathtt{B}^2(c)$ the corresponding Riemannian submersion, and let ξ be the associated unit vertical Killing vector field. To study Lancret helices with axis ξ, it is useful to consider the so called equivariant surfaces. For any profile curve, $\beta(s)$, in $\mathtt{B}^2(c)$, choose a horizontal lift, say $\bar{\beta}(s)$. Both curves have the same speed, so we will assume that both are parameterized by the arc-length. Now, consider the surface $S_\beta = \mathrm{p}^{-1}(\beta)$ in M^3, which can be nicely parameterized by horizontal lifts of β and ξ-orbits (from now on, simply orbits) via the following map

$$X(s,t) = \phi_t(\bar{\beta}(s)),$$

where $\{\phi_t : t \in \mathbb{R}\}$ is the one-parameter group generated by ξ. It is clear that S_β is an oriented flat surface, with the obvious induced orientation $X_s \times X_t$, which we will call the Hopf cylinder with profile curve $\beta(s)$. Now, it is easy to compute (see [10]) the second fundamental form of a Hopf cylinder, whose matrix in the above parameterization is given by

$$\begin{pmatrix} \kappa_g & -r \\ -r & 0 \end{pmatrix}$$

where κ_g stands for the geodesic curvature of the profile curve in the base $\mathtt{B}^2(c)$.

Let ∇' be the Levi-Civita connection of the induced metric on S_β and let $\gamma(s)$ be a unit speed geodesic in a certain Hopf cylinder, S_β. Then $\nabla'_{\gamma'}\gamma'(s) = 0$ and so we have

$$\frac{d}{ds}\langle \gamma'(s), \xi(\gamma(s))\rangle = \langle \overline{\nabla}_{\gamma'}\gamma'(s), \xi(\gamma(s))\rangle + r\langle \gamma'(s), \gamma'(s) \times \xi(\gamma(s))\rangle = 0,$$

which shows that the angle that $\gamma(s)$ makes with ξ is constant. In other words, *the geodesics of the Hopf cylinders are Lancret helices.*

To show the converse, let $\gamma(s)$ be a unit speed Lancret helix, so there exists $\varphi \in \mathbb{R}$, which we may suppose different from zero (otherwise the result is trivial) such that $\langle \gamma', \xi\rangle(s) = \cos\varphi$. Now, define the curve $\beta(s) = \mathrm{p}(\gamma(s))$ in $\mathtt{B}^2(c)$ and use it as a profile curve to get the Hopf cylinder $S_\beta = \mathrm{p}^{-1}(\beta)$. It is clear that the original Lancret helix lies in S_β, and, moreover, we have

$$0 = \frac{d}{ds}\langle \gamma', \xi\rangle(s) = \langle \overline{\nabla}_{\gamma'}\gamma', \xi\rangle(s) + r\langle \gamma', \gamma' \times \xi\rangle(s) = \langle \nabla'_{\gamma'}\gamma', \xi\rangle(s),$$

which proves that $\nabla'_{\gamma'}\gamma'(s) = 0$, and so $\gamma(s)$ is a geodesic in $S_\beta = \mathrm{p}^{-1}(\beta)$. Therefore we have the following

Proposition 3.1 *The Lancret helices in any homogeneous* 3*-space with rigidity of order four are just the geodesics of Hopf cylinders.*

Next, we wish to compute the Frenet apparatus, $\{T_\gamma, N_\gamma, B, \kappa, \tau\}(s)$ of a Lancret helix $\gamma(s)$ in M^3. Let $\beta(u)$ be the unit speed profile curve such that $\gamma(s)$ is a geodesic in $S_\beta = \mathrm{p}^{-1}(\beta)$. Furthermore, $\{T = \beta', N = JT, \kappa_g\}$ will be the Frenet apparatus of $\beta(u)$ in $\mathsf{B}^2(c)$. We also have the above standard parametrization $X(u,t) = \phi_t(\bar\beta(u))$ on the Hopf cylinder, so that $\bar T(u,t) = X_u(u,t) = d\phi_t(\bar\beta'(u))$ is a horizontal lift of T along the Hopf cylinder. In this context, since $\gamma(s)$ is a geodesic in the flat surface $S_\beta = \mathrm{p}^{-1}(\beta)$, it must be the image under X of a certain straight line in the (u,t)-plane. Therefore, there exists $\varphi \in (0, 2\pi)$ such that $\gamma(s) = X((\sin\varphi)\, s, (\cos\varphi)\, s)$. Then we have $T_\gamma = \sin\varphi\, \bar T + \cos\varphi\, \xi$. We also know that $N_\gamma = \bar T \times \xi$, because it must agree with the unit normal to the Hopf cylinder, and therefore $B = \cos\varphi\, \bar T - \sin\varphi\, \xi$. To compute the curvature function, $\kappa(s)$, we use that it should be the normal curvature of γ in S_β and then

$$\kappa = \begin{pmatrix} \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \kappa_g & -r \\ -r & 0 \end{pmatrix} \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix} = \frac{\kappa_g - 2rm}{m^2+1}, \tag{2}$$

where $m = \cot\varphi$. Finally, a direct computation allows us to obtain

$$\overline\nabla_{T_\gamma} B = [\kappa_g \cos\varphi \sin\varphi - r(\cos^2\varphi - \sin^2\varphi)]\, N_\gamma,$$

which gives

$$\tau = \frac{1}{2}\kappa_g \sin 2\varphi - r\cos 2\varphi = \frac{\kappa_g\, m - r(m^2-1)}{m^2+1}. \tag{3}$$

It is worth pointing out that, from (2) and (3), the curvature and torsion functions of Lancret helices in homogeneous 3-spaces are nicely related by

$$\tau = (\cot\varphi)\,\kappa + r. \tag{4}$$

Moreover, (2) and (3) allow us to compute the curvature, κ_g, of the profile curve and the slope, $m = \cot\varphi$, of a Lancret helix in terms of both its curvature function, κ, and its torsion τ. In fact, we get

$$\kappa_g = \kappa + \frac{\tau^2 - r^2}{\kappa}, \qquad m = \frac{\tau - r}{\kappa}.$$

As a consequence, we have the following

Proposition 3.2 *A Lancret helix in a homogeneous* 3*-space* $M^3 = \mathrm{E}(c,r)$ *is unique and equivalently determined by either*

(i) Its curvature and torsion functions (κ, τ)*; or*

(ii) *The curvature, κ_g, of the profile curve in the space of orbits and the slope of the helix, $m = \cot\varphi$, as a geodesic in the corresponding equivariant surface.*

The classical theorem of Lancret ensures that formula (4) with $r = 0$ provides a simple characterization of Lancret helices in the Euclidean space. This classical theorem of Lancret was discovered to be true in the 3-sphere (see [3]), where it was also observed that the only Lancret helices in the hyperbolic 3-space are those with curvature an torsion being both constant (in some sense they are like circular helices). Thus, it seems natural to ask for a Lancret-type theorem in homogeneous 3-spaces with rigidity of order four. Said otherwise

Problem 2: *Does equation* (4) *characterize Lancret helices in such homogeneous* 3-*spaces?*

However, a rotationally symmetric homogeneous space does not have enough isometries to guarantee the converse and so the theorem of Lancret. More precisely, we are going to show the existence of a curve, in a homogeneous 3-space with rigidity of order four, with the same curvature and the same torsion of a Lancret helix, but they are not congruent.

Proposition 3.3 *Let $M^3 = \mathrm{E}(c, r)$ be an homogeneous* 3-*space with $c \neq 4r^2$, and let $\gamma(s)$ be a Lancret helix in M^3 with curvature $\kappa(s)$ and torsion $\tau(s)$. Then there exists a curve $\delta(s)$ in M^3 with curvature function $\kappa_\delta = \kappa$ and torsion $\tau_\delta = \tau$ which is not congruent to any Lancret curve in M^3.*

Proof. The Frenet frame $\{T(s), N(s), B(s)\}$ of a curve parameterized by the arc-length, in M^3, with curvature $\kappa(s)$ and torsion $\tau(s)$ comes as a solution of the Frenet equations

$$\overline{\nabla}_s \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}$$

which is a first order linear system of differential equations. For any $p \in M^3$ and an orthonormal frame $\{\vec{t}, \vec{n}, \vec{b}\}$ in the tangent space $T_p(M^3)$, there exists a unique solution, $\{T_\delta(s), N_\delta(s), B_\delta(s)\}$, of the above system and then a unique arc-length parameterized curve, $\delta(s)$, such that $\delta(0) = p$ and $\{T_\delta(0), N_\delta(0), B_\delta(0)\} = \{\vec{t}, \vec{n}, \vec{b}\}$. Now, suppose that $\gamma(s)$ is a Lancret helix in M^3 with curvature $\kappa(s)$ and torsion $\tau(s)$ and let $m = \cot\varphi$ be its slope as a geodesic in a suitable Hopf cylinder. Now, by choosing an initial condition such that the angle between $\vec{t}$ and $\xi(p)$ is different from φ, then the corresponding solution $\delta(s)$ is not congruent to any Lancret helix. In fact, note that the slope of a Lancret helix is completely determined by

$$m = \cot\varphi = \frac{\tau - r}{\kappa}.$$

□

4 Extremals of the total curvature in homogeneous 3-spaces

The Riemannian curvature operator of a homogeneous 3-space $M^3 = \mathrm{E}(c,r)$ was computed in [9] to be

$$\begin{aligned} R(X,Y)Z &= (c-3r^2)(\langle Y,Z\rangle X - \langle X,Z\rangle Y) \\ &\quad + (-c+4r^2)[\langle Y,\xi\rangle\langle Z,\xi\rangle X - \langle X,\xi\rangle\langle Z,\xi\rangle Y + \langle X,\xi\rangle\langle Y,Z\rangle\xi - \langle Y,\xi\rangle\langle X,Z\rangle\xi]. \end{aligned}$$

In particular, along a curve $\gamma(s)$ with Frenet frame $\{T,N,B\}(s)$, we have

$$R(N,T)T = (c-3r^2)N + (-c+4r^2)[\langle T,\xi\rangle^2 N - \langle T,\xi\rangle\langle N,\xi\rangle T + \langle N,\xi\rangle\xi],$$

which can be combined with (1) to obtain the following equation whose solutions provide the extremals of the total curvature action on M^3 in $\mathrm{E}(c,r)$

$$\tau^2 = r^2 + (c-4r^2)\langle B,\xi\rangle^2, \qquad \tau_s = (c-4r^2)\langle N,\xi\rangle\langle B,\xi\rangle.$$

We can manipulate these equations to see that those extremals are just the solutions of the following field equations

$$\tau^2 = r^2 + (c-4r^2)\langle B,\xi\rangle^2, \qquad (r-2\tau)\langle N,\xi\rangle\langle B,\xi\rangle = 0. \tag{5}$$

The following result reduces the integration of the above equations to three fundamental cases.

Lemma 4.1 *Let $\gamma(s)$ be an extremal of the total curvature functional in $\mathrm{E}(c,r)$, then either*

(E1) *ξ lies in its rectifying plane along $\gamma(s)$, or*

(E2) *ξ lies in its osculating plane along $\gamma(s)$.*

Proof. According to the second equation in (5), we only need to study the case $r-2\tau=0$. Then the first equation of (5) shows that

$$\langle B,\xi\rangle^2 = -\frac{3\tau^2}{c-4r^2},$$

and so $\langle B,\xi\rangle$ is a constant. However $\frac{d}{ds}\langle B,\xi\rangle = (r-\tau)\langle N,\xi\rangle$ and consequently the extremals in this class have either horizontal normal or torsion $\tau = r$. However in the last case $\tau = 0$ and so the binormal is horizontal. $\square$

4.1 Extremals with horizontal normal

First of all, observe that a curve in $\mathrm{E}(c,r)$ has horizontal normal if and only if it is a Lancret helix. That follows from

$$\frac{d}{ds}\langle T,\xi\rangle = \kappa\langle N,\xi\rangle + r\langle T, T\times\xi\rangle = \kappa\langle N,\xi\rangle,$$

where geodesics can be considered trivial Lancret helices, as well as trivial extremals of the total curvature. In fact, they are minima, because the curvature function of a curve in dimension three does not change sign.

Consequently, the class of extremals with horizontal normal is a subclass of the family of Lancret helices. To determine those Lancret helices providing extremals, we only need to check the first equation of (5) because the second equation automatically holds for Lancret helices. Now, for a Lancret helix we see that $\langle B,\xi\rangle = \sin\varphi = \frac{1}{\sqrt{m^2+1}}$ is constant and therefore the curvature and torsion of extremal Lancret helices are both constant functions. Therefore, we obtain the following characterization of extremal Lancret helices

Proposition 4.2 *A Lancret helix in* $\mathrm{E}(c,r)$ *is an extremal of the total curvature if and only if it has constant torsion satisfying that*

$$\tau^2 = \frac{c + r^2(m^2-3)}{m^2+1}. \tag{6}$$

Remark 4.3 Since the only obstruction to the existence of extremal Lancret helices in $\mathrm{E}(c,r)$ is provided, according to the above proposition, by the positivity of $c+r^2(m^2-3)$, we have the following

Corollary 4.4 *Every homogeneous* 3*-space* $\mathrm{E}(c,r)$*, except* $\mathbb{H}^2\times\mathbb{R}$*, admits a real one-parameter class of Lancret helices which are extremal of the total curvature action.*

Proof. Homogeneous 3-spaces which are Riemannian products correspond to $r=0$. Then for $\mathbb{H}^2\times\mathbb{R}$, as $c<0$, the above proposition shows the non existence of extremal Lancret helices. Otherwise, we have the real one-parameter class of extremal Lancret helices with slope satisfying

$$m^2 > \frac{3r^2-c}{r^2}.$$

□

Now, helices in this class can be geometrically obtained as follows. The constancy of both torsion and curvature imply that they are geodesics in Hopf cylinders built over curves with constant geodesic curvature (circles) in $\mathtt{B}^2(c)$. According to (3) the torsion of any Lancret helix can be expressed in terms of both the geodesic curvature, κ_g, of the profile curve in $\mathtt{B}^2(c)$ and the slope, $m=\cot\varphi$, of the helix as a geodesic in the corresponding Hopf cylinder. Therefore, for any m satisfying the above inequality, just choose the geodesic with slope $m = \cot\varphi$ in the Hopf cylinder S_{β_m}, where β_m is the circle in $\mathtt{B}^2(c)$ with geodesic curvature

$$\kappa_g = \frac{\sqrt{(m^2+1)(c+r^2(m^2-3))} + r(m^2-1)}{m}, \tag{7}$$

to get a Lancret helix which is an extremal of the total curvature in $\mathrm{E}(c,r)$. $\square$

Example 4.5 *In Riemannian product homogeneous* 3*-spaces, we have* $r = 0$ *and then we observe that while* $\mathbb{H}^2(c)\times\mathbb{R}$ *has no extremal Lancret helices, in* $\mathbb{S}^2(c)\times\mathbb{R}$ *we can find a one-parameter class of Lancret helices which are extremal of the total curvature energy. In fact, for any* $m \in \mathbb{R} - \{0\}$*, we choose in* $\mathbb{S}^2(c)$ *the circle* β_m *with geodesic curvature*

$$\kappa_g = \frac{\sqrt{c(m^2+1)}}{m},$$

and then take γ_m *as the geodesic with slope* m *in the Hopf cylinder shaped on* β_m*. Then* γ_m *is an extremal of the total curvature in* $\mathbb{S}^2(c)\times\mathbb{R}$*.*

Example 4.6 *In nilmanifold geometries, which are associated to the Heisenberg group, we have* $c = 0$*. Therefore, for any* $\varphi \in (0, \pi/6)$ *there exists a Lancret helix which is an extremal of the total curvature action. These extremals appear as geodesic in Hopf cylinders built over circles in the Euclidean plane with curvature*

$$\kappa_g = \frac{r}{m}\,(\sqrt{m^4 - 2m^2 - 3} + m^2 - 1), \qquad m = \cot\varphi.$$

4.2 Extremals with horizontal binormal

In a three dimensional oriented Riemannian space one can define the cross product. Actually we are considering $\mathrm{E}(c,r)$ as an oriented space and have used the cross product to define the unit binormal of a curve, $\gamma(s)$, as $B(s) = T(s)\times N(s)$. In particular, the osculating planes along $\gamma(s)$ admit the obvious induced orientation $\{T(s), N(s)\}$.

In this section we focus on the study of extremals with horizontal binormal, $\langle B, \xi\rangle = 0$, and observe that any curve in $\mathrm{E}(c,r)$ satisfying this condition is automatically an extremal of the total curvature functional. Indeed, just note that

$$\frac{d}{ds}\langle B, \xi\rangle = (r-\tau)\langle N, \xi\rangle = 0,$$

and so $\tau = r$. Otherwise, the curve is an orbit. Thus the first equation of (5) holds. Consequently the whole family of curves with horizontal binormal in $\mathrm{E}(c,r)$ constitutes a class of extremals of the total curvature energy. Now, our immediate purpose is to understand this class of extremals.

Let $\gamma(s)$ be a unit speed curve with horizontal binormal. Then ξ lies in the osculating plane along $\gamma(s)$ and it allows us to define a unit horizontal vector field, η, along the curve, which also lies in the osculating plane, and satisfies that $\eta \times \xi = B$. We call $\{B, \eta, \xi\}$ a positive oriented orthonormal frame along $\gamma(s)$ which is adapted to the horizontal-vertical decomposition. Therefore, $\{T, N\}$ and $\{\eta, \xi\}$ provide the same orientation in the osculating plane along $\gamma(s)$. Let $\psi(s)$, $0 < \psi(s) < 2\pi$, be the angle between $T(s) = \gamma'(s)$ and ξ. Then it is a locally well defined differentiable function satisfying that

$$\begin{pmatrix} \eta(s) \\ \xi(s) \end{pmatrix} = \begin{pmatrix} \sin\psi(s) & -\cos\psi(s) \\ \cos\psi(s) & \sin\psi(s) \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \end{pmatrix} \tag{8}$$

Next we have the following characterization of the horizontality of the binormal:

Proposition 4.7 *Given a curve $\gamma(s)$ in $\mathrm{E}(c, r)$, the following statements are equivalent:*

(a) It has horizontal binormal;

(b) Its curvature function $\kappa(s)$, satisfies that $\kappa(s) + \psi'(s) = 0$;

(c) It is an extremal of the total curvature with $\tau = r$.

Proof. From (8) we have $\langle T(s), \xi \rangle = \cos\psi(s)$ and then $\kappa \langle N(s), \xi \rangle + \psi'(s) \sin\psi(s) = 0$. Moreover since $\xi = \cos\psi\, T + \sin\psi\, N$, we obtain $(\kappa + \psi') \sin\psi = 0$.

Conversely, if $\kappa(s) + \psi'(s) = 0$, then $\kappa(\langle N(s), \xi \rangle - \sin\psi) = 0$, and $\langle N(s), \xi \rangle = \sin\psi(s)$, which implies that $\langle B, \xi \rangle = 0$. This shows the equivalence between (1) and (2).

The equivalence between (1) and (3) follows from (5). □

Remark 4.8 (1). It should be observed that, as a consequence of the above proposition, curves in $\mathrm{E}(c, r)$ with curvature function satisfying $\kappa(s) = -\psi'(s)$, automatically have constant torsion $\tau = r$.

(2). Note also the existence of curves in $\mathrm{E}(c, r)$ with $\tau = r$ whose binormal is not horizontal. For example, the horizontal lifts of any curve in $\mathtt{B}^2(c)$.

To describe this class of curves from a geometric point of view and how they can be obtained, we proceed as follows. Let $\gamma(s)$ be a curve in $\mathrm{E}(c, r)$ whose binormal is horizontal and let $\gamma_1(s) = \pi(\gamma(s))$ be its projection over $\mathtt{B}^2(c)$, which, in general, need not be a unit speed curve. However, $B_1 = d\pi(B)$ defines a unit normal vector field along γ_1 and so $JB_1 = \gamma_1'/|\gamma_1'|$ is the unit tangent of that projection, where J denotes the usual complex structure on $\mathtt{B}^2(c)$. Let JB be the unit horizontal vector field along $\gamma(s)$ satisfying that $d\pi(JB) = JB_1$. Therefore, $\{B, JB, \xi\}$ provides a positive oriented orthonormal frame along $\gamma(s)$ which is adapted to the horizontal-vertical decomposition. In this framework, the unit tangent and the unit principal normal of $\gamma(s)$ in $\mathrm{E}(c, r)$ are given by

$$T(s) = \cos\psi\, \xi + \sin\psi JB, \qquad N(s) = \sin\psi\, \xi - \cos\psi JB.$$

Now, we compute the curvature function, $\kappa_g(s)$, of $\gamma_1(s)$ in $\mathtt{B}^2(c)$, which will determine, up to motions in $\mathtt{B}^2(c)$, the profile curve in whose Hopf cylinder $\gamma(s)$ lies. Then we have

$$\overline{\nabla}_T T = -\psi' \sin\psi\, \xi + r \cos\psi \sin\psi\, B + \psi' \cos\psi\, JB + \sin\psi\, \overline{\nabla}_T JB,$$

and

$$\overline{\nabla}_T \, JB = (r\,\cos\psi + \kappa_g\,\sin\psi)\,B.$$

Now, we use the first Frenet equation of $\gamma(s)$ in $\mathrm{E}(c,r)$, with $\kappa(s) = -\psi'(s)$, to see that

$$\kappa_g(s) = -2r\,\cot\psi(s).$$

At this point, the way to construct the class of extremals with horizontal binormal works as follows:

(1) First, we choose an arbitrary positive function $h(s)$, which will play the role of the curvature function of the extremal, and define the function

$$\psi(s) = -\int_0^s h(v)\,dv.$$

(2) Next, we take the curve $\beta(s)$ in $\mathtt{B}^2(c)$ (it is unique up to rigid motions) which is determined by its arc-length function $u(s)$ and its curvature function $\kappa(s)$ given, respectively, by

$$u(s) = \int_0^s \sin\psi(v)\,dv, \qquad \kappa_g(s) = -2r\,\cot\psi(s).$$

(3) Then, we use $\beta(s)$ as a profile curve to construct its Hopf cylinder S_β in $\mathrm{E}(c,r)$ and then choose in this flat surface the curve, $\gamma(s)$, with slope function (measured with respect to ξ) $\psi(s)$. We conclude that $\gamma(s)$ is an extremal of the total curvature functional on $\mathrm{E}(c,r)$ with horizontal binormal (consequently $\tau = r$) and curvature function $h(s)$. Moreover, all of extremals with horizontal binormal are obtained in this way.

It is well known that the curvature function of a plane curve is just the variation of an angle function measured with respect to a fixed direction and this is the key point to recover the curve, through quadratures, from its curvature function. So the solving natural equation works for plane curves. Now, the curvature function of a curve in $\mathrm{E}(c,r)$ with horizontal binormal is the variation of the angle that it makes with the Reeb vector field, ξ, and this is the main point to solve the so called solving natural equations problem for this kind of curve in $\mathrm{E}(c,r)$. For a better understanding of this problem, recall that for any curve $\beta(s)$ in $\mathtt{B}^2(c)$, we can choose a horizontal lift $\bar{\beta}(s)$ in $\mathrm{E}(c,r)$ and then parameterize the corresponding Hopf cylinder S_β by

$$X : \mathbb{R}^2 \to S_\beta \subset \mathrm{E}(c,r), \qquad X(u,t) = \phi_t(\bar{\beta}(u)),$$

where $\{\phi_t, \, : \, t \in \mathbb{R}\}$ is the one-parameter subgroup associated with ξ. Now, curves in $\mathbb{R}^2$ can be nicely obtained, using quadratures, from their curvature functions. In particular, the unit speed curve with curvature function $h(s)$ is given by

$$\alpha(s) = (u(s), t(s)) = \left(\int_0^s \sin\psi(v)\,dv, \int_0^s \cos\psi(v)\,dv\right),$$

where $\psi(s) = -\int_0^s h(v)\,dv$ is the angle that $\alpha(s)$ makes with the t-axis. Since X is an isometry, $\psi(s)$ is the angle that the curve

$$\gamma(s) = X(\alpha(s)) = \phi_{\int_0^s \cos\psi(v)\,dv}\left(\bar{\beta}\left(\int_0^s \sin\psi(v)\,dv\right)\right),$$

makes with the orbits. The curvature $\kappa(s)$ of $\gamma(s)$ in $\mathrm{E}(c,r)$ does not necessarily satisfy $\kappa(s)+\psi'(s)=0$. However, it holds if and only if the curve $\beta(s)$ is chosen to have arc-length and curvature functions given, respectively, by

$$u(s) = \int_0^s \sin\psi(v)\,dv, \qquad \kappa_g(s) = -2r\,\cot\psi(s).$$

Remark 4.9 It should be noted that this argument can be viewed as the solution of the so called solving natural equations for curves with horizontal binormal in $\mathrm{E}(c,r)$ problem. In fact, we obtain explicitly the curve from its curvature function through quadratures. Certainly, it is a surprising result keeping in mind that the spaces $\mathrm{E}(c,r)$ have not the highest rigidity. Moreover, the next theorem measures the size of this class of curves.

Even more, we can exploit the above argument to exhibit a one-to-one correspondence between the class of convex plane curves and the family of extremals with horizontal binormal in $\mathrm{E}(c,r)$. It can be obtained from the following result

Theorem 4.10 *For every convex curve α in the Euclidean plane, there exists a curve γ in $\mathrm{E}(c,r)$ with the same curvature function as α and torsion $\tau = r$, which is an extremal, with horizontal binormal, of the total curvature action on $\mathrm{E}(c,r)$.*

Proof. Let $\alpha(s) = (u(s), t(s))$ be any convex curve in the plane and denote by $\kappa(s) > 0$ its curvature function. Now, consider in $\mathtt{B}^2(c)$ a curve, $\beta(s)$ (unique up to motions) whose curvature function is $\kappa_g(s) = -2r\,\cot\psi(s)$ where

$$\psi(s) = -\int_0^s \kappa(v)\,dv.$$

Then consider the map $X : \mathbb{R}^2 \to S_\beta \subset \mathrm{E}(c,r)$ defined by

$$X(u,t) = \phi_t(\bar{\beta}(s),$$

where $\bar{\beta}$ is a horizontal lift of β. Finally the curve $\gamma(s) = X(\alpha(s))$ has curvature function $\kappa(s)$, not only in S_β but also in $\mathrm{E}(c,r)$, its torsion is $\tau = r$ and it is an extremal of the total curvature in $\mathrm{E}(c,r)$ which certainly has horizontal binormal. $\square$

A relevant special case appears when $r = 0$. In this case the homogeneous 3-space is a Riemannian product $\mathrm{E}(c,0) = \mathtt{B}^2(c) \times \mathbb{R}$, where (up to topology) $\mathtt{B}^2(c)$ is $\mathbb{S}^2(c)$ if $c > 0$ while it is $\mathbb{H}^2(c)$ when $c < 0$. Then, as a consequence of the above argument we have the following characterization for extremals with horizontal binormal

Corollary 4.11 *Every curve, $\gamma(s)$, with horizontal binormal in $\mathtt{B}^2(c)\times\mathbb{R}$ lies in a Hopf cylinder built on a geodesic of $\mathtt{B}^2(c)$. More precisely these curves can be explicitly parameterized as follows*

$$\gamma(s) = \left(\beta(u(s)), \int_0^s \cos\psi(v)\,dv\right),$$

where $\beta(u)$ is a unit speed geodesic in $\mathtt{B}^2(c)$ with arc-length function given by

$$u(s) = \int_0^s \sin\psi(v)\,dv \qquad \textit{with} \qquad \psi(s) = -\int_0^s \kappa(v)\,dv,$$

and where $\kappa(s)$ is an arbitrary function making the role of curvature function of $\gamma(s)$.

Observe that if β is a geodesic in $\mathtt{B}^2(c)$, then S_β is a totally geodesic surface in $\mathrm{E}(c,0) = \mathtt{B}^2(c)\times\mathbb{R}$. On the other hand, extremals with horizontal binormal have torsion zero. Therefore, the first claim of the above corollary can be viewed as a codimension reduction result, which is the classical behavior in spaces with constant curvature. In contrast, extremals in $\mathbb{S}^2(c)\times\mathbb{R}$ with horizontal normal do not lie in any totally geodesic surface (see Example 4.5).

5 Some examples and applications

In this section, we will show a few examples of curves which are extremal for the total curvature energy in $\mathrm{E}(c,r)$. They are intended to clarify the results which have been obtained in §4, however, we will not exhaust all possible cases, rather we will restrict ourselves to the two simplest cases, Riemannian products and nilmanifolds, where examples can be given explicitly and pictures can be exhibited. We follow the notation introduced in §4.

5.1 Extremals in Riemannian products

Remember that, as we have shown previously, the whole class of extremals in $\mathbb{H}^2(c)\times\mathbb{R}$ is made up of curves with horizontal binormal (see Corollary 4.11). On the contrary, in $\mathbb{S}^2(c)\times\mathbb{R}$, besides the class of curves with horizontal binormal (see Corollary 4.11), another one-parameter class of extremals with horizontal unit normal also appears (see Example 4.5). This additional class is formed by Lancret helices having both constant curvature and torsion given, respectively, by

$$\tau = \sqrt{\frac{c}{m^2+1}}, \qquad \kappa = \frac{1}{m}\sqrt{\frac{c}{m^2+1}}.$$

We want to use Corollary 4.11 to provide some explicit examples of curves with horizontal binormal, and so being extremal for the total curvature, in $\mathtt{B}^2(c)\times\mathbb{R}$ with either $c=1$ or $c=-1$.

Let us write $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2+y^2+z^2 = 1\}$ and $\mathbb{H}^2 = \{(x,y,z) \in \mathbb{L}^3 \,:\, x^2+y^2-z^2 = -1,\ z>0\}$ and consider, without loss of generality, the geodesics $\beta(u) = (\cos u, 0, \sin u) \subset \mathbb{S}^2$ and $\beta(u) = (\cosh u, 0, \sinh u) \subset \mathbb{H}^2$. Now, the algorithm to construct extremals with horizontal binormal in $\mathtt{B}^2 \times \mathbb{R}$, which is implicitly contained in Corollary 4.11, consists in building curves in the surface $\beta \times \mathbb{R}$ starting from a prescribed function that makes the role of curvature function. Applying this, one has, for instance

(1) Circles viewed as extremals. Consider a constant function $\kappa(s) = -a$, then we have $\psi(s) = as$ and therefore

$$\gamma(s) = (\cos(-\frac{1}{a}\cos(as)), 0, \sin(-\frac{1}{a}\cos(as)), \frac{1}{a}\sin(as))$$

is an extremal of the total curvature energy of curves in $\mathbb{S}^2 \times \mathbb{R}$, while

$$\gamma(s) = (\sinh(-\frac{1}{a}\cos(as)), 0, \cosh(-\frac{1}{a}\cos(as)), \frac{1}{a}\sin(as))$$

is an extremal in $\mathbb{H}^2 \times \mathbb{R}$.

(2) Clothoids regarded as extremals. Pick up the function $\kappa(s) = -s$, that represents the curvature function of a clothoid or Cornu spiral. We can solve the natural equations to obtain the clothoid in terms of the Fresnel integrals

$$\alpha(s) = (u(s), t(s)) = \left(\int_0^s \sin(v^2/2)\, dv, \int_0^s \cos(v^2/2)\, dv\right).$$

Since $\psi(s) = \frac{s^2}{2}$, we have that

$$\gamma(s) = \left(\cos\left(\int_0^s \sin(v^2/2)\, dv\right), 0, \sin\left(\int_0^s \sin(v^2/2)\, dv\right), \int_0^s \cos(v^2/2)\, dv\right),$$

is an extremal of the total curvature in $\mathbb{S}^2 \times \mathbb{R} \subset \mathbb{R}^4$, while

$$\gamma(s) = \left(\sinh\left(\int_0^s \sin(v^2/2)\, dv\right), 0, \cosh\left(\int_0^s \sin(v^2/2)\, dv\right), \int_0^s \cos(v^2/2)\, dv\right),$$

is an extremal of the total curvature in $\mathbb{H}^2 \times \mathbb{R} \subset \mathbb{L}^3 \times \mathbb{R}$.

(3) Catenaries as extremals. Now, take the function $\kappa(s) = \frac{1}{1+s^2}$. The unit speed curve having this curvature function is the catenary

$$\alpha(s) = (u(s), t(s)) = \left(\sqrt{1+s^2}, \lg(s+\sqrt{1+s^2}\right).$$

In this case, $\psi(s) = \arctan s$, and, consequently, we obtain that the following curves

$$\gamma(s) = \left(\cos\sqrt{1+s^2}, 0, \sin\sqrt{1+s^2}, \lg(s+\sqrt{1+s^2}\right),$$

$$\gamma(s) = \left(\sinh\sqrt{1+s^2}, 0, \cosh\sqrt{1+s^2}, \lg(s+\sqrt{1+s^2}\right).$$

are extremal for the total curvature in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, respectively,

5.2 Extremals in nilmanifolds

Now, we wish to add a few examples describing extremals in nilmanifolds geometries which are associated with the Heisenberg group. In dimension three they are one of the eight canonical Thurston geometries. According to the usual terminology, they are identified with $\mathrm{E}(0,r)$ and, consequently, appear as Riemannian submersions over a Euclidean plane.

The 3-dimensional Heisenberg group, N^3_r, is nothing but $\mathbb{R}^3$ endowed with the metric $ds^2 = dx^2 + dy^2 + \theta^2$, where θ is the one-form defined by

$$\theta = dz + r(y\,dx - x\,dy).$$

It is known that $\xi = \partial_z$ is an infinitesimal translation, which is, precisely, the Reeb vector field when (N^3_r, θ) is viewed as a contact manifold. The one-parameter subgroup associated with ξ is

$$\{\phi_t : \mathrm{N}^3_r \to \mathrm{N}^3_r \,:\, t \in \mathbb{R}\}, \qquad \phi_t(x,y,z) = (x,y,z+t).$$

Every orbit is principal and all of them are geodesics of N^3_r. The space of orbits is the Euclidean plane and the projection $\pi : \mathrm{N}^3_r \to \mathbb{R}^2$ is given by $\pi(x,y,z) = (x,y)$.

We recall again that we have two classes of extremals for the total curvature action on 3-dimensional homogeneous 3-spaces and that, in addition, we have provided algorithms and methods to construct extremals within both families. In particular, they can be described as curves lying in Hopf cylinders. This key fact will be used to give explicit extremals of the total curvature in the Heisenberg group.

Hopf cylinders shaped on a plane curve $\beta(s)$, S_β, are flat surfaces which can be obtained by specifying a Legendrian curve. So, choose a profile curve, $\beta(s) = (x(s), y(s))$, in the space of orbits, and assume that it is parameterized by the arc-length. Their horizontal lifts (Legendrian curves) $\bar\beta(s)$ project down over $\beta(s)$, and then $\bar\beta(s) = (x(s), y(s), z(s))$, where horizontality implies that the third coordinate must satisfy $\theta(\bar\beta'(s)) = 0$. Hence, horizontal lifts turn out to be determined by the solutions of the differential equation

$$\frac{dz}{ds} = r\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right). \tag{9}$$

Thus, a horizontal lift of the profile curve (which is obviously parameterized by the arc-length too) provides the following natural parameterisation of the equivariant surface S_β

$$X(s,t) = \phi_t(\bar\beta(s)) = (x(s), y(s), z(s)+t).$$

Now, the first family of extremals of the total curvature action we alluded to before (critical curves with horizontal unit normal) are Lancret helices. Then, we apply the corresponding algorithm, given in §4, to obtain a real one-parameter family of extremals in the 3-dimensional Heisenberg group, namely we have

Corollary 5.1 *For any $\varphi \in (0, \pi/6)$, there exists a Lancret helix in N^3_r, with slope $m = \cot\varphi$ in a suitable Hopf cylinder, which is an extremal of the total curvature.*

Lancret helices in N_r^3 are curves, $\gamma(s)$, forming a constant angle with ξ and they coincide precisely with the set of geodesics of the Hopf cylinders of N_r^3 (Propositon 3.1). In particular, those Lancret helices which provide extremals for the total curvature must be geodesics of Hopf cylinders shaped on circles of the Euclidean plane. So, let $\beta(s) = \left(R\cos\frac{s}{R}, R\sin\frac{s}{R}\right)$ be the unit speed circle with radius R and centered at the origin. In this case, it is very easy to solve the equation (9) to see that horizontal lifts of the above circles are just circular helices when they are viewed in the Euclidean 3-space, namely

$$\bar{\beta}(s) = \left(R\cos\frac{s}{R}, R\sin\frac{s}{R}, rRs + c\right), \qquad \forall c \in \mathbb{R}.$$

Remark 5.2 Notice that Euclidean horizontal lifts of $\beta(s)$ are just circles, with radius R and centers on the z-axis, located in planes parallel to $z = 0$.

The corresponding Hopf cylinders can be parameterized by

$$X : \mathbb{R}^2 \to \mathrm{N}^3, \qquad X(s,t) = \left(R\cos\frac{s}{R}, R\sin\frac{s}{R}, rRs + t\right).$$

Now, we use Proposition 3.1 to see that extremal Lancret helices in the Heisenberg group are those whose torsion satisfies

$$\tau^2 = r^2(4\cos^2\varphi - 3) = \frac{r^2(m^2-3)}{m^2+1},$$

and, according to (3), they are geodesics in Hopf cylinders over circles with radius

$$R = \frac{1}{\kappa_g} = \frac{m}{r\left(\sqrt{(m^2+1)(m^2-3)} + m^2 - 1\right)}, \qquad m = \cot\varphi.$$

Then, for any $\varphi \in (0, \pi/6)$ we obtain the extremal Lancret helices parameterized by

$$\gamma_m(s) = X\left(\frac{s}{\sqrt{1+m^2}}, \frac{ms}{\sqrt{1+m^2}}\right) = \left(R\cos\frac{s}{R\sqrt{1+m^2}}, R\sin\frac{s}{R\sqrt{1+m^2}}, \frac{(rR+m)s}{\sqrt{1+m^2}}\right).$$

Now, in order to provide examples of extremals in the second class, that is, extremals with horizontal binormal, we can use the following alternative algorithm obviously derived from the results in §4.2. It works on any $\mathrm{E}(c,r)$ and, in particular, on the Heisenberg group:

(i) Consider any unit speed curve $\beta(u)$ in $\mathtt{B}^2(c)$ with curvature function $\kappa_g(u)$.

(ii) Define functions $\psi(u) = -\mathrm{arcot}\frac{\kappa_g(u)}{2r}$ and $s(u) = \int_0^u \csc\psi(v)\,dv$.

(iii) Now, in the Hopf cylinder $S_\beta = \pi^{-1}(\beta(u(s)))$, choose the curve $\gamma(s)$ with slope $\psi(u(s))$ to obtain an extremal with horizontal binormal in $\mathrm{E}(c,r)$.

Following this procedure we give a few more examples of critical curves belonging to the second class.

(1) An extremal built on a catenary. By solving the natural equations one gets a parameterization of the unit speed curve, unique up to motions in $\mathbb{R}^2$, determined by the curvature function

$$\kappa_g(u) = \frac{1}{1+u^2}\,.$$

The trace of this curve is a catenary which, therefore, can be parameterized as

$$\beta(u) = (x(u), y(u)) = (\operatorname{arcsinh}(u), \sqrt{1+u^2}).$$

Using this and integrating (9) one has

$$z(u) = r\left(\sqrt{1+u^2}\operatorname{arcsinh}(u) - u\right),$$

which means that a horizontal lift, $\bar{\beta}$, of β is given by

$$\bar{\beta}(u) = \left(\operatorname{arcsinh}(u), \sqrt{1+u^2}, r\left(\sqrt{1+u^2}\operatorname{arcsinh}(u) - u\right)\right).$$

Now, it is easy to write a parameterization $X(u,t)$ for the Hopf cylinder shaped on β, $S_\beta = \pi^{-1}(\beta)$

$$X(u,t) = \left(\operatorname{arcsinh}(u), \sqrt{1+u^2}, r\left(\sqrt{1+u^2}\operatorname{arcsinh}(u) - u\right) + t\right).$$

Finally, in the flat surface S_β we choose the curve $\gamma(s)$ whose slope, measured with respect to the vertical axis, is given by

$$\psi(u) = -\operatorname{arcot}\frac{1}{2r(1+u^2)}\,.$$

A parameterization of $\gamma(s)$ is

$$\gamma(s) = X(u(s), t(s))\,,$$

where

$$u(s) = -\int_0^s \frac{2r(1+v^2)}{\sqrt{1+4r^2(1+v^2)^2}}\,dv, \qquad t(s) = \int_0^s \frac{1}{\sqrt{1+4r^2(1+v^2)^2}}\,dv\,,$$

which can be expressed in terms of Elliptic functions. Then, $\gamma(s)$ provides an extremal for the total curvature action (with horizontal binormal) on the Heisenberg group (Fig. 1.a).

(2) An extremal built on a logarithmic spiral. Proceeding analogously, we see that a unit speed parameterization of the planar curve (unique up to motions in $\mathbb{R}^2$) determined by the curvature function

$$\kappa_g(u) = \frac{1}{u}\,,$$

is given by $\beta(u) = (x(u), y(u))$, where

$$x(u) = \frac{1}{2}u\left(\cos(\log(u)) + \sin(\log(u))\right)\,, \quad y(u) = \frac{1}{2}u\left(\sin(\log(u)) - \cos(\log(u))\right)\,. \tag{10}$$

Moreover, solving (9) we obtain

$$z(u) = \frac{ru^2}{4}\,. \tag{11}$$

Hence, using (10) and (11) one sees that a horizontal lift of β, $\bar{\beta}(u)$, and the Hopf cylinder shaped on β, $S_\beta = \pi^{-1}(\beta)$, can be explicitly and respectively parameterized by

$$\bar{\beta}(s) = (x(s), y(s), z(s))\,, \quad X(u,t) = (x(u), y(u), z(u) + t)\,.$$

Again, in S_β we take the curve $\gamma(s)$ whose slope, measured with respect to the vertical axis, is given by

$$\psi(u) = -\mathrm{arcot}\frac{1}{u}\,.$$

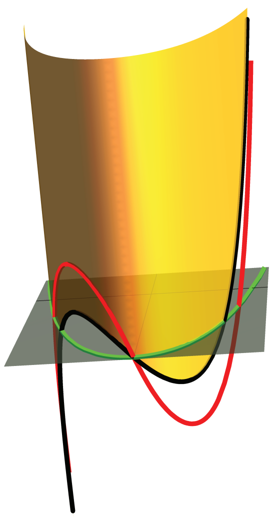
A parameterization of $\gamma(s)$ is

$$\gamma(s) = X(u(s), t(s))\,,$$

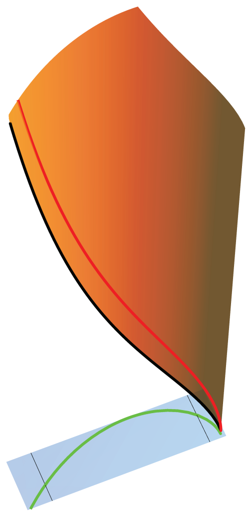
where

$$u(s) = -\int_0^s \frac{2rv}{\sqrt{1+4r^2v^2}}\,dv = \sqrt{1+s^2}\,, \qquad t(s) = \int_0^s \frac{1}{\sqrt{1+4r^2v^2}}\,dv = \operatorname{arcsinh}(s)\,.$$

Then, $\gamma(s)$ is also an extremal for the total curvature action (with horizontal binormal) on the Heisenberg group (Fig. 1.b).



(a) β is a catenary.



(b) β is a logarithmic spiral.

Figure 1. A critical curve γ (curve in red) for the total curvature in Nil_3 with $r = 1$. Here, γ lies on a Hopf cylinder shaped on a planar curve β (curve in green), whose horizontal lift $\bar{\beta}$ is shown in black.

(3) An extremal constructed out of an involute of a circle. As a final example, let us consider the unit speed planar curve, β, determined by the curvature function

$$\kappa_g(u) = \frac{1}{\sqrt{u}}\,.$$

Solving the natural equation we obtain the following unit speed parameterization of the involute of a circle $\beta(u) = (x(u), y(u))$, where

$$x(u) = \frac{1}{2}\left(\cos(2\sqrt{u}) + \sqrt{u}\sin(2\sqrt{u})\right)\,, \quad y(u) = \frac{1}{2}\left(-2\sqrt{u}\cos(2\sqrt{u}) + \sin(2\sqrt{u})\right)\,. \quad (12)$$

Using these values of $x(u)$ and $y(u)$ in (9) and solving, we have

$$z(u) = r\sqrt{u}\,. \quad (13)$$

Formulae (12) and (13) allow us to obtain a parameterization of both, a horizontal lift of β, $\bar{\beta}(u)$, and of the Hopf cylinder shaped on β, $S_\beta = \pi^{-1}(\beta)$, which can be explicitly given by

$$\bar{\beta}(s) = (x(s), y(s), z(s))\,, \quad X(u,t) = (x(u), y(u), z(u) + t)\,.$$

Then, the curve $\gamma(s)$ parameterized by $\gamma(s) = X(u(s), t(s))$, with

$$\begin{aligned} u(s) &= -\int_0^s \frac{2r\sqrt{v}}{\sqrt{1+4r^2v}}\,dv = -\frac{1}{4}\left(2\sqrt{s(1+4s)} - \operatorname{arcsinh}\left(2\sqrt{s}\right)\right)\,, \\ t(s) &= \int_0^s \frac{1}{\sqrt{1+4r^2v}}\,dv = \frac{1}{2}\sqrt{1+4s}\,, \end{aligned}$$

lies in S_β forming an angle ψ with respect to the vertical axis given by

$$\psi(u) = -\operatorname{arcot}\frac{1}{2r\sqrt{u}}\,.$$

Therefore, $\gamma(s)$ is another extremal for the total curvature action (with horizontal binormal) on the Heisenberg group (Fig. 2).

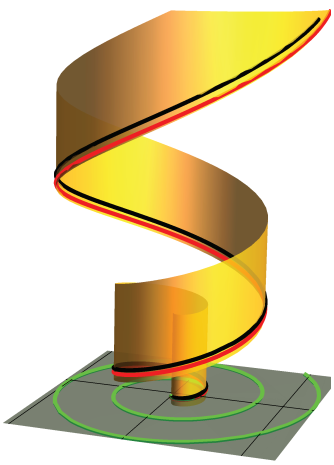


Figure 2. A critical curve γ (curve in red) for the total curvature in Nil_3 with $r = 1$. Here, γ lies on a Hopf cylinder shaped on an involute of a circle β (curve in green), whose horizontal lift $\bar{\beta}$ is shown in black.

6 Closed extremals and critical levels

When one studies problems related with the differential geometry of curves, besides the solving natural equations problem, a second fundamental and open problem appears. It is called the closed curve problem. In this section we will focus on the solution to this problem for extremals of the total curvature in rotationally symmetric homogeneous 3-spaces. However, since, as we have shown, the space of extremals is made up of two families whose members are quite different from a qualitative point of view, we will treat the closed curve problem separately in both families.

Let us consider first the family of extremals which are Lancret helices, that is, those with horizontal normal. According to Corollary 4.4, every $\mathrm{E}(c,r)$, except $\mathbb{H}^2(c)\times\mathbb{R}$, admits a real one-parameter class of Lancret helices providing extremals. In addition, we also know that these extremals are realized as geodesics of Hopf cylinders built over circles in $\mathtt{B}^2(c)$. Consequently, in order to get closed solutions, it is necessary that Hopf cylinders become tori, which happens just when the orbits close up. Otherwise, we can take a quotient of $\mathrm{E}(c,r)$ via a suitable vertical translation ϕ_t. In this context we have the following quantization principle à la Dirac.

Theorem 6.1 *Every $\mathrm{E}(c,r)$ with compact orbits, except $\mathbb{H}^2(c)\times\mathbb{S}^1$, admits a rational one-parameter class of closed Lancret helices which are extremals of the total curvature energy.*

Proof. Since $\pi:\mathrm{E}(c,r)\to\mathtt{B}^2(c)$ is a circle bundle, then for every closed curve β in $\mathtt{B}^2(c)$ its Hopf cylinder, S_β, is actually a torus, which is embedded when the curve has no self-intersections. Moreover, it is a flat torus whose isometry type is known to be described as follows (see [1, 5] for special cases).

First, note that if $r=0$, then $\mathrm{E}(c,0)$ is the Riemannian product $\mathbb{S}^2(c)\times\mathbb{S}^1$, where we have assumed, without loss of generality, that the orbits are unit circles. Since the holonomy is trivial, if β is a simple closed curve in $\mathbb{S}^2(c)$, with length L, then S_β is a flat torus generated by a rectangular lattice in the Euclidean plane, namely, it is $\mathbb{R}^2/\Gamma$, where Γ is the lattice generated by $\{(0,2\pi),(L,0)\}$. Then a Lancret helix (a geodesic in S_β) closes up if and only if its slope is a rational multiple of $2\pi/L$. Now the class of extremal Lancret helices in $\mathbb{S}^2(c)\times\mathbb{S}^1$ is the real one-parameter class of curves $\{\gamma_m : m\neq 0\}$, where γ_m is the geodesic with slope m in the Hopf torus built on the circle β_m in $\mathbb{S}^2(c)$, whose curvature is given by (7) with $r=0$. As a consequence, the class of closed Lancret helices that are extremals of the total curvature in $\mathbb{S}^2(c)\times\mathbb{S}^1$ is the rational one-parameter family

$$\left\{\gamma_m \,:\, m=\frac{2\pi}{L}\,q,\;\; q\in\mathbb{Q}\right\}.$$

If $r\neq 0$, then the holonomy is not trivial, which implies that the lattice describing the isometry type of S_β is not rectangular. In fact, it is generated by $\{(0,2\pi),(L,2A)\}$, where A is the area enclosed by β in $\mathtt{B}^2(c)$. Now a straight line $\alpha(s)$ with slope m provides a closed geodesic in the Hopf torus if and only if there exists $s_o > 0$ such that $\alpha(s_o) \in \Gamma$, which is equivalent to the following quantization formula

$$m = \frac{2}{L}(A + \pi\, r\, q), \quad q \in \mathbb{Q}.$$

As above, let γ_m be the geodesic (Lancret helix) in the Hopf torus constructed on the circle, in $\mathtt{B}^2(c)$, with curvature given by (7). Then γ_m is an extremal of the total curvature energy if and only if

$$m^2 > 3 - \frac{c}{r^2}.$$

Consequently, the class of closed Lancret helices which are extremals coincides with the following rational one-parameter family

$$\left\{ \gamma_m \,:\, m^2 > 3 - \frac{c}{r^2}, \quad m = \frac{2}{L}(A + \pi\, r\, q), \;\; q \in \mathbb{Q} \right\}.$$

$\square$

Finally, we turn our attention to the case of extremals with horizontal binormal.

Theorem 6.2 *For every convex closed curve α in the Euclidean plane, there exists a closed curve γ in $\mathrm{E}(c, r)$ with the same curvature function as α and torsion $\tau = r$, which is an extremal, with horizontal binormal, of the total curvature action on $\mathrm{E}(c, r)$.*

Proof. It follows from the proof of Theorem 4.10 for closed curves in the Euclidean plane.

$\square$

Remark 6.3 (1). The above theorem provides, up to congruences, a map from the space of convex closed curves in the Euclidean plane in the space of closed extremals with horizontal binormal in $\mathrm{E}(c, r)$. Even more, it is a one-to-one correspondence between both spaces of curves. In fact, if the Euclidean curve α is not closed then its image γ also is not, because S_β is not a torus. Observe that the profile curve has curvature function $\kappa_g(s) = -2r \cot \psi(s)$.

(2). If we consider the total curvature energy on closed curves in $\mathrm{E}(c, r)$, then, except in $\mathbb{H}^2(c) \times \mathbb{S}^1$, we obtain two families of extremals. On the one hand, a rational one-parameter class, $\mathbf{F}_N$, of Lancret helices (equivalently, curves with horizontal normal). On the other hand, the family of extremals, $\mathbf{F}_B$, with horizontal binormal, which has been identified with the class of convex closed curves in the Euclidean plane. The first family is discrete and consequently the critical values in $\mathbf{F}_N$ are quantized à la Dirac. The second family is certainly far from being discrete. However, it should be noted that according to the above one-to-one correspondence, a curve in $\mathbf{F}_B$ and its partner in the Euclidean plane have the same curvature function in the corresponding spaces. Consequently the critical values of the total curvature energy on closed extremals with horizontal binormal are not arbitrary, but they are integer multiples of a fixed value. Namely, the critical levels are

$$\mathcal{F}(\gamma) = \int_\gamma \kappa(s)\, ds = 2\pi\, \mathrm{i}(\alpha),$$

$\mathrm{i}(\alpha)$ standing for the rotation index of the corresponding closed curve in the Euclidean plane. This proves an interesting principle of quantization for the total curvature energy acting on closed curves in $\mathrm{E}(c, r)$.

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