On the energy density of helical proteins

Manuel Barros[1] and Angel Ferrández[2*]

[1]Departamento de Geometría y Topología, Facultad de Ciencias
Universidad de Granada, 18071-Granada, Spain.
E-mail address: `mbarros@ugr.es`

[2]Departamento de Matemáticas, Universidad de Murcia
Campus de Espinardo, 30100-Murcia, Spain.
E-mail address: `aferr@um.es`

Abstract

We solve the problem of determining the energy actions whose moduli space of extremals contains the class of Lancret helices with a prescribed slope. We first see that the energy density should be linear both in the total bending and in the total twisting, such that the ratio between the weights of them is the prescribed slope. This will give an affirmative answer to the conjecture stated in [2]. Then, we normalize to get the best choice for the helical energy. It allows us to show that the energy, for instance of a protein chain, does not depend on the slope and is invariant under homotopic changes of the cross section which determines the cylinder where the helix is lying. In particular, the energy of a helix is not arbitrary, but it is given as natural multiples of some basic quantity of energy.

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1 Motivation

The least action principle (also known as the Maupertuis principle) states that *when a change occurs in Nature, the quantity of action necessary for the change is the least possible.* It is certainly one of the fundamental props of the modern science, so the shapes in Nature must be stable, and so extremals, for a suitable action. Often

(*) Corresponding author A. Ferrández.

it turns out to be complicated to choose the action, a typical example to illustrate this claim is provided by the string theory. On the other hand, the choice of the action is not arbitrary, it involves some requirements. The first obvious one is that it must be invariant under rigid motions. Since we are interested in one-dimensional configurations (see [6]), the action must be also invariant under reparameterizations. Consequently, admissible Lagrangian densities must be functions of the following three geometrical invariants: length, bending (curvature) and twisting (torsion).

In [7] the authors proposed a model to describe protein chains which is governed by an energy action whose density does not depend on the torsion and is linear in the curvature. The motivation of that choice was based on the following main facts:

(1) The helical structure of proteins implies the choice of an energy action whose only extremals would be helices.

(2) The main result of [7], which was stated as follows: *If the extremals of* $\mathcal{F}(\gamma) = \int_\gamma F(\kappa(s))\, ds$ *are circular helices, then* $F(\kappa) = m + n\kappa$*, with* $m\, n \neq 0$.

Nevertheless, these arguments seem to provide a certain incoherence regarding the concept of helix. While the former concerns the helical nature of the protein chains, the latter asks for the extremals which are circular helices. However, it is obvious that helical structures appearing in nature are far from being circular helices. Therefore, the model proposed in [7], a priori, is not suited to describe other helical structures as important as elliptical, conical and spherical helices that might be of remarkable interest not only in proteins folding, but also in other contexts such as antennas or nanotechnology.

On the other hand, several strong arguments could be given in order for the torsion to be included in the energy density governing the protein model. Let us mention a few of them:

(i) It is well known that the length, the curvature and the torsion are the natural geometric invariants which allow us to characterize the congruence class of curves (center lines of protein chains), that is, the corresponding moduli space. Then, there is no reason to exclude the torsion from the energy density.

(ii) It is worth pointing out that the torsion is an essential ingredient in the equation of Calugareanu [5] and White [19], which becomes quite important when studying the theory of DNA supercoiling.

(iii) Circular helices geometrically appear as geodesics in right cylinders shaped on circles. An obvious extension of this picture is provided when changing the circle by another plane curve. The geodesics of a right cylinder, with arbitrary cross section, are called general or Lancret helices. They can be viewed as curves making a constant angle with a fixed direction, the axis, and they are well known in the literature as curves with constant slope, or *Böschungslinien* (see for instance [1] and references therein). From now on, we will denote by $\mathcal{L}_\omega$ the class of Lancret helices with slope ω. It is clear that in this family we can find, among others, spherical, conical and elliptical helices.

(iv) Helices appear at every level across the different orders of magnitude that span the range of side between molecules and galaxies. Therefore, if we wish to construct a geometrical model to describe helices in nature, it seems natural to explore the relationship between both the helix as an abstract mathematical idea, with its elegance and simplicity, and the real helical configurations that contributes to the richness and complexity of nature. In this respect, it is important to consider the history of what helix means along the biological literature (see for example [9] and references therein).

As far as we know, the starting point in this study is the work of Pauling, [15, 16, 17], where the importance and ubiquity of helices, in particular in biology, is due to the fact that *identical objects, regularly assembled, form a helix*. This is a simple and elegant theorem which is well known in the biological community. However, it is less familiar to mathematicians and physicists. Throughout the biological literature, this theorem is often motivated by different pictures, though its first proof was provided by K. Cahill, [4], using the differential geometry of Lancret. There, that theorem is illustrated by and applied to nucleic acids, protein secondary structures, protein folding and viral capsids, which are regarded as Lancret helices. Certainly any structure that is straight or rod like (including fibres when length greatly exceeds diameter) is one having repetition along a screw axis, that is, a helix (see [6]). So helix means a coiled form that advances around a central axis. This history takes us to the idea of *general helix or Lancret helix*. Therefore many of the helices in nature are Lancret ones (see [12, 13], where general (or Lancret) helices were used in connection with proteins).

On the other hand, from a generic and geometrical point of view, to study helices in $\mathbb{R}^3$, we can proceed as follows. Start from a vector field, say X, in $\mathbb{R}^3$; integrate it to obtain the corresponding flow and then look at those curves that evolve making a constant angle with that flow. Thus, we obtain the idea of helix with axis X. In particular, those general helices in nature correspond to the case where the axis, X, is an infinitesimal translation. Of course, this is only the first step of a series of problems which arise when relaxing the rigidity of the axis X.

Now it seems natural to study the following problem which was first stated in [2]: *Determine the energy density $F(\kappa, \tau)$ in order for the class of extremals of the action $\mathcal{F}(\gamma) = \int_\gamma F(\kappa(s), \tau(s))\, ds$ to be precisely $\mathcal{L}_\omega$, for a prescribed ω.*

2 The uniqueness of the energy action

A curve of constant slope or general helix in Euclidean space $\mathbb{R}^3$ is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. To highlight the importance of this result, throughout this paper, those curves will be called Lancret helices.

The family $\mathcal{L}_\omega$ of Lancret helices with slope ω is nicely characterized by the condition

$$\frac{\tau}{\kappa} = \omega, \tag{1}$$

where the slope is $\omega = \cot\theta$ and θ is the angle that the curve makes with its axis. Geometrically, these curves are viewed as geodesics on right cylinders whose cross sections are curves lying in planes orthogonal to the axis. Then a Lancret curve is completely determined, up to motions in $\mathbb{R}^3$, by either

(i) A pair of functions κ (curvature) and τ (torsion) satisfying (1) for some constant ω; or

(ii) A function $\rho > 0$ (standing for the curvature of its plane cross section of the right cylinder where the helix is a geodesic) and the slope $\omega = \cot\theta$.

Both moduli are related by

$$\kappa = \rho \sin^2\theta, \quad \tau = \rho \sin\theta\cos\theta. \tag{2}$$

Therefore, once the slope ω is fixed, the moduli space $\mathcal{L}_\omega$ of Lancret helices with that slope is identified with the space of differentiable functions of a real variable. Certainly $\mathcal{L}_\omega$ admits a notable subspace, $\mathcal{C}_\omega \subset \mathcal{L}_\omega$, made up of circular helices. Geometrically circular helices correspond to geodesics of circular right cylinders, those whose cross sections are circles. On the other hand, $\mathcal{L}_{\omega=0}$, the Lancret curves with slope zero, correspond to plane curves. In this case helix and cross section agree. $\mathcal{C}_{\omega=0}$ is of course the class of circles.

Admissible helical structures in nature, in particular helical proteins, should be extremals of a reasonable elastic energy action. The choice of that energy action involves some requirements. Thus, it must be invariant not only by reparameterizations, but also by motions in the Euclidean space. Then the energy density should be a certain function of the geometrical invariants of curves: *the arc length, the curvature and the torsion*

$$\mathcal{F}(\gamma) = \int_\gamma F(\kappa(s), \tau(s))\, ds, \tag{3}$$

F standing for any differentiable function. The Euler-Lagrange equations, acting on a suitable space of curves, can be obtained by using standard arguments that involve several integrations by parts. For fixed points p and q, and frames f_p and f_q at these points, let Ω be the space of Frenet curves connecting them with Frenet frames f_p and f_q at those points. The extremals of $\mathcal{F}$ on Ω, in a general background, were computed in [8] (see equations (4) and (5)) which now yields

$$-\kappa F + (\kappa^2 - \tau^2)F_\kappa + +2\kappa\tau F_\tau + F_\kappa'' + \left(\frac{\tau}{\kappa}F_\tau'\right)' + \tau\left(\frac{F_\tau'}{\kappa}\right)' = 0, \tag{4}$$

$$\tau F_\kappa' + \frac{\tau^2}{\kappa}F_\tau' + (\tau F_\kappa - \kappa F_\tau)' - \left(\frac{F_\tau'}{\kappa}\right)'' = 0, \tag{5}$$

where $F_\kappa = \partial F/\partial\kappa$, $F_\tau = \partial F/\partial\tau$ and prime means differentiation with respect to the arc length parameter. These equations were obtained later in [14] and then manipulated in [14, 18], but no significant progress was achieved, even in special cases.

The key assumption is that $\mathcal{F}(\gamma) = \int_\gamma F(\kappa(s), \tau(s))\, ds$ admits an extremal which is a Lancret helix with slope ω. Then, from (5) we obtain

$$\omega(\kappa F_\kappa' + \tau F_\tau') + (\tau F_\kappa - \kappa F_\tau)' - \left(\frac{F_\tau'}{\kappa}\right)'' = 0.$$

However

$$\kappa F_\kappa' + \tau F_\tau' = (\kappa F_\kappa + \tau F_\tau)' - (\kappa' F_\kappa + \tau' F_\tau) = (\kappa F_\kappa + \tau F_\tau)' - F',$$

and then we obtain

$$\left(\frac{F_\tau'}{\kappa}\right)' = -\omega F + (\omega\kappa + \tau)F_\kappa + (\omega\tau - \kappa)F_\tau + K, \tag{6}$$

K being a constant.

We combine (6) with (5) to obtain

$$(1+\omega^2)\kappa F = (1+\omega^2)\kappa^2 F_\kappa + (1+\omega^2)\kappa\tau F_\tau + \omega K\kappa + F_\kappa'' + \omega F_\tau''.$$

Now define the function $h(\kappa) = F(\kappa, \omega\kappa)$ along the Lancret helix. As $h_\kappa = F_\kappa + \omega F_\tau$ we obtain

$$(1+\omega^2)\kappa F = (1+\omega^2)\kappa^2 F_\kappa + (1+\omega^2)\kappa\tau F_\tau + \omega K\kappa + h_\kappa''. \tag{7}$$

We assume that the extremal is chosen to be a solution of $h_\kappa'' = 0$ (see the remark below). Then we get

$$F(\kappa, \tau) = \kappa F_\kappa + \tau F_\tau + A_1, \qquad A_1 = \frac{\omega K}{1+\omega^2}. \tag{8}$$

Remark 1 *It is worth noting that the energy density should be of the form (8) provided the action $\mathcal{F}(\gamma) = \int_\gamma F(\kappa(s), \tau(s))\, ds$ admits an extremal being a Lancret helix which solves the equation $h_\kappa'' = 0$. Using (2), that equation, for Lancret helices, can be viewed as a differential equation in the curvature function $\rho(s)$ of the corresponding cross section of the cylinder. Therefore, when choosing a solution we get the cross section as well as the Lancret helix with slope ω. Alternatively, circular helices always solve the equation $h_\kappa'' = 0$. In particular, equation (8) holds if there exists an extremal which is a circular helix with slope ω. In particular, that holds provided the space of extremals is $\mathcal{L}_\omega$.*

Now, from (8), we directly obtain

$$\kappa F_{\kappa\kappa} + \tau F_{\kappa\tau} = \kappa F_{\kappa\tau} + \tau F_{\tau\tau} = 0,$$

and use again the existence of a Lancret helix, with slope ω, as an extremal to conclude that

$$F_\kappa + \omega F_\tau = A_2, \tag{9}$$

A_2 being a certain constant. On the other hand, from (4), the existence of an extremal satisfying $h''_\kappa = 0$ yields the following energy density

$$F(\kappa,\tau) = (1-\omega^2)\kappa F_\kappa + 2\tau F_\tau, \tag{10}$$

which we compare with (8) to deduce

$$-\omega^2 \kappa F_\kappa + \tau F_\tau = A_1,$$

A_1 being a constant. Finally, we can solve this equation using (9) to obtain

$$\kappa F_\kappa = \frac{A_2}{1+\omega^2}\kappa - \frac{A_1}{1+\omega^2}, \qquad \tau F_\tau = \frac{A_2\omega}{1+\omega^2}\tau + \frac{A_1}{1+\omega^2},$$

which shows that the energy density

$$F(\kappa,\tau) = \frac{A_2}{1+\omega^2}\kappa + \frac{A_2\omega}{1+\omega^2}\tau + A_1$$

is affine in κ and τ.

Summarizing, we have shown the following

Proposition 2 *If the extremals of $\mathcal{F}(\gamma) = \int_\gamma F(\kappa(s),\tau(s))\,ds$ are Lancret helices, then $F(\kappa) = a + b\kappa + c\tau$.*

3 The extremals of the linear energy action

For any real numbers $m, n, p \in \mathbb{R}$, we consider the action

$$\mathcal{F}_{mnp} : \Omega \to \mathbb{R}, \qquad \mathcal{F}_{mnp}(\gamma) = \int_\gamma (m + n\kappa + p\,\tau)ds.$$

The field equations (4) and (5) become

$$m\kappa + (n\tau - p\,\kappa)\tau = 0, \quad n\tau' - p\,\kappa' = 0,$$

which can be easily solved. In particular, if $m \neq 0$, the solutions are circular helices as stated in [18]. However, when $m = 0$ and $n \neq 0$, then the space of extremals is $\mathcal{L}_\omega$ with $\omega = p/n$.

The solutions of the Euler-Lagrange equations of $\mathcal{F}_{mnp}$ are summarized in the following table. For simplicity of interpretation, we show different cases according to the values of the three coupling parameters specifying the energy of the model.

m	n	p	Moduli space of trajectories
$\neq 0$	$=0$	$=0$	(i) Straightlines, $\kappa = 0$
$=0$	$=0$	$\neq 0$	(ii) $\mathcal{C}_{\omega=0}$, κ constant
$=0$	$\neq 0$	$=0$	(iii) $\mathcal{L}_{\omega=0}$, $\tau = 0$
$\neq 0$	$\neq 0$	$=0$	(iv) Circular helices with $\kappa = \frac{-n\tau^2}{m}$
$\neq 0$	$=0$	$\neq 0$	(v) Circular helices with arbitrary κ and $\tau = \frac{m}{p}$
$=0$	$\neq 0$	$\neq 0$	(vi) $\mathcal{L}_\omega$, with $\omega = p/n$
$\neq 0$	$\neq 0$	$\neq 0$	(vii) Circular helices with $\kappa = \frac{-na^2}{m+ap}$, $\tau = \frac{ma}{m+ap}$, $a \in \mathbb{R} - \{-\frac{m}{p}\}$

It should be noted that, regardless of the values of the coupling parameters, the space of extremals is always a space of Lancret helices. However, except in cases (iii) and (vi), the solutions are circular helices, and they reduce to one, just because the former is a special case of the latter. Thus, case (vi) turns into the best choice of the energy action to model helical protein chains as well as other helical configurations in nature. Therefore, given a helical structure viewed as a Lancret helix with slope ω, its energy is a linear combination of both the bending and the twisting, and the ratio between their weights is given by ω.

For a better understanding of the uniqueness of the helical energy, we start from an arbitrary energy action $\mathcal{F}(\gamma) = \int_\gamma F(\kappa(s), \tau(s))\, ds$, and a real number ω. Let $\mathcal{H}_\omega \subset \mathcal{L}_\omega$ be the space of helices which are solutions of $h''_\kappa = 0$, where $h(\kappa) = F(\kappa, \omega\kappa)$. Obviously $\mathcal{H}_\omega$ contains the space of circular helices with slope ω. On the other hand, let $\mathcal{E}_F$ be the space of extremals of $\mathcal{F}(\gamma) = \int_\gamma F(\kappa(s), \tau(s))\, ds$. Then we have

(1) *If there exists $\gamma \in \mathcal{H}_\omega$ with $\gamma \in \mathcal{E}_F$, then the energy density must be $F(\kappa,\tau) = m + n\kappa + p\tau$, with $\omega = \frac{p}{n}$.*

(2) *$\mathcal{L}_\omega \subset \mathcal{E}_F$ if and only if $F(\kappa,\tau) = n\kappa + p\tau$, with $\omega = \frac{p}{n}$. Moreover, in this case $\mathcal{L}_\omega = \mathcal{E}_F$.*

It is worth noting that the above result reduces the choice of the energy action for helical structures to the space of linear combinations of both the total bending and the total twisting. Then $\mathcal{L}_\omega$ is the space of extremals of the following series of energy actions

$$\mathcal{F}_{np}(\gamma) = n \int_\gamma \kappa(s)\, ds + p \int_\gamma \tau(s)\, ds, \qquad p/n = \omega.$$

We can normalize the helices in $\mathcal{L}_\omega$ to have unit speed, and then determine uniquely its energy action by choosing

$$\mathcal{F}_\theta : \Omega \to \mathbb{R}, \qquad \mathcal{F}_\theta(\gamma) = \sin\theta \int_\gamma \kappa(s)\, ds + \cos\theta \int_\gamma \tau(s)\, ds, \qquad \cot\theta = \omega.$$

Thus, that energy is encoded in the map $\mathbf{E} : \mathbb{S}^1 \times \Omega \to \mathbb{R}$ defined by

$$\mathbf{E}(e^{i\theta}, \gamma) = \mathcal{F}_\theta(\gamma) = \sin\theta \int_\gamma \kappa(s)\, ds + \cos\theta \int_\gamma \tau(s)\, ds,$$

where θ is the angle that the helix makes with the axis.

Now, the space of extremals can be geometrically built according to the following algorithm:

(1) Choose any unit speed plane curve $\alpha : [0, L] \to \mathbb{R}^2$ and construct the right cylinder $X(t, v) = \alpha(t) + v\,\vec{\xi}$, where $t \in [0, L]$ and $v \in \mathbb{R}$.

(2) Take in that cylinder the geodesic with slope $\omega = \cot\theta$, that is $\alpha_\theta(s) = \alpha(\sin\theta\, s) + \cos\theta\, s\,\vec{\xi}$, with $s \in [0, \frac{L}{\sin\theta}]$.

(3) Then $\alpha_\theta \in \mathcal{L}_\omega$ is an extremal of $\mathcal{F}_\theta : \Omega \to \mathbb{R}$ and all of that extremals can be constructed in this way.

Even more, an easy computation allows us to obtain the energy of a helical protein as well as any other helical structure. To do it, we first observe that the unit normal of a Lancret helix is defined independently of the acceleration vector field. This is because it coincides with the normal of the corresponding plane cross section. It allows us (see (2)) to define the curvature and the torsion functions of a Lancret helix with the same meaning as the curvature function of a plane curve, where the sign is important. However, that can not be done for general curves in Ω, where the unit normal is defined after differentiating twice to get the acceleration. This implies that the curvature function should be signed, usually taken nonnegative. Therefore, the critical values (or the energy of helices) of $\mathcal{F}_\theta$ are given by

$$\mathcal{F}_\theta(\alpha_\theta) = \sin\theta \int_0^{L/\sin\theta} \kappa(s)\, ds + \cos\theta \int_0^{L/\sin\theta} \tau(s)\, ds = \sin\theta \int_0^{L/\sin\theta} |\rho(s)|\, ds,$$

which yields

$$\mathcal{F}_\theta(\alpha_\theta) = \int_0^L |\rho(t)|\, dt = \int_\alpha |\rho(t)|\, dt.$$

Therefore, *the critical values of the energy, which are reached on helical structures, are provided by the total absolute curvature of the corresponding cross section.*

This result has important consequences, among which we will mention the following:

(1) The energy of a helical structure does not depend on its slope. All helices lying in the same right cylinder provide the same critical value of the energy.

(2) The energy of a helical structure only depends on the corresponding cross section, and is just computed as the total absolute curvature of that plane curve. To evaluate it, we consider the convex envelope $\tilde{\alpha}$ of the cross section. This curve is geometrically obtained from α by symmetrization, namely reflecting concave parts by using straight lines at the inflection points of α. In other words, $\tilde{\alpha}$ is the arclength parameterized curve with curvature function $\rho_{\tilde{\alpha}} = |\rho_\alpha|$. Now, $\mathcal{F}_\theta(\alpha_\theta)$ is nothing but the total curvature of the convex envelope of the cross section.

(3) As a consequence, the energy of a helical structure is given by

$$\mathcal{F}_\theta(\alpha_\theta) = 2\pi\, i(\tilde{\alpha}) + \phi_o,$$

where $i(\tilde{\alpha})$ is the rotation number of $\tilde{\alpha}$ and ϕ_o is a constant which measures the angle between the tangent vectors $\alpha'_\theta(0)$ and $\alpha'_\theta(L)$ of α_θ at the ending points.

Thus, we get the following Dirac quantization principle for extremals: *The energy of a helical configuration is not arbitrary, but it is given, up to a constant, as natural multiples of a basic energy value. In particular it only depends on the homotopy class of the corresponding cross section.*

4 Some examples

As an illustration, in this section we give some examples of usual helical structures, which are often found in nature and are described as solutions of the above variational model. Besides circular helices, there are many different shapes of helical configurations in nature that might be of considerable interest. Let us focus on the following examples:

(1) *Conical helices.* Nature is plenty of coiled forms on cones. The so called concho-spirals, which describes the shape of gastropod shells; sheep, goat and antelope horns; bacterial filamentous viruses and the cochlea of the ear (see [9]). These helical shapes appear as solutions of our model because they are Lancret helices built as geodesics of right cylinders whose transversal section is either a logarithmic or an Archimedean spiral. For example, the former one can be explicitly parameterized by

$$\alpha(u) = (r\,u\cos(c\ln u), r\,u\sin(c\ln u))\,,\quad u>0,$$

u standing for the arc length parameter. Now, on the cylinder $\phi(u,v)=\alpha(u)+v\,\vec{\partial}_z$, consider the geodesic with slope $h=\cot\theta=p/n$ to find the curve

$$\gamma_h(t) = (r\,nt\cos(c\ln(nt)), r\,nt\sin(c\ln(nt)), p\,t)\,,$$

which is a conical helix lying on a cone of equation $x^2+y^2=\frac{r^2n^2}{p^2}\,z^2$. The same can be done starting from an Archimedean spiral.

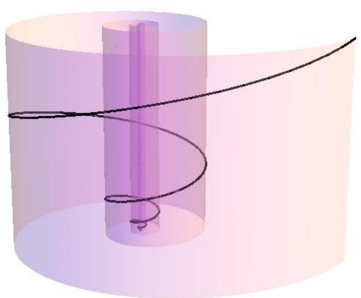

Fig. 1: *Lancret over a logarithmic spiral*

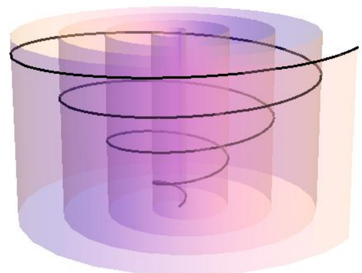

Fig. 2: *Lancret over an Archimedean spiral*

(2) *Elliptical helices.* These helices are mainly used in technology, from building antennas, [20], to nanotechnology, [11]. They appear in our model as solutions associated with Lancret helices constructed as geodesics of cylinders with cross section being an ellipse.

(3) *Spherical helices.* They are Lancret helices lying on two spheres. A direct computation yields that the cross section must be an epicycloid, a planar curve traced

out by a point on a circle rolling outside another circle. These solutions model helices which have been widely used in a range of applications running from technology to the gyroscopic force theory (see [3], and references therein, for details).

(4) *Helices over a Poleni's syntractrix.* In 1729 Giovanni Poleni studied a family of curves related to the tractrix which are known as syntractrices. A syntractrix is the locus of a point on the tangent to a tractrix at a constant distance, L, from its intersection with the axis. When L is twice the constant length of the segment that generates the tractrix, one obtains the Poleni's syntractrix (also called *la courbe des forçats* or galley slaves). The Poleni's curve can be viewed in many ways, perhaps the most usual is that related to the elastica of James Bernoulli. This variational problem was proposed by Daniel Bernoulli in 1742 as follows: find those plane curves which are extremals of the following elastic action

$$\mathcal{F}_\lambda(\alpha) = \int_\alpha (\kappa^2(s) + \lambda)\, ds, \qquad \lambda \in \mathbb{R}.$$

This problem was solved by L. Euler in 1744 (see [10] and references therein). In particular the only non-periodic solution is given by the function

$$\kappa(s) = 2\sqrt{2}r\ \mathrm{sech}(\sqrt{2}\, rs),$$

which is the curvature of the Poleni's syntractrix. Now, the natural equations can be solved to obtain the following unit speed parametrization of the Poleni's curve, $\alpha : \mathbb{R} \to \mathbb{R}^2$,

$$\alpha(s) = \left(s - \frac{\sqrt{2}}{r} \tanh(\sqrt{2}\, rs), \frac{\sqrt{2}}{r} \mathrm{sech}(\sqrt{2}\, rs) \right).$$

To compute the critical values of the energy corresponding to pieces of helices built on this curve, we only need to measure the angle between the tangent at the ending points. Therefore, in $[s_1, s_2]$, the energy is just the angle $\phi(s_1, s_2)$ between $\alpha'(s_1)$ and $\alpha'(s_2)$ (see Fig. 3). It is not difficult to see that

$$0 < \phi(s_1, s_2) < 2\pi, \qquad \lim_{(s_1,s_2)\to(-\infty,\infty)} \phi(s_1, s_2) = 2\pi.$$

(5) *Helices over a Cornu spiral.* The Cornu spiral (also known as Euler spiral, clothoid, or simply spiros) has many applications in engineering. Clothoids are widely used in transition curve design in railroad and highway engineering for connecting and transiting the geometry between a tangent and a circular curve. Design standards for modern highways and railways require a smooth transition between straight line segments and circles. In fact, the curvature of a Cornu spiral changes linearly with its arclength. Therefore, for simplicity we may assume that $\kappa(s) = s$. Now, we can use the Fresnel integrals to solve the natural equations of a Cornu spiral and get the unit speed parametrization

$$\alpha(s) = \left(\int_0^s \cos\frac{u^2}{2}\, du, \int_0^s \sin\frac{u^2}{2}\, du \right), \quad s \in \mathbb{R}.$$

It should be noted that it presents an inflection point at the origin, where the tangent is horizontal. Moreover, the curvature is positive when $s > 0$, while it is negative when $s < 0$. To compute the critical values of the energy functional for pieces of helices shaped on a Cornu spiral, we only need to compute the number of loops as well as the angles at the ending points. For example, the energy of $\alpha([0, L])$ is given by $2\pi r + \phi_o$, where r is the number of times that the tangent becomes horizontal in $[0, L]$ and ϕ_o is the angle that $\alpha'(L)$ makes with the horizontal axis (see Fig. 4).

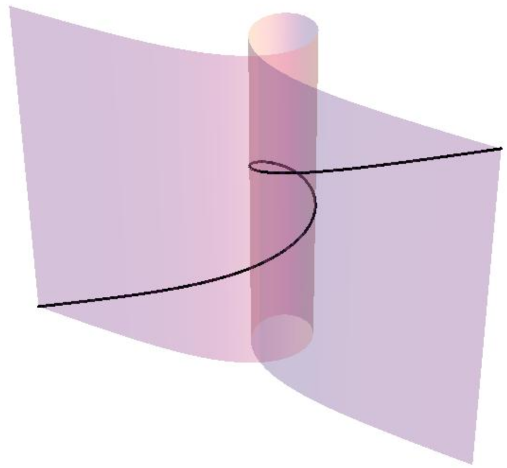

Fig. 3: *Lancret over a Poleni's curve*

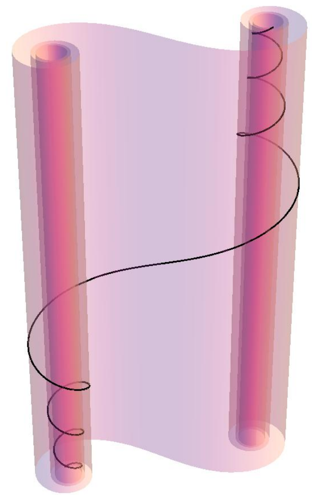

Fig. 4: *Lancret over a Cornu spiral*

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