An extension of Bianchi-Cartan-Vranceanu spaces

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1 Introduction (see [7])

Dido's problem is a variant of the isoperimetric problem. It was formulated in the *Aeneid*, Virgil's epic poem glorifying the beginnings of Rome.

Queen Dido had to flee across the Mediterranean in a ship with friends and servants. She had what we would nowadays call a dysfunctional family. Her brother, Pygmalion, had just murdered her husband and taken most of her possessions. Dido landed, nearly penniless, on a part of the African coast ruled by King Jarbas. After dickering and begging, Dido persuaded Jarbas to give her as much land as she could enclose with an ox hide. Dido told her servants to cut an ox hide into a single long, narrow strip. They turned the ox hide into a single leather string.

Dido had in this way reformulated her difficult situation into the following geometric problem. Given a string of fixed length ℓ and a fixed line L (the Mediterranean coastline), place the ends of the string on L and determine the shape of the curve c for which the figure enclosed by c together with L has the maximum possible area. This is Dido's problem. It is also sometimes referred to as the *problem of Pappus*. Dido found the solution - a half-circle - and thus founded the semicircular city of Carthage.

Take the one-form $\alpha = \frac{1}{2}(xdy - ydx)$ which satisfies $d\alpha = dx \wedge dy$ and $\alpha|_L = 0$ for any ray L through the origin.

According to Stokes' theorem, the area Φ enclosed by a closed planar curve c is

$$\Phi(c) = \int_c \alpha. \tag{1}$$

As $\alpha|_L = 0$, if c is a non-closed curve beginning at the origin, $\Phi(c)$ represents the area enclosed by the closed curve obtained by traversing c and then returning to the origin along the ray that connects the endpoint of c to the origin.

The length ℓ of $c = (x(t), y(t))$ is

$$\ell(c) = \int_c ds, \tag{2}$$

where ds $= \sqrt{dx^2 + dy^2} = ||c'||dt$ is the usual element of arc length. In this manner Dido's problem, and the (dual) isoperimetric problem, becomes the following constrained variational problem:

Problem 1. Minimize the length $\ell(c)$ of a closed rectifiable curve c, subject to the constraint that the signed area $\Phi(c)$ of the curve be a fixed constant.

The introduction of α lets us extend the problem to non-closed curves. The ray used to close up c corresponds to the coastline L in Dido's problem.

Now Montgomery constructs the three-dimensional geometry whose geodesics correspond to the solutions to the isoperimetric problem. Add a third direction z whose motion

is linked to that of x and y according to

$$dz = \frac{1}{2}(xdy - ydx) \tag{3}$$

In this way we associate a family of curves $\gamma(t) = (x(t), y(t), z(t))$ to a single planar curve $c(t) = (x(t), y(t))$, the family being parameterized by the initial value z_0 of the height z. We will call any one of these paths a *horizontal lift* of c, and more generally, any path γ in $\mathbb{R}^3$ that satisfies the differential constraint (3) a *horizontal path.* Set

$$\mathrm{ds}^2 = dx^2 + dy^2$$

and define the length of any horizontal path in $\mathbb{R}^3$ to be $\int_\gamma ds$. In other words, we have defined the length of γ to be equal to the usual length of its planar projection c.

Problem 2. Minimize the length $\int_\gamma ds$ over all horizontal paths γ that join two fixed points in three-space.

To see that this is a reformulation of the dual to Dido's problem, or the isoperimetric problem, observe that

$$z(1) - z(0) = \int_c \frac{1}{2}(xdy - ydx),$$

where $c(t) = (x(t), y(t))$ is the projection of the curve $\gamma(t) = (x(t), y(t), z(t))$ to the plane. Observe that, according to Stokes' theorem, if c joins the origin to (x_1, y_1) and if we take $z(0) = 0$, then the endpoints of γ are $(0, 0, 0)$ and $(x_1, y_1, \Phi(c))$, where $\Phi(c)$ denotes the

signed area defined by the closed curve given by traversing c and then returning to the origin along a line segment.

Defining the differential 1-form $\omega = dz - \frac{1}{2}(xdy - ydx)$ we can write

$$\begin{aligned}\mathcal{H} &= \operatorname{Ker}\omega = \{\omega(x,y,z) = 0\} \\ &= \{(v_1, v_2, v_3) : v_3 - \frac{1}{2}(xv_2 - yv_1) = 0\} \subset \mathbb{R}^3.\end{aligned}$$

This $\mathcal{H}$ is a field of two-planes in three-space, or what it is called a *distribution*: a linear subbundle of the tangent bundle. The restriction of ds^2 to these two-planes defines a smoothly varying family of inner products $\langle \cdot\,, \cdot \rangle$ on the planes $\mathcal{H}$. Thus if $v, w \in \mathcal{H}_{(x,y,z)}$, then $\langle v, w \rangle = v_1 w_1 + v_2 w_2$.

Definition 1 $\mathbb{R}^3$ *endowed with the structure of this distribution* $\mathcal{H}$ *and this family of inner products* ds^2 *on* $\mathcal{H}$ *is called the Heisenberg group (complex contact group), which is the first nontrivial example of subriemannian geometry* [9].

The Bianchi-Cartan-Vranceanu (BCV) spaces (see [4])

For real numbers l and m, with $l \geq 0$, consider the set

$$BCV(l,m) = \{(x,y,z) \in \mathbb{R}^3 \,:\, 1 + m(x^2+y^2) > 0\}$$

equipped with the metric

$$\mathrm{ds}^2_{l,m} = \frac{dx^2+dy^2}{(1+m(x^2+y^2))^2} + \left(dr + \frac{l}{2}\,\frac{xdy - ydx}{1+m(x^2+y^2}\right)^2.$$

Observe that this metric is obtained as a conformal deformation of the planar Euclidean metric by adding the imaginary part of $zd\bar{z}$, for a complex number z.

Take the vector fields E_i and its corresponding dual 1-forms ω^j

$$E_1 = (1+m(x^2+y^2))\,\partial_x - \tfrac{l}{2}y\partial_z \qquad \omega^1 = \frac{dx}{1+m(x^2+y^2)}$$

$$E_2 = (1+m(x^2+y^2))\,\partial_y + \tfrac{l}{2}x\partial_z \qquad \omega^2 = \frac{dy}{1+m(x^2+y^2)}$$

$$E_3 = \partial_z \qquad \omega^3 = dz + \frac{l}{2}\,\frac{ydx - xdy}{(1+m(x^2+y^2)}$$

Let $\mathcal{D}$ be the distribution generated by $\{E_1, E_2\}$. The manifold $\left(BCV(l,m), \mathcal{D}, \mathrm{ds}^2_{l,m}\right)$ is called a Bianchi-Cartan-Vranceanu (BCV for short) space ([1, 2, 3, 11]), which is an

example of sub-riemannian geometry (see [4, 9]) and the horizontal distribution is a 2-step breaking-generating distribution everywhere.

2 Extended Bianchi-Cartan-Vranceanu spaces

Observe that letting $z = x + iy$, we see that $\mathrm{Im}(z\, d\bar{z}) = ydx - xdy$, which reminds us the map $\mathbb{C} \times \mathbb{C} \to \mathbb{R} \times \mathbb{C}$ given by $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2(z_1\bar{z}_2))$, that easily leads to the classical Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$, where coordinates in $\mathbb{S}^2$ are given by $(|z_1|^2 - |z_2|^2, 2\mathrm{Re}(z_1 z_2), 2\mathrm{Im}(z_1 z_2))$.

In the same line we get the fibration $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$, but using quaternions $\mathbb{H}$ instead of complex numbers. Quaternions are usually presented with the imaginary units i, j, k in the form $q = x_0 + x_1 i + x_2 j + x_3 k$, $x_0, x_1, x_2, x_3 \in \mathbb{R}$ with $i^2 = j^2 = k^2 = ijk = -1$. They can also be defined equivalently, using the complex numbers $c_1 = x_0 + x_1 i$ and $c_2 = x_2 + x_3 i$, in the form $q = c_1 + c_2 j$. Then for a point $(q_1 = \alpha + \beta j, q_2 = \gamma + \delta j) \in \mathbb{S}^7$, we get the following coordinate expressions $(|q_1|^2 - |q_2|^2, 2\mathrm{Re}(\bar{\alpha}\gamma + \bar{\beta}\delta), 2\mathrm{Im}(\bar{\alpha}\gamma + \bar{\beta}\delta), 2\mathrm{Re}(\alpha\delta - \beta\gamma), 2\mathrm{Im}(\alpha\delta - \beta\gamma))$.

For any $q = w + xi + yj + zk \in \mathbb{H}$ we find that $qd\bar{q} = wdw + xdx + ydy + zdz + (xdw - wdx + zdy - ydz)i + (ydw - wdy + xdz - zdx)j + (zdw - wdz + ydx - xdy)k$. As the quaternionic contact group $\mathbb{H} \times \mathrm{Im}\mathbb{H}$, with coordinates (w, x, y, z, r, s, t) can be equipped

with the metric

$$\mathrm{ds}^2 = (dw^2 + dx^2 + dy^2 + dz^2) + \left(dr + \frac{1}{2}(xdw - wdx + zdy - ydz)\right)^2$$
$$+ \left(ds + \frac{1}{2}(ydw - wdy + xdz - zdx)\right)^2 + \left(dt + \frac{1}{2}(zdw - wdz + ydx - xdy)\right)^2.$$

Then, by extending this metric, it seems natural to find a 7-dimensional generalization of the 3-dimensional BCV spaces endowed with the two-parameter family of metrics

$$\mathrm{ds}^2_{l,m} = \frac{dw^2 + dx^2 + dy^2 + dz^2}{K^2} + \left(dr + \frac{l}{2}\frac{wdx - xdw + ydz - zdy}{K}\right)^2$$
$$+ \left(ds + \frac{l}{2}\frac{wdy - ydw + zdx - xdz}{K}\right)^2 + \left(dt + \frac{l}{2}\frac{wdz - zdw + xdy - ydx}{K}\right)^2,$$

where l, m are real numbers and $K = 1 + m(w^2 + x^2 + y^2 + z^2)$.

Then $(EBCV, \mathrm{ds}^2_{l,m})$ will be called extended BCV spaces ($EBCV$ for short).

That metric is obtained as a conformal deformation of the Euclidean metric of $\mathbb{R}^4$ by adding three suitable terms which depend on l and m concerning the imaginary part of $q\bar{q}$, for a quaternion q. When $m = 0$ we get a one-parameter of Riemannian metrics depending on l. Furthermore, if $l = 1$, we find the 7-dimensional quaternionic Heisenberg group (see

[6] and [12]). The manifold $EBCV$ provides another example of sub-riemannian geometry and the horizontal distribution is a 2-step breaking-generating distribution everywhere.

That metric can also be written as

$$\mathrm{ds}^2_{l,m} = \sum_{i=1}^{7} \omega^i \otimes \omega^i$$

where

$$\begin{aligned}
\omega^1 &= dr + \tfrac{l}{2K}(wdx - xdw + ydz - zdy), & \omega^4 &= dw/K,\\
\omega^2 &= ds + \tfrac{l}{2K}(wdy - ydw + zdx - xdz), & \omega^5 &= dx/K,\\
\omega^3 &= dt + \tfrac{l}{2K}(wdz - zdw + xdy - ydx), & \omega^6 &= dy/K,\\
& & \omega^7 &= dz/K,
\end{aligned}$$

$$X_1 = \partial_r, \qquad X_2 = \partial_s, \qquad X_3 = \partial_t,$$

$$X_4 = K\partial_w + \frac{lx}{2}\partial_r + \frac{ly}{2}\partial_s + \frac{lz}{2}\partial_t, \qquad X_5 = K\partial_x - \frac{lw}{2}\partial_r - \frac{lz}{2}\partial_s + \frac{ly}{2}\partial_t,$$

$$X_6 = K\partial_y + \frac{lz}{2}\partial_r - \frac{lw}{2}\partial_s - \frac{lx}{2}\partial_t, \qquad X_7 = K\partial_z - \frac{ly}{2}\partial_r + \frac{lx}{2}\partial_s - \frac{lw}{2}\partial_t.$$

Then we find that

Lemma 2 $\{X_1, X_2, \cdots, X_7\}$ *is an orthonormal basis of vector fields whit respect to the given metric* $\mathrm{ds}^2_{l,m} := \langle\, , \rangle$.

Writing $1 \leq a, b \leq 3$; $4 \leq u, v \leq 7$, we find that

$$[X_a, X_b] = 0; \qquad [X_a, X_u] = 0$$

as well as

$[\ ,\ ]$	X_4	X_5	X_6	X_7
X_4	0	$-lm(x^2+y^2+\frac{1}{m})X_1+lm(wz+xy)X_2-lm(wy-xz)X_3-2mxX_4+2mwX_5$	$-lm(wz-xy)X_1-lm(x^2+z^2+\frac{1}{m})X_2+lm(wx+yz)X_3-2myX_4+2mwX_6$	$lm(wy+xz)X_1-lm(wx-yz)X_2-lm(x^2+y^2+\frac{1}{m})X_3-2mzX_4+2mwX_7$
X_5		0	$-lm(wy+xz)X_1+lm(wx-yz)X_2-lm(\frac{1}{m}+w^2+z^2)X_3-2myX_5+2mxX_6$	$lm(xy-wz)X_1+lm(w^2+y^2+\frac{1}{m})X_2+lm(wx+yz)X_3-2mzX_5+2mxX_7$
X_6			0	$-lm(w^2+x^2+\frac{1}{m})X_1-lm(wz+xy)X_2+lm(wy-xz)X_3-2mzX_6+2myX_7$
X_7				0

For later use, when $m = 0$ brackets reduce to

$[\ ,\ ]$	X_4	X_5	X_6	X_7
X_4	0	$-lX_1$	$-lX_2$	$-lX_3$
X_5		0	$-lX_3$	lX_2
X_6			0	$-lX_1$
X_7				0

Remark 3 *When $l = 1$, we have the brackets of the quaternionic contact manifold.*

As for the Levi-Civita connection in a Riemannian manifold ([5], p. 160) we find

$$\nabla_{X_a} X_b = 0; \qquad \nabla_{X_a} X_u = \nabla_{X_u} X_a$$

and

$\nabla_{X_i} X_j$	X_4	X_5	X_6	X_7
X_1	$\frac{lm}{2}(y^2 + z^2 + \frac{1}{m})X_5 + \frac{lm}{2}(wz - xy)X_6 - \frac{lm}{2}(wy + xz)X_7$	$-\frac{lm}{2}(y^2 + z^2 + \frac{1}{m})X_4 + \frac{lm}{2}(wy + xz)X_6 + \frac{lm}{2}(wz - xy)X_7$	$-\frac{lm}{2}(wz - xy)X_4 - \frac{lm}{2}(wy + xz)X_5 + \frac{lm}{2}(w^2 + x^2 + \frac{1}{m})X_7$	$\frac{lm}{2}(wy + xz)X_4 - \frac{lm}{2}(wz - xy)X_5 - \frac{lm}{2}(w^2 + x^2 + \frac{1}{m})X_6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_7	$-\frac{lm}{2}(wy + xz)X_1 + \frac{lm}{2}(wx - yz)X_2 + \frac{lm}{2}(x^2 + y^2 + \frac{1}{m})X_3 - 2mwX_7$	$\frac{lm}{2}(wz - xy)X_1 - \frac{lm}{2}(w^2 + y^2 + \frac{1}{m})X_2 - \frac{lm}{2}(wx + yz)X_3 - 2mxX_7$	$\frac{lm}{2}(w^2 + x^2 + \frac{1}{m})X_1 + \frac{lm}{2}(wz + xy)X_2 - \frac{lm}{2}(wy - xz)X_3 - 2myX_7$	$2m(wX_4 + xX_5 + yX_6)$

When $m = 0$, the Levi-Civita connection reduces to

$\nabla_{X_i} X_j$	X_4	X_5	X_6	X_7
X_1	$\frac{l}{2}X_5$	$-\frac{l}{2}X_4$	$\frac{l}{2}X_7$	$-\frac{l}{2}X_6$
X_2	$\frac{l}{2}X_6$	$-\frac{l}{2}X_7$	$-\frac{l}{2}X_4$	$\frac{l}{2}X_5$
X_3	$\frac{l}{2}X_7$	$\frac{l}{2}X_6$	$-\frac{l}{2}X_5$	$-\frac{l}{2}X_4$
X_4	0	$-\frac{l}{2}X_1$	$-\frac{l}{2}X_2$	$-\frac{l}{2}X_3$
X_5	$\frac{l}{2}X_1$	0	$-\frac{l}{2}X_3$	$\frac{l}{2}X_2$
X_6	$\frac{l}{2}X_2$	$\frac{l}{2}X_3$	0	$-\frac{l}{2}X_1$
X_7	$\frac{l}{2}X_3$	$-\frac{l}{2}X_2$	$\frac{l}{2}X_1$	0

Remark 4 *When $l = 1$, we find the Levi-Civita connection of the quaternionic contact manifold.*

If R denotes the curvature tensor we can prove that

$\frac{4}{l^2}R$	(X_1, X_4)	$\vdots$	(X_6, X_7)
(X_1, X_4)	$m^2[(y^2 + z^2 + 1/m)^2 + (wz - xy)^2 + (wy + xz)^2]$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
(X_6, X_7)	$\vdots$	$\vdots$	$\frac{16m}{l^2} - 3m^2[(w^2 + x^2 + 1/m)^2 + (wz+xy)^2 + (wy-xz)^2]$

Remark 5 *When $m = 0$, the curvature of the quaternionic contact manifold reduces to*

R	(X_1, X_4)	$\vdots$	(X_6, X_7)
(X_1, X_4)	$\frac{l^2}{4}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
(X_6, X_7)	$\vdots$	$\vdots$	$-\frac{3l^2}{4}$

3 The Ricci tensor

Proposition 6 *The matrix representing the Ricci tensor is given by*

$$
\left[\begin{array}{ccc|ccccc}
l^2/2(K^2+1) & 0 & 0 & -lmx(K+2) & lmw(K+2) & -lmz(K+2) & lmy(K+2) \\
0 & l^2/2(K^2+1) & 0 & -lmy(K+2) & lmz(K+2) & lmw(K+2) & -lmx(K+2) \\
0 & 0 & l^2/2(K^2+1) & -lmz(K+2) & -lmy(K+2) & lmx(K+2) & lmw(K+2) \\
-lmx(K+2) & -lmy(K+2) & -lmz(K+2) & A(K-1-mw^2)+B & l^2m(K+1)wx & l^2m(K+1)wy & l^2m(K+1)wz \\
lmw(K+2) & lmz(K+2) & -lmy(K+2) & l^2m(K+1)wx & A(K-1-mx^2)+B & l^2m(K+1)xy & l^2m(K+1)xz \\
-lmz(K+2) & lmw(K+2) & lmx(K+2) & l^2m(K+1)wy & l^2m(K+1)xy & A(K-1-my^2)+B & l^2m(K+1)yz \\
lmy(K+2) & -lmx(K+2) & lmw(K+2) & l^2m(K+1)wz & l^2m(K+1)xz & l^2m(K+1)yz & A(K-1-mz^2)+B
\end{array}\right].
$$

where $A = -l^2(K+1)$ *and* $B = 12m - 3/2l^2$.

Some particular cases could be interesting, for instance we get the following Ricci matrix when $K = 1$ (or $m = 0$)

$$\mathrm{Ric}_1 = \begin{pmatrix} l^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & l^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3/2l^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3/2l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3/2l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3/2l^2 \end{pmatrix}$$

Remark 7 *When* $l = 1$, *we find the Ricci curvature of the quaternionic contact manifold.*

An easy computation leads to

Corollary 8 *The EBCV manifold has constant scalar curvature* $S = 48m$.

4 Killing vector fields in $EBCV$

Remember that a Killing vector field is a vector field on a Riemannian manifold that preserves the metric. Killing vector fields are the infinitesimal generators of isometries, that is, flows generated by Killing fields are continuous isometries of the manifold. Specifically, a vector field X is a Killing vector field if the Lie derivative with respect to X of the metric g vanishes: $\mathcal{L}_X g = 0$ or equivalently

$$\mathcal{L}_X \mathrm{ds}^2_{l,m} = (\mathcal{L}_X \omega^i) \otimes \omega^i = 0, \tag{4}$$

where

$$\mathcal{L}_X \omega^i = \iota_X d\omega^i + d(\iota_X \omega^i).$$

In terms of the Levi-Civita connection, Killing's condition is equivalent to

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0. \tag{5}$$

It is easy to prove that

Proposition 9 *$\mathcal{L}_X g(Y, Z) = 0$ if and only if $\mathcal{L}_X g(X_i, X_j) = 0$ for basic vector fields X_i, X_j.*

We know that the dimension of the Lie algebra of the Killing vector fields is

$$m \leq n(n+1)/2,$$

and the maximum is reached on constant curvature manifolds ([5], p. 238, Vol. II) , then for our manifold $m < 28$. Then obviously

Proposition 10 *The basic vertical vector fields X_1, X_2, X_3 are Killing fields.*

From (5) it is easy to prove that the horizontal basic vector fields $X_4, \cdots, X_7$ are not Killing vector fields.

In her thesis, Profir [8] proved that the Lie algebra of Killing vector fields is 4-dimensional. Our problem now is to determine the space of Killing vector fields in $EBCV$. By using (4) and the values of ω^i and $d\omega^i$ we obtain that the Killing vector fields are characterized by the following system of partial differential equations (28 equations).

The Killing equations

In the usual coordinate system (r, s, t, w, x, y, z) on $EBCV$, a vector field $X = \sum_{i=1}^{7} f_i X_i$ will be a Killing field if and only if the real functions f_i satisfy the following system of 28-partial differential equations:

$$
\begin{array}{c|l}
01 & \partial_r(f_1) = 0 \\
\vdots & \vdots \\
04 & \partial_r(f_2) + \partial_s(f_1) = 0 \\
\vdots & \vdots \\
07 & \partial_r(f_4) + K\partial_w(f_1) + \frac{ly}{2}\partial_s(f_1) + \frac{lz}{2}\partial_t(f_1) - lm\{\frac{1}{m} + (y^2 + z^2)\}f_5 - lm(wz - xy)f_6 + lm(wy + xz)f_7 = 0 \\
\vdots & \vdots \\
19 & K\partial_w(f_4) + \frac{lx}{2}\partial_r(f_4) + \frac{ly}{2}\partial_s(f_4) + \frac{lz}{2}\partial_t(f_4) - 2mxf_5 - 2myf_6 - 2mzf_7 = 0 \\
\vdots & \vdots \\
28 & \partial_z(f_7) - \frac{ly}{2}\partial_r(f_7) + \frac{lx}{2}\partial_s(f_7) - \frac{lw}{2}\partial_t(f_7) = 0
\end{array}
$$

It seems that the solution of the system is very difficult, so that we focus on solving the system for $m = 0$, that is:

01	$\partial_r(f_1) = 0$
$\vdots$	$\vdots$
04	$\partial_r(f_2) + \partial_s(f_1) = 0$
$\vdots$	$\vdots$
07	$\partial_r(f_4) + \partial_w(f_1) + \frac{ly}{2}\partial_s(f_1) + \frac{lz}{2}\partial_t(f_1) - lf_5 = 0$
$\vdots$	$\vdots$
19	$\partial_w(f_4) + \frac{lx}{2}\partial_r(f_4) + \frac{ly}{2}\partial_s(f_4) + \frac{lz}{2}\partial_t(f_4) = 0$
$\vdots$	$\vdots$
28	$\partial_z(f_7) - \frac{ly}{2}\partial_r(f_7) + \frac{lx}{2}\partial_s(f_7) - \frac{lw}{2}\partial_t(f_7) = 0$

Proceeding as in [8], Profir considered a harder condition $L_X\omega^i = 0$, then we find the following result

Proposition 11 *When $m = 0$, the following vector fields*

$$
\begin{aligned}
K_1 &= X_1 \\
K_2 &= X_2 \\
K_3 &= X_3 \\
K_4 &= -lxX_1 - lyX_2 - lzX_3 + X_4 \\
K_5 &= lwX_1 + lzX_2 - lyX_3 + X_5 \\
K_6 &= -lzX_1 + lwX_2 + lxX_3 + X_6 \\
K_7 &= lyX_1 - lxX_2 + lwX_3 + X_7
\end{aligned}
$$

are Killing ones.

Remark 12 *(1) If $l = 1$, we obtain Killing fields for the quaternionic Heisenberg group.*

(2) We have just known that $K_i = X_i, i = 1, 2, 3$ were Killing vector fields, however the Lie brackets of K_i do not produce new Killing fields.

5 BCV as a submanifold of $EBCV$

We define a basis of vector fields in BCV, seen as a submanifold of $EBCV$, adapted to the coordinates (r, s, t, w, x, y, z) as follows:

$$M_1 = \partial_r, \qquad M_2 = L\partial_w + \frac{lx}{2}\partial_r, \qquad M_3 = L\partial_x - \frac{lw}{2}\partial_r,$$

where $L = 1 + m(w^2 + x^2)$

We complete this basis to obtain a new one $\mathcal{B} = \{M_1, M_2, M_3, M_4, M_5, M_6, M_7\}$ for $EBCV$, given by:

$$\begin{aligned}
M_1 &= \partial_r, \\
M_2 &= L\partial_w + \tfrac{lx}{2}\partial_r, \\
M_3 &= L\partial_x - \tfrac{lw}{2}\partial_r, \\
M_4 &= \partial_s, \\
M_5 &= \partial_t, \\
M_6 &= K\partial_y + \tfrac{lz}{2}\partial_r - \tfrac{lw}{2}\partial_s - \tfrac{lx}{2}\partial_t, \\
M_7 &= K\partial_z - \tfrac{ly}{2}\partial_r + \tfrac{lx}{2}\partial_s - \tfrac{lw}{2}\partial_t.
\end{aligned}$$

This basis is well defined and $\{M_a, a = 1, 2, 3\}$ span the tangent space of the submanifold and $\{M_u, u = 4, 5, 6, 7\}$ span an orthonormal basis of the normal space.

We define the metric

$$\mathrm{ds}^2_{l,m|_{BCV}} = \frac{dw^2 + dx^2}{L^2} + \left(dr + \frac{l}{2L}(wdx - xdw)\right)^2.$$

We can also consider $\mathcal{B}$ as an orthonormal basis of $EBCV$. Then we are going to get the induced Levi-Civita connection to study the geometry of BCV as a submanifold of $EBCV$.

The only non null brackets are

$[M_2, M_3] = -lM_1 - 2mxM_2 + 2mwM_3,$

$[M_2, M_6] = -\frac{L}{2K}\{2lmwzM_1 + (Kl - 2lmw^2)M_4 - 2lmwxM_5 - 4mwM_6\},$

$[M_2, M_7] = \frac{L}{2K}\{2lmwyM_1 - 2lmwxM_4 - (Kl - 2lmw^2)M_5 + 4mwM_7\},$

$[M_3, M_6] = -\frac{L}{2K}\{2lmxzM_1 - 2lmwxM_4 + (Kl - 2lmx^2)M_5 - 4mxM_6\},$

$[M_3, M_7] = \frac{L}{2K}\{2lmxyM_1 + (Kl - 2lmx^2)M_4 + 2lmwxM_5 + 4mxM_7\},$

$[M_6, M_7] = -LlM_1 - lm(xy + wz)M_4 + lm(wy - xz)M_5 - 2mzM_6 + 2myM_7.$

Gauss and Weingarten formulas

Let us write the Gauss and Weingarten formulas (see [5])

$$\nabla'_X Y = \nabla_X Y + \alpha(X,Y),$$
$$\nabla'_X \xi = -A_\xi X + D_X \xi,$$

where $X, Y \in \mathfrak{X}(BCV)$, $\xi \in \mathfrak{X}^\perp(BCV)$, ∇', ∇ are the Levi-Civita connections on $EBCV$ and BCV, respectively, and D is the normal connection. Then, for instance, we find that

$$\nabla'_{M_1} M_1 = 0$$
$$\nabla'_{M_1} M_3 = -\tfrac{l}{2} M_2 + \tfrac{Llmx}{2K}(zM_6 - yM_7)$$
$$\nabla'_{M_3} M_1 = -\tfrac{l}{2} M_2 + \tfrac{Llmx}{2K}(zM_6 - yM_7)$$
$$\nabla'_{M_2} M_2 = 2mxM_3$$
$$\nabla'_{M_3} M_3 = 2mwM_2$$
$$\nabla'_{M_1} M_2 = \tfrac{l}{2} M_3 + \tfrac{Llmw}{2K}(zM_6 - yM_7)$$
$$\nabla'_{M_2} M_1 = \tfrac{l}{2} M_3 + \tfrac{Llmw}{2K}(zM_6 - yM_7)$$
$$\nabla'_{M_2} M_3 = -\tfrac{l}{2} M_1 - 2mxM_2$$
$$\nabla'_{M_3} M_2 = \tfrac{l}{2} M_1 - 2mwM_3$$

so that

$$
\begin{aligned}
\nabla_{M_1} M_1 &= 0, & \alpha(M_1, M_1) &= 0,\\
\nabla_{M_1} M_2 &= \tfrac{l}{2} M_3, & \alpha(M_1, M_2) &= \tfrac{Llmw}{2K}(zM_6 - yM_7),\\
\nabla_{M_1} M_3 &= -\tfrac{l}{2} M_2, & \alpha(M_1, M_3) &= \tfrac{Llmx}{2K}(zM_6 - yM_7),\\
\nabla_{M_2} M_3 &= -\tfrac{l}{2} M_1 - mxM_2, & \alpha(M_2, M_3) &= 0\\
&\vdots & &\vdots
\end{aligned}
$$

Therefore we get

Corollary 13 *Only when $m = 0$, BCV is a totally geodesic submanifold of $EBCV$.*

Using the theory of submanifolds of a Riemannian manifold, we can now study a lot of problems such that the equations of Gauss, Codazzi and Ricci and their consequences.

6 Talking about geodesics (see [7])

In [7] it is shown that

Theorem 14 *The geodesics for the Heisenberg group are exactly the horizontal lifts of arcs of circles, including line segments as a degenerate case.*

A Riemannian metric on a manifold M is defined by a covariant two-tensor, which is to say, a section of the bundle $S^2(T^*M)$. There is no such object in subriemannian geometry. Instead, a subriemannian metric can be encoded as a contravariant symmetric two-tensor, which is a section of $S^2(TM)$. This two-tensor has rank $k < n$, where k is the rank of the distribution, so it cannot be inverted to obtain a Riemannian metric. We call this contravariant tensor the *cometric*.

Definition 15 *A* cometric *is a section of the bundle $S^2(TM) \subset TM \otimes TM$ of symmetric bilinear forms on the cotangent bundle of M.*

Since TM and T^*M are dual, any cometric defines a fiber-bilinear form $((\cdot\,,\cdot)) : T^*M \otimes T^*M \to \mathbb{R}$, i.e. a kind of inner product on covectors. This form in turn defines a symmetric bundle map $\beta : T^*M \to TM$ by $p(\beta_q(\mu)) = ((p,\mu))_q$, for $p, \mu \in T_q^*M$ and $q \in M$. Thus $\beta_q(\mu) \in T_qM$. The adjective symmetric means that β equals its adjoint $\beta^* : T^*M \to T^{**}M = TM$.

The cometric β for a subriemannian geometry is uniquely defined by the following conditions:

(1) $\mathrm{im}(\beta_q) = \mathcal{H}_q$;

(2) $p(v) = \langle \beta_q(p), v \rangle$, for $v \in \mathcal{H}_q$, $p \in T_q M$,

where $\langle \beta_q(p), v \rangle_q$ is the subriemannian inner product on $\mathcal{H}_q \cdot$ Conversely, any cometric of constant rank defines a subriemannian geometry whose underlying distribution has that rank.

Definition 16 *The fiber-quadratic function $H(q,p) = \frac{1}{2}(p,p)_q$, where $(\cdot\,,\cdot)_q$ is the cometric on the fiber $T_q^* M$, is called the* subriemannian Hamiltonian*, or the* kinetic energy*.*

The Hamiltonian H is related to length and energy as follows. Suppose that γ is a horizontal curve. Then, $\dot{\gamma}(t) = \beta_{\gamma(t)}(p)$, for same covector $p \in T^*_{\gamma(t)} M$, and

$$\frac{1}{2}||\dot{\gamma}||^2 = H(q,p).$$

H uniquely determines β by polarization, and β uniquely determines the subriemannian structure. This proves the following proposition:

Proposition 17 *The subriemannian structure is uniquely determined by its Hamiltonian. Conversely, any nonnegative fiber-quadratic Hamiltonian of constant fiber rank k gives rise to a subriemannian structure whose underlying distribution has rank k .*

To compute the subriemannian Hamiltonian we can start with a local frame $\{X_a\}, a = 1, \ldots, k$, of vector fields for $\mathcal{H}$. We think of the X_a as fiber-linear functions on the cotangent bundle. In so doing, we rename them P_a. Thus

$$P_a(q,p) = p(X_a(q)), \quad q \in M, p \in T_q^*M.$$

Definition 18 *Let X be a vector field on the manifold M. The fiber-linear function on the cotangent bundle $P_X : T^*M \to \mathbb{R}$, defined by $P_X(q,p) = p(X(q))$ is called the* momentum function *for X.*

Thus the $P_a = P_{X_a}$ are the momemtum functions for our horizontal frame. If $X_a = \sum X_a^i(x)\frac{\partial}{\partial x^i}$ is the expression for X_a relative to coordinates x^i, then $P_{X_a}(x,p) = \sum X_a^i(x)p_i$, where $p_i = P_{\frac{\partial}{\partial x^i}}$ are the momentum functions for the coordinate vector fields. The x^i and p_i together form a coordinate system on T^*M. They are called *canonical coordinates.*

Let $g_{ab}(q) = \langle X_a(q), X_b(q)\rangle_q$ be the matrix of inner products defined by our horizontal frame. Let $g^{ab}(q)$ be its inverse matrix. Then g^{ab} is a $k \times k$ matrix-valued function defined in some open set of M.

Proposition 19 *Let P_a and g^{ab} be the functions on T^*M that are induced by a local horizontal frame X_a as just described. Then*

$$H(q,p) = \frac{1}{2}\sum g^{ab}(q)P_a(q,p)P_b(q,p). \tag{6}$$

Indeed,

$$\begin{aligned} H(q,p) &= \frac{1}{2}(p,p)_q = \frac{1}{2}(\sum p_a dx^a, \sum p_b dx^b) = \frac{1}{2}\sum g^{ab}(q)(p_a,p_b) \\ &= \frac{1}{2}\sum g^{ab}(q)(p(X_a)(q), p(X_b)(q)) = \frac{1}{2}\sum g^{ab}(q)P_a(q,p)P_b(q,p). \end{aligned}$$

Note, in particular, that if the X_a are an orthonormal frame for $\mathcal{H}$ relative to the subriemannian inner product, then $H = \frac{1}{2}P_a^2$.

Normal geodesics. Like any smooth function on the cotangent bundle, our function H generates a system of Hamiltonian differential equations. In terms of canonical coordinates (x^i, p_i), these differential equations are

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \tag{7}$$

Definition 20 *The Hamiltonian differential equations (7) are called the* normal geodesic equations.

Riemannian geometry can be viewed as a special case of subriemannian geometry, one in which the distribution is the entire tangent bundle. The cometric is the usual inverse metric, written g^{ij} in coordinates. The normal geodesic equations in the Riemannian case are simply the standard geodesic equations, rewritten on the cotangent bundle.

Geodesics of the complex contact manifold $\mathbb{C} \times \mathrm{Im}\mathbb{C}$ (see [7])

Going back to the complex contact manifold $\mathbb{C} \times \mathrm{Im}\mathbb{C}$, we know that vector fields

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z},$$

form an orthonormal frame. This means that they frame the two plane $\mathcal{H}$ and that they are orthonormal with respect to the inner product $\mathrm{ds}^2 = (dx^2 + dy^2)|_{\mathcal{H}}$ on the distribution. According to the above discussion, the subriemannian Hamiltonian is given by

$$H = \frac{1}{2}(P_X^2 + P_Y^2), \tag{8}$$

P_X, P_Y standing for the momentum functions of X and Y, respectively. Thus

$$P_X = p_x - \frac{1}{2} y p_z, \qquad P_Y = p_y + \frac{1}{2} x p_z,$$

where p_x, p_y, p_z are the fiber coordinates on the cotangent bundle of $\mathbb{R}^3$ corresponding to the Cartesian coordinates (x, y, z) on $\mathbb{R}^3$. Again, these fiber coordinates are defined

by writing a covector as $p = p_x dx + p_y dy + p_z dz$. Together, (x, y, z, p_x, p_y, p_z) are global coordinates on the cotangent bundle $T^*\mathbb{R}^3 = \mathbb{R}^3 \oplus \mathbb{R}^3$.

Hamilton's equations can be written

$$\frac{df}{dt} = \{f, H\}, f \in C^\infty(T^*M), \tag{9}$$

which holds for any smooth function f. The function H defines a vector field X_H, called the Hamiltonian vector field, which has a flow $\Phi_t : T^*M \to T^*M$. Let $f : T^*\mathbb{R}^3 = T^*M \to \mathbb{R}$ be any smooth function on the cotangent bundle. Form the time-dependent function $f_t = \Phi_t^* f$ by pulling f back via the flow. Thus $f_t(x, y, z, p_x, p_y, p_z) = f(\Phi_t(x, y, z, p_x, p_y, p_z))$. In other words, $\frac{df}{dt} = X_H[f_t]$, which gives meaning to the left-hand side of Hamilton's equations.

To define the right hand side, which is to say the vector field X_H, we will need the Poisson bracket. The Poisson bracket on the cotangent bundle T^*M of a manifold M is a canonical Lie algebra structure defined on the vector space $C^\infty(T^*M)$ of smooth functions on T^*M. The Poisson bracket is denoted $\{\cdot, \cdot\} : C^\infty \times C^\infty \to C^\infty$, where $C^\infty = C^\infty(T^*M)$, and can be defined by the coordinate formula

$$\{f, g\} = \sum \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p_i}.$$

This formula is valid in any canonical coordinate system, and can be shown to be coordinate independent. The Poisson bracket satisfies the Leibniz identity

$$\{f, gh\} = g\{f, h\} + h\{f, g\},$$

which means that the operation $\{\cdot\,, H\}$ defines a vector field X_H, called the *Hamiltonian vector field*.

By letting the functions f vary over the collection of coordinate functions x^i and p_i we get the more common form of Hamilton's equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}.$$

Indeed, for the first one we take $f = x$ and $g = H$. Then

$$\{x, H\} = \frac{\partial x}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial x^i}\frac{\partial x}{\partial p_i}$$

if and only if

$$\dot{x} = \frac{\partial H}{\partial p_x}.$$

Similarly, $\dot{y} = \frac{\partial H}{\partial p_y}$. (These equations are in turn equivalent to the above formulation (9). It is more convenient to use la formulation (9), because the momentum function $X \to P_X$ is a Lie algebra anti-homomorphism from the Lie algebra of all smooth vector fields on M to $C^\infty(T^*M)$ with its Poisson bracket:

$$\{P_X, P_Y\} = -P_{[X,Y]}. \tag{10}$$

Proof of (10). With the above notations

$$\{P_X, P_Y\} = \{p_x - \frac{1}{2}yp_z, p_y + \frac{1}{2}xp_z\} = -\frac{1}{2}p_z - \frac{1}{2}p_z = -p_z := P_Z,$$

because $[X,Y]=Z$. For the complex contact group, with our choose of X and Y as a frame for $\mathcal{H}$, we compute

$$[X,Y]=Z:=\frac{\partial}{\partial z},\quad [X,Z]=[Y,Z]=0.$$

Thus

$$\{P_X,P_Y\}=-p_z:=P_Z,\quad \{P_X,P_Z\}=\{P_Y,P_Z\}=0.$$

Let me prove that

$$\{P_X,P_Z\}=\{p_x-\frac{1}{2}yp_z,p_z\}=0.$$

These relations can also easily be computed by hand, from our formulae for P_X, P_Y and the bracket in terms of $\{x,y,z,p_x,p_y,p_z\}$. By letting f vary over the collection of functions $\{x,y,z,P_X,P_Y,P_Z\}$, using the bracket relations and (10), we find that Hamilton's equations are equivalent to the system $\dot{x}=P_X$.

Indeed, remember that $H=\frac{1}{2}(P_X^2+P_Y^2)$. Then $\dot{x}=\{x,H\}=P_X\frac{\partial P_X}{\partial p_x}=P_X$. Similarly, $\dot{y}=P_Y$. As for $\dot{z}=\frac{1}{2}P_X+\frac{1}{2}P_Y$, we have that $\dot{z}=\{z,H\}=P_X\frac{\partial P_X}{\partial p_z}+P_Y\frac{\partial P_Y}{\partial p_z}=-\frac{1}{2}yP_X+\frac{1}{2}xP_Y$. Finally, $\dot{P}_X=-P_ZP_X$. In fact, $\dot{P}_X=\{P_X,H\}=\{p_x-\frac{1}{2}yp_z,\frac{1}{2}(P_X^2+P_Y^2)\}=(\frac{\partial(p_x-\frac{1}{2}yp_z)}{\partial y})P_X\frac{\partial P_y}{\partial y}=-P_ZP_X$.

In a similar way we obtain $\dot{P}_Y=P_ZP_Y$.

$\dot{P}_Z = 0$ is a consequence of the fact $\dot{P}_Z = \{P_Z, H\} = \{p_z, \frac{1}{2}(P_X^2 + P_Y^2)\} = 0$.

Summing up, the equations are:

$$\begin{aligned}
\dot{x} &= P_X \\
\dot{y} &= P_Y \\
\dot{z} &= \frac{1}{2}(P_X + P_Y) \\
\dot{P}_X &= -P_Z P_X \\
\dot{P}_Y &= P_Z P_Y \\
\dot{P}_Z &= 0
\end{aligned}$$

The last equation asserts that $P_Z = p_z$ is constant. The variable z appears nowhere in the right-hand sides of these equations. It follows that the variables x, y, P_X, P_Y evolve independently of z, and so we can view the system as defining a one-parameter family of dynamical systems on $\mathbb{R}^4$ parameterized by the constant value of P_Z. Combine x and y into a single complex variable $\omega = x + iy$.

Note that the first two equations say that $\frac{d\omega}{dt} = P_X + iP_Y$. The fourth and fifth equations say that the time derivative of $P_X + iP_Y$ is $P_X + iP_Y$. All together, then, we have $\frac{d^2\omega}{dt^2} = ip_z\omega$, p_z constant. These are the famed Lorentz equations for the motion of a particle in a constant magnetic field. To convert the electromagnetic notation, we set the parameter p_z to $\frac{Be}{m}$, where e is the particle's charge, B is the magnetic field strength, and m is the mass of the particle.

Finally note that the third of our Hamilton's equations, the z equation, is just the differential constraint $\dot{z} = \frac{1}{2}(x\dot{y} - y\dot{x})$. A first integration of the Lorentz equations yields the evolution of the planar velocity: $P_X + iP_Y = P(0)\exp(ip_z t)$, where the complex vector $P(0) = P_X(0) + iP_Y(0)$ describes the initial velocity.

A second integration yields the general form of the geodesics on the complex contact group:

$$x(t) + iy(t) = w(t) = \frac{P(0)}{ip_z}(\exp(ip_z t) - 1) + (x(0) + iy(0)),$$
$$z(t) = z(0) + \frac{1}{2}\int_0^t \mathrm{Im}(\bar{w}dw).$$

Geodesic equations of the quaternionic contact manifold $\mathbb{H}\times \mathrm{Im}\mathbb{H}$

The following vector fields, which are the old $X_4, \cdots, X_7$ ones provided $m = 0$,

$$
\begin{aligned}
W &= \partial_w + \frac{1}{2}(x\partial_r + y\partial_s + z\partial_t),\\
X &= \partial_x - \frac{1}{2}(w\partial_r + z\partial_s - y\partial_t),\\
Y &= \partial_y + \frac{1}{2}(z\partial_r - w\partial_s - x\partial_t),\\
Z &= \partial_z - \frac{1}{2}(y\partial_r - x\partial_s - w\partial_t),
\end{aligned}
$$

along with $\{\partial_r, \partial_s, \partial_t\}$ form an orthonormal frame for the quaternionic contac manifold $\mathbb{H}\times\mathrm{Im}\mathbb{H}$. This means that $\{W, X, YZ\}$ frame the fourth plane $\mathcal{H}$ and they are orthonormal with respect to the inner product $\mathrm{ds}^2 = (dw^2 + dx^2 + dy^2 + dz^2)|_{\mathcal{H}}$ on the distribution. According to the above discussion, the subriemannian Hamiltonian is

$$
H = \frac{1}{2}(P_W^2 + P_X^2 + P_Y^2 + P_Z^2), \tag{11}
$$

where P_W, P_X, P_Y, P_Z are the momentum functions of the vector fields W, X, Y, Z, respectively. Thus

$$\begin{aligned} P_W &= p_w + \frac{1}{2}(xp_r + yp_s + zp_t), \\ P_X &= p_x - \frac{1}{2}(wp_r + zp_s - yp_t), \\ P_Y &= p_y + \frac{1}{2}(zp_r - wp_s - xp_t), \\ P_Z &= p_z - \frac{1}{2}(yp_r - xp_s + wp_t), \end{aligned}$$

where $p_w, p_x, p_y, p_z, p_r, p_s, p_t$ are the fiber coordinates on the cotangent bundle of $\mathbb{R}^7$ corresponding to the Cartesian coordinates w, x, y, z, r, s, t on $\mathbb{R}^7$. Again, these fiber coordinates are defined by writing a covector as $p = p_w dw + p_x dx + p_y dy + p_z dz + p_r dr + p_s ds + p_t dt$. Together, $(w, x, y, z, r, s, t, p_w, p_x, p_y, p_z, p_r, p_s, p_t)$ are global coordinates on the cotangent bundle $T^*\mathbb{R}^7 = \mathbb{R}^7 \oplus \mathbb{R}^7$. Hamilton's equations can be written

$$\frac{df}{dt} = \{f, H\}, \quad f \in C^\infty(T^*\mathbb{R}^7), \tag{12}$$

which holds for any smooth function f. The function H defines a vector field X_H, called the Hamiltonian vector field, which has a flow $\Phi_t : T^*\mathbb{R}^7 \to T^*\mathbb{R}^7$. Let $f : T^*\mathbb{R}^7 \to \mathbb{R}$ be any smooth function on the cotangent bundle. Form the time-dependent function $f_t = \Phi_t^* f$ by pulling f back via the flow. Thus $f_t(w, x, y, z, r, s, t, p_w, p_x, p_y, p_z, p_r, p_s, p_t) =$

$f(\Phi_t(w,x,y,z,r,s,t,p_w,p_x,p_y,p_z,p_r,p_s,p_t))$. In other words $\frac{df}{dt} = X_H[f_t]$, which gives meaning to the left-hand side of Hamilton's equations.

To define the right hand side, which to say the vector field X_H, we will need the Poisson bracket. The Poisson bracket on the cotangent bundle $T^*\mathbb{R}^7$ of a manifold $\mathbb{R}^7$ is a canonical Lie algebra structure defined on the vector space $C^\infty(T^*\mathbb{R}^7)$ of smooth functions on $T^*\mathbb{R}^7$. The Poisson bracket is denoted $\{\cdot\,,\cdot\} : C^\infty \times C^\infty \to C^\infty$, where $C^\infty = C^\infty(T^*\mathbb{R}^7)$, and can be defined by the coordinate formula

$$\{f,g\} = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p_i}.$$

This formula is valid in any canonical coordinate system, and can be shown to be coordinate independent. The Poisson bracket satisfies the Leibniz identity

$$\{f,gh\} = g\{f,h\} + h\{f,g\},$$

which means that the operation $\{.,H\}$ defines a vector field X_H, called the Hamiltonian vector field. By letting the functions f vary over the collection of coordinate functions x^i and we get the more common form of Hamilton's equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}.$$

Indeed, for the first one we take $f = w$ and $g = H$. Then $\{w, H\} = \frac{\partial w}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial x^i}\frac{\partial w}{\partial p_i}$ if and only if $\dot{w} = \frac{\partial H}{\partial p_w}$. Also we have

$$\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{y} = \frac{\partial H}{\partial p_y}, \quad \dot{z} = \frac{\partial H}{\partial p_z}.$$

These equations are in turn equivalent to the above formulation (13). It is more convenient to use la formulation (13), because the momentum function $W \to P_W$ is a Lie algebra anti-homomorphism from the Lie algebra of all smooth vector fields on $\mathbb{R}^7$ to $C(T^*\mathbb{R}^7)$ with the Poisson brackets:

$$\begin{aligned} \{P_W, P_X\} = -P_{[W,X]}, \quad \{P_W, P_Y\} = -P_{[W,Y]}, \quad \{P_W, P_Z\} = -P_{[W,Z]}, \\ \{P_X, P_Y\} = -P_{[X,Y]}, \quad \{P_X, P_Z\} = -P_{[X,Z]}, \quad \{P_Y, P_Z\} = -P_{[Y,Z]}. \end{aligned} \tag{13}$$

Since all calculations are similar, we only prove the first one:

$\{P_W, P_X\} = \{p_w + \frac{x}{2}p_r + \frac{y}{2}p_s + \frac{z}{2}p_t, p_x - \frac{w}{2}p_r - \frac{z}{2}p_s + \frac{y}{2}p_t\} = p_r = -P_{[W,X]}$.

For the quaternionic contact group, with our choose of W, X, Y, Z as a frame for $\mathcal{H}$, we compute

$$[W, X] = -\partial_r, \quad [W, Y] = -\partial_s, \quad [W, Z] = -\partial_t,$$

$$[X, Y] = -\partial_t, \quad [X, Z] = \partial_s, \quad [Y, Z] = -\partial_t,$$

$$[W, \partial_r] = [W, \partial_r s] = [W, \partial_t] = [X, \partial_r] = [X, \partial_s] = [X, \partial_t] = 0,$$

$$[Y, \partial_r] = [Y, \partial_s] = [Y, \partial_t] = [Z, \partial_r] = [Z, \partial_s] = [Z, \partial_t] = 0.$$

Thus

$$\{P_W, P_X\} = \partial_r := P_r, \quad \{P_W, P_Y\} = \partial_s := P_s, \quad \{P_W, P_Z\} = \partial_t := P_t,$$

$$\{P_X, P_Y\} = P_t, \quad \{P_X, P_Z\} = -p_s = -P_s, \quad \{P_Y, P_Z\} = p_r = P_r$$

We can prove that

$$\{P_W, P_r\} = \{P_W, P_s\} = \{P_W, P_t\} = \{P_X, P_r\} = \{P_X, P_s\} = \{P_X, P_t\} = 0,$$

$$\{P_Y, P_r\} = \{P_Y, P_s\} = \{P_Y, P_t\} = \{P_Z, P_r\} = \{P_Z, P_s\} = \{P_Z, P_t\} = 0.$$

These relations can also easily be computed by hand, from our formulae for P_W, P_X, P_Y, P_Z and the bracket in terms of $w, x, y, z, r, s, r, p_w, p_x, p_y, p_z, p_r, p_s, p_t$.

Lemma 21 *By letting f vary over the functions $w, x, y, z, r, s, r, P_W, P_X, P_Y, P_Z, P_r, P_s, P_t$, using the bracket relations and equation* (13)*, we find that Hamilton's equations are equivalent to the system*

$$
\begin{aligned}
\dot{w} &= P_W, \\
\dot{x} &= P_X, \\
\dot{y} &= P_Y, \\
\dot{z} &= P_Z, \\
\dot{r} &= \frac{1}{2}(xP_W - wP_X + zP_Y - yP_Z), \\
\dot{s} &= \frac{1}{2}(yP_W - zP_X + xP_Y - wP_Z), \\
\dot{t} &= \frac{1}{2}(zP_W + yP_X - xP_Y - wP_Z), \\
\dot{P}_W &= p_r P_X + p_s P_Y + p_t P_Z, \\
\dot{P}_X &= -p_r P_W - p_s P_Z + p_t P_Y, \\
\dot{P}_Y &= p_r P_Z - p_s P_W - p_t P_X, \\
\dot{P}_Z &= -p_r P_Y + p_s P_X - p_t P_W, \\
\dot{P}_r &= 0, \\
\dot{P}_s &= 0, \\
\dot{P}_t &= 0.
\end{aligned}
$$

To see it, remember that $H = \frac{1}{2}(P_W^2 + P_X^2 + P_Y^2 + P_z^2)$. Then

$$\begin{aligned}
\dot{w} &= \{w, H\} = P_w \frac{\partial P_W}{\partial p_w} = P_W, \\
\dot{x} &= \{x, H\} = P_X \frac{\partial P_X}{\partial p_x} = P_X, \\
\dot{y} &= P_Y, \\
\dot{z} &= P_Z.
\end{aligned}$$

Also, considering that:

$$\frac{\partial P_W}{\partial p_r} = \frac{x}{2}, \qquad \frac{\partial P_W}{\partial p_s} = \frac{y}{2}, \qquad \frac{\partial P_W}{\partial p_t} = \frac{z}{2},$$

$$\frac{\partial P_X}{\partial p_r} = -\frac{w}{2}, \qquad \frac{\partial P_X}{\partial p_s} = -\frac{z}{2}, \qquad \frac{\partial P_X}{\partial p_t} = \frac{y}{2},$$

$$\frac{\partial P_Y}{\partial p_r} = \frac{z}{2}, \qquad \frac{\partial P_Y}{\partial p_s} = -\frac{w}{2}, \qquad \frac{\partial P_Y}{\partial p_t} = -\frac{x}{2},$$

$$\frac{\partial P_Z}{\partial p_r} = -\frac{y}{2}, \qquad \frac{\partial P_Z}{\partial p_s} = \frac{x}{2}, \qquad \frac{\partial P_Z}{\partial p_t} = -\frac{w}{2},$$

we have

$$\dot{r} = \frac{1}{2}(xP_W - wP_X + zP_Y - yP_Z).$$

Indeed,

$$\begin{aligned}
\dot{r} = \{r, H\} &= P_W \frac{\partial P_W}{\partial p_r} + P_X \frac{\partial P_X}{\partial p_r} + P_Y \frac{\partial P_Y}{\partial p_r} + P_Z \frac{\partial P_Z}{\partial p_r} \\
&= \frac{1}{2}(xP_W - wP_X + zP_Y - yP_Z) \\
\dot{s} = \{s, H\} &= P_W \frac{\partial P_W}{\partial p_s} + P_X \frac{\partial P_X}{\partial p_s} + P_Y \frac{\partial P_Y}{\partial p_s} + P_Z \frac{\partial P_Z}{\partial p_s} \\
&= \frac{1}{2}(yP_W - zP_X + xP_Y - wP_Z) \\
\dot{t} = \{t, H\} &= P_W \frac{\partial P_W}{\partial p_t} + P_X \frac{\partial P_X}{\partial p_t} + P_Y \frac{\partial P_Y}{\partial p_t} + P_Z \frac{\partial P_Z}{\partial p_t} \\
&= \frac{1}{2}(zP_W + yP_X - xP_Y - wP_Z).
\end{aligned}$$

Working as above we obtain

$$\begin{aligned}
\dot{P}_W &= \{P_W, H\} = p_r P_X + p_s P_Y + p_t P_Z, \\
\dot{P}_X &= \{P_X, H\} = -p_r P_W - p_s P_Z + p_t P_Y, \\
\dot{P}_Y &= \{P_Y, H\} = p_r P_Z - p_s P_W - p_t P_X, \\
\dot{P}_Z &= \{P_Z, H\} = -p_r P_Y + p_s P_X - p_t P_W.
\end{aligned}$$

Finally, it is not difficult to see that $\dot{P}_r = \dot{P}_s = \dot{P}_t = 0$.

The last three equations assert that $P_r = p_r$, $P_s = p_s$ and $P_t = p_t$ are constant. The variables r, s, t appears nowhere in the right-hand sides of these equations. It follows that the variables $w, x, y, z, P_W, P_X, P_Y, P_Z$ evolve independently of r, s, t, and so we can view the system as defining a one-parameter family of dynamical systems on $\mathbb{R}^8$ parameterized by the constant value of P_r, P_s, P_t.

Combine w, x, y, z into a single quaternionic variable $\omega = w + ix + jy + kz$ and taking into account the fourteen equations one has

$$\frac{d\omega}{dt} = P_W + iP_X + jP_Y + kP_Z$$

The time derivative of $P_W + iP_X + jP_Y + kP_Z$ is $-(ip_r + jp_s + kp_t)(P_W + iP_X + jP_Y + kP_Z)$. Then we have $\frac{d^2\omega}{dt^2} = -(ip_r + jp_s + kp_t)\frac{d\omega}{dt}$, where p_r, p_s and p_t are constant.

By integrating the above expression we get

$$P_W + iP_X + jP_Y + kP_Z = P(0)\exp(-(ip_r + jp_s + kp_t)t),$$

where $P(0) = P_W(0) + iP_X(0) + jP_Y(0) + kP_Z(0)$.

A second integration yields the general form of the geodesics on the quaternionic contact group:

$$
\begin{aligned}
\omega(t) &= w(t) + ix(t) + jy(t) + kz(t) = \\
&\quad \frac{P(0)}{ip_r + jp_s + kp_t}\left(\exp(-(ip_r + jp_s + kp_t)t - 1) + w(0) + ix(0) + jy(0) + kz(0)\right), \\
r(t) &= r(0) + \frac{1}{2}\int_0^t \mathrm{Im}_I(\bar{\omega}\, d\omega), \\
s(t) &= s(0) + \frac{1}{2}\int_0^t \mathrm{Im}_J(\bar{\omega}\, d\omega), \\
r(t) &= t(0) + \frac{1}{2}\int_0^t \mathrm{Im}_K(\bar{\omega}\, d\omega).
\end{aligned}
$$

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