## Some geometric properties of Extended Bianchi-Cartan-Vranceanu spaces

Angel Ferrández ${ }^{1}$, Antonio M. Naveira ${ }^{2}$ and Ana D. Tarrío ${ }^{3}$

${ }^{1}$ Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain. E-mail address: aferr@um.es
${ }^{2}$ Departamento de Matemáticas, Universidad de Valencia (Estudi General), Campus de Burjassot, 46100 Burjassot, Spain. E-mail address: naveira@uv.es
${ }^{3}$ Departamento de Matemáticas, E.U. Arquitectura Técnica, Campus de A Zapateira, 15001 A Coruña, Spain. E-mail address: madorana@udc.es

## 1 Introduction (see [14])

Dido's problem is a variant of the isoperimetric problem. It was formulated in the Aeneid, Virgil's epic poem glorifying the beginnings of Rome.

Queen Dido had to flee across the Mediterranean in a ship with friends and servants. She had what we would nowadays call a dysfunctional family. Her brother, Pygmalion, had just murdered her husband and taken most of her possessions. Dido landed, nearly penniless, on a part of the African coast ruled by King Jarbas. After dickering and begging, Dido persuaded Jarbas to give her as much land as she could enclose with an ox hide. Dido told her servants to cut an ox hide into a single long, narrow strip. They turned the ox hide into a single leather string.

Dido had in this way reformulated her difficult situation into the following geometric problem. Given a string of fixed length $\ell$ and a fixed line $L$ (the Mediterranean coastline), place the ends of the string on $L$ and determine the shape of the curve $c$ for which the figure enclosed by $c$ together with $L$ has the maximum possible area. This is Dido's problem. It is also sometimes referred to as the problem of Pappus. Dido found the solution - a half-circle - and thus founded the semicircular city of Carthage.

Take the one-form $\alpha=\frac{1}{2}(x d y-y d x)$ which satisfies $d \alpha=d x \wedge d y$ and $\left.\alpha\right|_{L}=0$ for any ray $L$ through the origin.

According to Stokes' theorem, the area $\Phi$ enclosed by a closed planar curve $c$ is

$$
\begin{equation*}
\Phi(c)=\int_{c} \alpha . \tag{1}
\end{equation*}
$$

As $\left.\alpha\right|_{L}=0$, if $c$ is a non-closed curve beginning at the origin, $\Phi(c)$ represents the area enclosed by the closed curve obtained by traversing $c$ and then returning to the origin along the ray that connects the endpoint of $c$ to the origin.

The length $\ell$ of $c=(x(t), y(t))$ is

$$
\begin{equation*}
\ell(c)=\int_{c} d s \tag{2}
\end{equation*}
$$

where ds $=\sqrt{d x^{2}+d y^{2}}=\left\|c^{\prime}\right\| d t$ is the usual element of arc length. In this manner Dido's problem, and the (dual) isoperimetric problem, becomes the following constrained variational problem:

Problem 1. Minimize the length $\ell(c)$ of a closed rectifiable curve $c$, subject to the constraint that the signed area $\Phi(c)$ of the curve be a fixed constant.

The introduction of $\alpha$ lets us extend the problem to non-closed curves. The ray used to close up corresponds to the coastline $L$ in Dido's problem.

Now Montgomery constructs the three-dimensional geometry whose geodesics correspond to the solutions to the isoperimetric problem. Add a third direction $z$ whose motion
is linked to that of $x$ and $y$ according to

$$
\begin{equation*}
d z=\frac{1}{2}(x d y-y d x) \tag{3}
\end{equation*}
$$

In this way we associate a family of curves $\gamma(t)=(x(t), y(t), z(t))$ to a single planar curve $c(t)=(x(t), y(t))$, the family being parameterized by the initial value $z_{0}$ of the height $z$. We will call any one of these paths a horizontal lift of $c$, and more generally, any path $\gamma$ in $\mathbb{R}^{3}$ that satisfies the differential constraint (3) a horizontal path. Set

$$
\mathrm{ds}^{2}=d x^{2}+d y^{2}
$$

and define the length of any horizontal path in $\mathbb{R}^{3}$ to be $\int_{\gamma} d s$. In other words, we have defined the length of $\gamma$ to be equal to the usual length of its planar projection $c$.

Problem 2. Minimize the length $\int_{\gamma} d s$ over all horizontal paths $\gamma$ that join two fixed points in three-space.

To see that this is a reformulation of the dual to Dido's problem, or the isoperimetric problem, observe that

$$
z(1)-z(0)=\int_{c} \frac{1}{2}(x d y-y d x)
$$

where $c(t)=(x(t), y(t))$ is the projection of the curve $\gamma(t)=(x(t), y(t), z(t))$ to the plane. Observe that, according to Stokes' theorem, if $c$ joins the origin to $\left(x_{1}, y_{1}\right)$ and if we take $z(0)=0$, then the endpoints of $\gamma$ are $(0,0,0)$ and $\left(x_{1}, y_{1}, \Phi(c)\right)$, where $\Phi(c)$ denotes the
signed area defined by the closed curve given by traversing $c$ and then returning to the origin along a line segment.

Defining the differential 1-form $\omega=d z-\frac{1}{2}(x d y-y d x)$ we can write

$$
\begin{aligned}
\mathcal{H} & =\operatorname{Ker} \omega=\{\omega(x, y, z)=0\} \\
& =\left\{\left(v_{1}, v_{2}, v_{3}\right): v_{3}-\frac{1}{2}\left(x v_{2}-y v_{1}\right)=0\right\} \subset \mathbb{R}^{3} .
\end{aligned}
$$

This $\mathcal{H}$ is a field of two-planes in three-space, or what it is called a distribution: a linear subbundle of the tangent bundle. The restriction of $d s^{2}$ to these two-planes defines a smoothly varying family of inner products $\langle\cdot, \cdot\rangle$ on the planes $\mathcal{H}$. Thus if $v, w \in \mathcal{H}_{(x, y, z)}$, then $\langle v, w\rangle=v_{1} w_{1}+v_{2} w_{2}$.
Definition $1 \mathbb{R}^{3}$ endowed with the structure of this distribution $\mathcal{H}$ and this family of inner products $\mathrm{ds}^{2}$ on $\mathcal{H}$ is called the Heisenberg group (complex contact group), which is the first nontrivial example of subriemannian geometry $[2,6,18]$.

## 2 Extended Bianchi-Cartan-Vranceanu spaces

We can see in ([3, 4, 5, 6, 21])the Bianchi-Cartan-Vranceanu (BCV for short) space, which is an example of sub-riemannian geometry (see $[6,16]$ ) and the horizontal distribution is a 2 -step breaking-generating distribution everywhere.

We generalize these manifolds in the following way
Observe that letting $z=x+i y$, we see that $\operatorname{Im}(z d \bar{z})=y d x-x d y$, which reminds us the map $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R} \times \mathbb{C}$ given by $\left(z_{1}, z_{2}\right) \mapsto\left(\left|z_{1}\right|^{2}-\left|z_{2}\right|^{2}, 2\left(z_{1} \bar{z}_{2}\right)\right)$, that easily leads to the classical Hopf fibration $\mathbb{S}^{1} \hookrightarrow \mathbb{S}^{3} \rightarrow \mathbb{S}^{2}$, where coordinates in $\mathbb{S}^{2}$ are given by $\left(\left|z_{1}\right|^{2}-\left|z_{2}\right|^{2}, 2 \operatorname{Re}\left(z_{1} z_{2}\right), 2 \operatorname{Im}\left(z_{1} z_{2}\right)\right)$.

In the same line we get the fibration $\mathbb{S}^{3} \hookrightarrow \mathbb{S}^{7} \rightarrow \mathbb{S}^{4}$, but using quaternions $\mathbb{H}$ instead of complex numbers. For a point $\left(q_{1}=\alpha+\beta j, q_{2}=\gamma+\delta j\right) \in \mathbb{S}^{7}$, we get the following coordinate expressions $\left(\left|q_{1}\right|^{2}-\left|q_{2}\right|^{2}, 2 \operatorname{Re}(\bar{\alpha} \gamma+\bar{\beta} \delta), 2 \operatorname{Im}(\bar{\alpha} \gamma+\bar{\beta} \delta), 2 \operatorname{Re}(\alpha \delta-\beta \gamma), 2 \operatorname{Im}(\alpha \delta-\right.$ $\beta \gamma)$ ).

For any $q=w+x i+y j+z k \in \mathbb{H}$ we find that $q d \bar{q}=w d w+x d x+y d y+z d z+(x d w-$ $w d x+z d y-y d z) i+(y d w-w d y+x d z-z d x) j+(z d w-w d z+y d x-x d y) k$. As the quaternionic contact group $\mathbb{H} \times \operatorname{Im} \mathbb{H}$, with coordinates $(w, x, y, z, r, s, t)$ can be equipped
with the metric

$$
\begin{aligned}
\mathrm{ds}^{2}= & \left(d w^{2}+d x^{2}+d y^{2}+d z^{2}\right)+\left(d r+\frac{1}{2}(x d w-w d x+z d y-y d z)\right)^{2} \\
& +\left(d s+\frac{1}{2}(y d w-w d y+x d z-z d x)\right)^{2}+\left(d t+\frac{1}{2}(z d w-w d z+y d x-x d y)\right)^{2} .
\end{aligned}
$$

Then, by extending this metric, it seems natural to find a 7 -dimensional generalization of the 3 -dimensional $B C V$ spaces endowed with the two-parameter family of metrics

$$
\begin{aligned}
\mathrm{ds}_{l, m}^{2}= & \frac{d w^{2}+d x^{2}+d y^{2}+d z^{2}}{K^{2}}+\left(d r+\frac{l}{2} \frac{w d x-x d w+y d z-z d y}{K}\right)^{2} \\
& +\left(d s+\frac{l}{2} \frac{w d y-y d w+z d x-x d z}{K}\right)^{2}+\left(d t+\frac{l}{2} \frac{w d z-z d w+x d y-y d x}{K}\right)^{2},
\end{aligned}
$$

where $l, m$ are real numbers and $K=1+m\left(w^{2}+x^{2}+y^{2}+z^{2}\right)$.
Then ( $E B C V, \mathrm{ds}_{l, m}^{2}$ ) will be called extended $B C V$ spaces ( $E B C V$ for short).
That metric is obtained as a conformal deformation of the Euclidean metric of $\mathbb{R}^{4}$ by adding three suitable terms which depend on $l$ and $m$ concerning the imaginary part of $q \bar{q}$, for a quaternion $q$. When $m=0$ we get a one-parameter of Riemannian metrics depending on $l$. Furthermore, if $l=1$, we find the 7 -dimensional quaternionic Heisenberg group (see
[8] and [23]). The manifold $E B C V$ provides another example of sub-riemannian geometry and the horizontal distribution is a 2-step breaking-generating distribution everywhere.

That metric can also be written as

$$
\mathrm{ds}_{l, m}^{2}=\sum_{i=1}^{7} \omega^{i} \otimes \omega^{i}
$$

where

$$
\begin{array}{rlrl}
\omega^{1}=d r+\frac{l}{2 K}(w d x-x d w+y d z-z d y), & & \omega^{4}=d w / K, \\
\omega^{2}=d s+\frac{l}{2 K}(w d y-y d w+z d x-x d z), & \omega^{5}=d x / K, \\
\omega^{3}=d t+\frac{l}{2 K}(w d z-z d w+x d y-y d x), & \omega^{6}=d y / K, \\
\omega^{7}=d z / K, \\
X_{1}=\partial_{r}, \quad X_{2}=\partial_{s}, & X_{3}=\partial_{t}, & \\
X_{4}=K \partial_{w}+\frac{l x}{2} \partial_{r}+\frac{l y}{2} \partial_{s}+\frac{l z}{2} \partial_{t}, \quad X_{5}=K \partial_{x}-\frac{l w}{2} \partial_{r}-\frac{l z}{2} \partial_{s}+\frac{l y}{2} \partial_{t}, \\
X_{6}=K \partial_{y}+\frac{l z}{2} \partial_{r}-\frac{l w}{2} \partial_{s}-\frac{l x}{2} \partial_{t}, \quad X_{7}=K \partial_{z}-\frac{l y}{2} \partial_{r}+\frac{l x}{2} \partial_{s}-\frac{l w}{2} \partial_{t} .
\end{array}
$$

Then we find that

Lemma $2\left\{X_{1}, X_{2}, \cdots, X_{7}\right\}$ is an orthonormal basis of vector fields whit respect to the given metric $\mathrm{ds}_{l, m}^{2}:=\langle$,$\rangle .$

Observe that
when $m=l=0, E B C V$ is nothing but $\mathbb{R}^{7}$,
when $m>0, l=0, E B C V \cong \mathbb{S}^{4}(4 m) \times \mathbb{R}^{3}$, and
when $m<0, l=0, E B C V \cong \mathbb{H}^{4}(4 m) \times \mathbb{R}^{3}$.

## 3 The Levi-Civita connection and curvature tensor

Writing $1 \leq a, b \leq 3,4 \leq u, v \leq 7$, we find that

$$
\left[X_{a}, X_{b}\right]=0 ; \quad\left[X_{a}, X_{u}\right]=0
$$

We have obtained the values of all the Lie brackets non null (see [8]), for example
$\left[X_{4}, X_{5}\right]=-\operatorname{lm}\left(x^{2}+y^{2}+\frac{1}{m}\right) X_{1}+\operatorname{lm}(w z+x y) X_{2}-\operatorname{lm}(w y-x z) X_{3}-2 m x X_{4}+2 m w X_{5}$
For later use, when $m=0$ brackets reduce to

| $[]$, | $X_{4}$ | $X_{5}$ | $X_{6}$ | $X_{7}$ |
| :---: | :---: | :---: | :---: | :---: |
| $X_{4}$ | 0 | $-l X_{1}$ | $-l X_{2}$ | $-l X_{3}$ |
| $X_{5}$ |  | 0 | $-l X_{3}$ | $l X_{2}$ |
| $X_{6}$ |  |  | 0 | $-l X_{1}$ |
| $X_{7}$ |  |  |  | 0 |

Remark 3 When $l=1$, we have the brackets of the quaternionic contact manifold.
As for the Levi-Civita connection in a Riemannian manifold ([12], p. 160) we find

$$
\nabla_{X_{a}} X_{b}=0 ; \quad \nabla_{X_{a}} X_{u}=\nabla_{X_{u}} X_{a}
$$

and for the other nonvanishing ones we get, for instance,

$$
\begin{aligned}
& \nabla_{X_{1}} X_{4}=\frac{l m}{2}\left(y^{2}+z^{2}+\frac{1}{m}\right) X_{5}+\frac{l m}{2}(w z-x y) X_{6}-\frac{l m}{2}(w y+x z) X_{7}, \\
& \nabla_{X_{4}} X_{4}=2 m\left(x X_{5}+y X_{6}+z X_{7}\right), \\
& \nabla_{X_{4}} X_{5}=-\frac{l}{2}\left(1+m\left(y^{2}+z^{2}\right)\right) X_{1}+\frac{l m}{2}(w z+x y) X_{2}-\frac{l m}{2}(w y-x z) X_{3}-2 m x X_{4} .
\end{aligned}
$$

When $m=0$, the Levi-Civita connection reduces to

| $\nabla_{X_{i}} X_{j}$ | $X_{4}$ | $X_{5}$ | $X_{6}$ | $X_{7}$ |
| :---: | :---: | :---: | :---: | :---: |
| $X_{1}$ | $\frac{l}{2} X_{5}$ | $-\frac{l}{2} X_{4}$ | $\frac{l}{2} X_{7}$ | $-\frac{l}{2} X_{6}$ |
| $X_{2}$ | $\frac{l}{2} X_{6}$ | $-\frac{l}{2} X_{7}$ | $-\frac{l}{2} X_{4}$ | $\frac{l}{2} X_{5}$ |
| $X_{3}$ | $\frac{l}{2} X_{7}$ | $\frac{l}{2} X_{6}$ | $-\frac{l}{2} X_{5}$ | $-\frac{l}{2} X_{4}$ |
| $X_{4}$ | 0 | $-\frac{l}{2} X_{1}$ | $-\frac{l}{2} X_{2}$ | $-\frac{l}{2} X_{3}$ |
| $X_{5}$ | $\frac{l}{2} X_{1}$ | 0 | $-\frac{l}{2} X_{3}$ | $\frac{l}{2} X_{2}$ |
| $X_{6}$ | $\frac{l}{2} X_{2}$ | $\frac{l}{2} X_{3}$ | 0 | $-\frac{l}{2} X_{1}$ |
| $X_{7}$ | $\frac{l}{2} X_{3}$ | $-\frac{l}{2} X_{2}$ | $\frac{l}{2} X_{1}$ | 0 |

Remark 4 When $l=1$, we find the Levi-Civita connection of the quaternionic contact manifold.

If $R$ denotes the curvature tensor we can prove that

| $\frac{4}{l^{2}} R$ | $\left(X_{1}, X_{4}\right)$ | $\vdots$ | $\left(X_{6}, X_{7}\right)$ |
| :---: | :--- | :--- | :--- |
| $\left(X_{1}, X_{4}\right)$ | $m^{2}\left[\left(y^{2}+z^{2}+1 / m\right)^{2}+\right.$ <br> $\left.(w z-x y)^{2}+(w y+x z)^{2}\right]$ | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\left(X_{6}, X_{7}\right)$ | $\vdots$ | $\vdots$ | $\frac{16 m}{l^{2}}-3 m^{2}\left[\left(w^{2}+x^{2}+\right.\right.$ <br> $\left.x z)^{2}\right]$ |

Remark 5 When $m=0$, the curvature of the quaternionic contact manifold reduces to

| $R$ | $\left(X_{1}, X_{4}\right)$ | $\vdots$ | $\left(X_{6}, X_{7}\right)$ |
| :---: | :--- | :--- | :--- |
| $\left(X_{1}, X_{4}\right)$ | $\frac{l^{2}}{4}$ | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\left(X_{6}, X_{7}\right)$ | $\vdots$ | $\vdots$ | $-\frac{3 l^{2}}{4}$ |

## 4 The Ricci tensor

Proposition 6 The matrix representing the Ricci tensor is given by

$$
\left[\begin{array}{lllll}
l^{2} / 2\left(K^{2}+1\right) & 0 & 0 & \vdots & \\
0 & l^{2} / 2\left(K^{2}+1\right) & 0 & \vdots & \\
0 & 0 & l^{2} / 2\left(K^{2}+1\right) & \vdots & \\
-l m x(K+2) & -l m y(K+2) & -l m z(K+2) & \vdots & \\
l m w(K+2) & l m z(K+2) & -l m y(K+2) & \vdots & \\
-l m z(K+2) & l m w(K+2) & l m x(K+2) & \vdots & \\
l m y(K+2) & -l m x(K+2) & l m w(K+2) & \vdots & \\
\vdots & -l m x(K+2) & l m w(K+2) & -l m z(K+2) & l m y(K+2) \\
\vdots & -l m y(K+2) & l m z(K+2) & l m w(K+2) & -l m x(K+2) \\
\vdots & -l m z(K+2) & -l m y(K+2) & l m x(K+2) & l m w(K+2) \\
\vdots & A\left(K-1-m w^{2}\right)+B & l^{2} m(K+1) w x & l^{2} m(K+1) w y & l^{2} m(K+1) w z \\
\vdots & l^{2} m(K+1) w x & A\left(K-1-m x^{2}\right)+B & l^{2} m(K+1) x y & l^{2} m(K+1) x z \\
\vdots & l^{2} m(K+1) w y & l^{2} m(K+1) x y & A\left(K-1-m y^{2}\right)+B & l^{2} m(K+1) y z \\
\vdots & l^{2} m(K+1) w z & l^{2} m(K+1) x z & l^{2} m(K+1) y z & A\left(K-1-m z^{2}\right)+B
\end{array}\right]
$$

where $A=-l^{2}(K+1)$ and $B=12 m-3 / 2 l^{2}$.

Using Mathematica we can obtain the eigenvalues of this matrix.

Proposition 7 The eigenvalues for the matrix of Ricci tensor $\lambda_{1}, \lambda_{2}, \lambda_{3}$ whit multiplicity 1,3 and 3 respectively are given by:

$$
\begin{aligned}
& \lambda_{1}=12 m-3 l^{2} / 2 \\
& \lambda_{2}=-\frac{\sqrt{576 m^{2}+\left(16 K^{3}-96 K^{2}-160\right) l^{2} m+\left(9 K^{4}+12 K^{2}+4\right) l^{4}}-24 m+K^{2} l^{2}}{4} \\
& \lambda_{3}=\frac{\sqrt{576 m^{2}+\left(16 K^{3}-96 K^{2}-160\right) l^{2} m+\left(9 K^{4}+12 K^{2}+4\right) l^{4}}+24 m-K^{2} l^{2}}{4}
\end{aligned}
$$

Open question: We are interested in computing the corresponding eigenvectors.

Remark: For arbitrary $m$ and $l$ these eingenvalues are different, but in some special cases they can coincide, for example
a) If $m=0, \lambda_{1}=\lambda_{2}=-\left(3 l^{2}\right) / 2, \lambda_{3}=l^{2}$ and then we obtain the Ricci curvature of the quaternionic contact manifold.
b) If $l=0, \lambda_{1}=12 m, \lambda_{2}=6 m-6|m|$ and $\lambda_{3}=6|m|+m$.
c) If $l=0$ and $m=0$ we have the trivial case: $E B C V$ is nothing but $\mathbb{R}^{7}$.

Corollary 8 The EBCV manifold has constant scalar curvature $S=48 \mathrm{~m}$.

## 5 Killing vector fields in $E B C V$

Remember that a Killing vector field is a vector field on a Riemannian manifold that preserves the metric. Killing vector fields are the infinitesimal generators of isometries, that is, flows generated by Killing fields are continuous isometries of the manifold. Specifically, a vector field $X$ is a Killing vector field if the Lie derivative with respect to $X$ of the metric $g$ vanishes: $\mathcal{L}_{X} g=0$ In terms of the Levi-Civita connection, Killing's condition is equivalent to

$$
\begin{equation*}
g\left(\nabla_{Y} X, Z\right)+g\left(Y, \nabla_{Z} X\right)=0 \tag{4}
\end{equation*}
$$

It is easy to see that $\mathcal{L}_{X} g(Y, Z)=0$ if and only if $\mathcal{L}_{X} g\left(X_{i}, X_{j}\right)=0$ for basic vector fields $X_{i}, X_{j}$. Furthermore, the vertical vector fields $X_{1}, X_{2}, X_{3}$ are Killing ones, while horizontal vector fields $X_{4}, X_{5}, X_{6}, X_{7}$ are not Killing vector fields.

It is well known (see [12], vol. I, p. 238) that an upper bound for the dimension $m$ of the Lie algebra of the Killing vector fields is

$$
m \leq \frac{n(n+1)}{2}
$$

and the maximum is reached on constant curvature manifolds. Then for our manifold $m<28$.

Using results of Wang, [22], Kobayashi proved the following result ([11], p. 47): Let $M$ be an $n$-dimensional Riemannian manifold with $n \neq 4$. Then the group of isometries contains no closed subgroup of dimension $r$ for $\frac{1}{2} n(n-1)+1<r<\frac{1}{2} n(n+1)$. Since in our case $n=7$, then between the dimensions 22 and 28 do not exist closed subgroup of isometries. Just in her thesis Profir [16] proved that the Lie algebra of Killing vector fields is 4-dimensional.

Problem: Determine the space of Killing vector fields in $E B C V$.
By using Killing condition and the values of $\omega^{i}$ and $d \omega^{i}$ we obtain that the Killing vector fields are characterized by a system of partial differential equations ( 28 equations).

We think that it is a difficult problem.

### 5.1 The Killing vector fields for $m=0$

In the usual coordinate system $(r, s, t, w, x, y, z)$ on $E B C V$, a vector field $X=\sum_{i=1}^{7} f_{i} X_{i}$ will be a Killing field if and only if the real functions $f_{i}$ satisfy a difficult system of 28partial differential equations. It seems that the solution of the system is very difficult, so that we focus on solving the system for $m=0$, in such case the solution is given by

$$
\begin{aligned}
& f_{1}(r, s, t, w, x, y, z)=(P+R) s+(S-N) t+\frac{l}{2}\left\{-M\left(w^{2}+x^{2}\right)-U\left(y^{2}+z^{2}\right)+(R-P)(w y+x z)\right. \\
&+(N+S)(w z-x y)+2 T w-2 Q x+2 W y-2 V z\}+C_{1}, \\
& f_{2}(r, s, t, w, x, y, z)=-(P+R) r+(M+U) t-\frac{l}{2}\left\{N\left(w^{2}+y^{2}\right)-S\left(x^{2}+z^{2}\right)+(R-P)(w x-y z)\right. \\
&+(M-U)(w z+x y)-2 V w+2 W x+2 Q y-2 T z\}+C_{2}, \\
& f_{3}(r, s, t, w, x, y, z)=-(S-N) r-(M+U) s-\frac{l}{2}\left\{P\left(w^{2}+z^{2}\right)+R\left(x^{2}+y^{2}\right)+(N+S)(w x+y z)\right. \\
&\quad+(U-M)(w y-x z)-2 W w-2 V x+2 T y+2 Q z\}+C_{3}, \\
& f_{4}(r, s, t, w, x, y, z)=M x+N y+P z+Q, \\
& f_{5}(r, s, t, w, x, y, z)=-M w+R y+S z+T \\
& f_{6}(r, s, t, w, x, y, z)=-N w-R x+U z+V \\
& f_{7}(r, s, t, w, x, y, z)=-P w-S x-U y+W
\end{aligned}
$$

where $M, N, P, Q, R, S, T, U, V, W, C_{1}, C_{2}, C_{3} \in \mathbb{R}$.

As a consequence, when $m=0$, we obtain
Proposition 9 The Lie algebra of Killing vector fields is 13-dimensional.

## 6 The characteristic connection in the EBVC manifold

We now consider in $E B C V$ the characteristic connection $D$. It is well known in the literature and has been extensively used by different authors (see for example [10], [17]):

$$
D_{L} M=\nabla_{L} M+A(L, M)
$$

where $A(L, M)=\frac{P}{2}\left(\nabla_{L} P\right) M$, for any vector fields $L, M$, and $P$ stands for the natural almost product structure. Remember that $P=\mathcal{V}-\mathcal{H}, I d=\mathcal{V}+\mathcal{H}, \mathcal{V}$ and $\mathcal{H}$ being the natural vertical and horizontal projections, respectively. Observe that the vertical distribution in $E B C V$ is spanned by $X_{1}, X_{2}, X_{3}$ and the horizontal distribution by $X_{4}, X_{5}, X_{6}, X_{7}$. Then we have

$$
\begin{aligned}
& D_{X_{a}} X_{b}=\mathcal{V}\left(\nabla_{X_{i}} X_{j}\right)=0, a, b=1,2,3 \\
& D_{X_{u}} X_{a}=\mathcal{V}\left(\nabla_{X_{a}} X_{a}\right)=0, u=4, \ldots, 7 ; a=1,2,3 \\
& D_{X_{a}} X_{u}=\mathcal{H}\left(\nabla_{X_{a}} X_{u}\right)=\nabla_{X_{a}} X_{u}, a=1,2,3 ; u=4, \ldots, 7, \\
& D_{X_{u}} X_{v}=\mathcal{H}\left(\nabla_{X_{u}} X_{v}\right), u, v=4, \ldots, 7
\end{aligned}
$$

Proposition 10 The almost product structure verifies several interesting properties (see for instance [7], [10], [15] among others)
a) $\left\langle\left(\nabla_{L} P\right) P M, P N\right\rangle=-\left\langle\left(\nabla_{L} P\right) M, N\right\rangle$,
b) $\langle P M, P N\rangle=\langle M, N\rangle$,
c) $O\left\langle\left(\nabla_{L} P\right) M, P N\right\rangle=\left\langle\left(\nabla_{O L}^{2} P\right) M, P N\right\rangle+\left\langle\left(\nabla_{\nabla_{O} L} P\right) M, P N\right\rangle$

$$
+\left\langle\left(\nabla_{L} P\right) \nabla_{O} M, P N\right\rangle+\left\langle\left(\nabla_{L} P\right) M, \nabla_{O}(P N)\right\rangle
$$

d) $\left\langle\left(\nabla_{L} P\right) M, N\right\rangle=\left\langle\left(\nabla_{L} P\right) N, M\right\rangle$,
e) $\left\langle\left(\nabla_{L} P\right) A, B\right\rangle=0 ; \quad\left\langle\left(\nabla_{L} P\right) X, Y\right\rangle=0$,
f) $\left\langle\left(\nabla_{L} P\right) A, P B\right\rangle=0 ; \quad\left\langle\left(\nabla_{L} P\right) X, P Y\right\rangle=0$,
$L, M, N, O$ being arbitrary vector fields, $A, B$ vertical vector fields and $X, Y$ horizontal vector fields.

Following the classification given by A.M. Naveira for almost product structures, [15], we have:

Proposition $11(E B C V, P)$ is in $(T G F, A F)$ class.

We have to prove that $\nabla_{L}(P) M=0$ when $L, M$ are vertical and $\nabla_{X}(P) X=0$ if $X$ is horizontal. The result follows using the tables giving in [8] for the Levi-Civita connection.

Proposition $12 D$ is a metric connection.

Proof $D$ is a metric connection if and only if for any basic vector fields $L, M, N$ in $E B C V$ we have

$$
\left\langle D_{L} M, N\right\rangle+\left\langle D_{L} N, M\right\rangle=0 .
$$

Since $\nabla$ is a metric connection, that condition is equivalent to

$$
\langle A(L, M), N\rangle+\langle A(L, N), M\rangle=0
$$

The result follows by we using the results for the Levi-Civita connection.

Proposition 13 The tensor field A satisfies that

$$
\begin{aligned}
& A\left(X_{i}, X_{i}\right)=0, \quad i=1, \ldots, 7, \\
& A\left(X_{a}, X_{i}\right)=0, \quad a=1,2,3, \quad i=1, \ldots, 7 \\
& A\left(X_{4}, X_{1}\right)=-\frac{l}{2}\left\{1+m\left(y^{2}+z^{2}\right)\right\} X_{5}-\frac{m l}{2}(w z-x y) X_{6}+\frac{m l}{2}(w y+x z) X_{7} \\
& \ldots \ldots \ldots
\end{aligned} \begin{aligned}
& A\left(X_{7}, X_{6}\right)=-\frac{l}{2}\left\{\left(1+m\left(w^{2}+x^{2}\right)\right) X_{1}+m(w z+x y) X_{2}-m(w y-x z) X_{3}\right\} .
\end{aligned}
$$

Let $T^{D}$ denote the torsion tensor of $D$, that is,

$$
T_{L}^{D} M \equiv T^{D}(L, M)=D_{L} M-D_{M} L-[L, M]
$$

or even

$$
T^{D}(L, M)=\frac{P}{2}\left(\left(\nabla_{L} P\right) M-\left(\nabla_{M} P\right) L\right)=A(L, M)-A(M, L) .
$$

By using the previous results we find
Corollary 14 The torsion tensor $T^{D}$ satisfies:

$$
\begin{aligned}
T^{D}\left(X_{i}, X_{i}\right) & =0, \quad i=1, \ldots, 7 \\
T^{D}\left(X_{a}, X_{b}\right) & =0, \quad a, b=1,2,3 \\
T^{D}\left(X_{a}, X_{u}\right) & =-T^{D}\left(X_{u}, X_{a}\right), \quad a=1,2,3, \quad u=4, \ldots, 7, \\
T^{D}\left(X_{u}, X_{v}\right) & =2 A\left(X_{u}, X_{v}\right), \quad u, v=4, \ldots, 7 .
\end{aligned}
$$

Now from [1] and [20] we obtain
Corollary 15 The tensor field $A$ is in the class $\mathfrak{T}_{2} \oplus \mathfrak{T}_{3}$.
Proof: It is enough to realize that

$$
c_{12}(A)=\sum A_{X_{i}} X_{i}=0, i=1, \ldots, 7 .
$$

Proposition 16 The connection $D$ does not parallelize $A$.
It is sufficient to prove that $\left(D_{L} A\right)(M, N)$ does not vanish for a particular choice of the vector fields $L, M, N$. Then, for example, we see that

$$
\begin{gathered}
\left(D_{X_{6}} A\right)\left(X_{4}, X_{5}\right)= \\
=\left(K l m y-l m^{2} w^{2} y-l^{2} x^{2} y\right) X_{1}+\left(-\frac{l m x}{2}+\operatorname{lm}^{2} w(w x-y z)+\operatorname{lm} x\left(1+m\left(x^{2}+z^{2}\right)\right) X_{2}\right. \\
+\left(\frac{l m w}{2}-\operatorname{lm} w\left(1+m\left(w^{2}+z^{2}\right)\right)-l m^{2} x(w x+y z)\right) X_{3} \neq 0 .
\end{gathered}
$$

As for the curvature tensor of $D$ we have
Proposition 17 The curvature $R^{D}$ of the connection $D$ is given by

$$
\begin{aligned}
R^{D}\left(X_{1}, X_{2}, X_{1}, X_{2}\right) & =0, \\
\ldots \ldots \ldots & R^{D}\left(X_{1}, X_{2}, X_{3}, X_{4}\right)=0, \\
R^{D}\left(X_{1}, X_{2}, X_{4}, X_{5}\right) & =-\frac{m l^{2}}{4}(K-1)(w y-x z), \\
\ldots \ldots \cdots & \\
R^{D}\left(X_{1}, X_{4}, X_{4}, X_{5}\right) & =-l m w, \\
\ldots \ldots & R^{D}\left(X_{1}, X_{4}, X_{5}, X_{6}\right)=\frac{K l m}{2} y, \\
R^{D}\left(X_{4}, X_{5}, X_{4}, X_{5}\right) & =4 m, \quad R^{D}\left(X_{4}, X_{5}, X_{6}, X_{7}\right)=0 .
\end{aligned}
$$

Proposition 18 The connection $D$ does not parallelize $R^{D}$.

It is enough to show that for a particular choice of vector fields $D R^{D} \neq 0$. Indeed,

$$
\left(D_{X_{4}} R^{D}\right)\left(X_{1}, X_{4}, X_{5}, X_{6}\right)=-\operatorname{lm}\left[m\left(K x z-y^{2}+2(K+3) w y\right)-\frac{K}{2}\right] \neq 0 .
$$

OPEN QUESTION: We are interested in proving that $E B C V$ is a homogeneous manifold. To do that using the results of Tricerri-Vanhecke (see [20], [21]) we have to find a tensor $A$ such that $D=\nabla+A$ must satisfy the following three conditions

$$
D g=0, \quad D A=0, \quad D R^{D}=0
$$

In [20] Tricerri and Vanhecke solve this question for the 3-dimensional Heisenberg space through the existence of an antisymmetric tensor $T$ of type (1,2). In our case we have tried to generalize this method by using different tensors, such as the antisymmetric tensor $A$, but we still have not succeeded. However, an affirmative answer is obtained for the 7-dimensional Heisenberg group which corresponds to $m=0$.

It seems that this is a difficult problem. The given natural tensor $A=$ $\frac{P}{2}\left(\nabla_{L} P\right) M$ does not verify these conditions.

As for the geodesics we obtain

Proposition 19 The connections $\nabla$ and $D$ have not the same geodesics.

Proof: Having the same geodesics is equivalent to say that A is antisymmetric, but does not hold, since $A\left(X_{2}, X_{6}\right) \neq A\left(X_{6}, X_{2}\right)$.

Remark By considering just either the vertical distribution or the horizontal distribution we obtain:
(i) If $M$ is vertical then $\left(\nabla_{M} P\right) M=0$.
(ii) If $M$ is horizontal, since $\nabla_{M} M$ is also horizontal, we have

$$
\left(\nabla_{M} P\right) M=-\nabla_{M} M+\nabla_{M} M=0 .
$$

(iii) However, for mixed vector fields $M=A+X$, where $A$ is vertical and $X$ is horizontal, then above result does not hold. Indeed, taking $M=X_{2}+X_{5}$ we get

$$
\left\langle\left(D_{X_{2}+X_{5}} P\right)\left(X_{2}+X_{5}\right), X_{1}+X_{4}\right\rangle \neq 0 .
$$

Remark: In [9] the authors have calculated the horizontal geodesic equations and the corresponding solutions for the quaternionic contact manifold in dimension 7.

## 7 Curvature in sub-Riemannian geometry

Let $\nabla^{\mathcal{H}}$ denote the horizontal connection, the restriction of the Levi-Civita connection $\nabla$ to the horizontal distribution generated by $\left\{X_{4}, X_{5}, X_{6}, X_{7}\right\}$, which is metric. After some easy calculations we obtain

$$
\begin{aligned}
\nabla_{X_{4}}^{\mathcal{H}} X_{4} & =2 m x X_{5}+2 m y X_{6}+2 m z X_{7}, \\
\nabla_{X_{4}}^{\mathcal{H}} X_{5} & =-2 m x X_{4}, \\
\nabla_{X_{4}}^{\mathcal{H}} X_{6} & =-2 m y X_{4}, \\
\nabla_{X_{4}}^{\mathcal{H}} X_{7} & =-2 m z X_{4},
\end{aligned}
$$

and so on.
We also have

$$
\begin{array}{ll}
{\left[X_{4}, X_{5}\right]^{\mathcal{H}}=-2 m x X_{4}+2 m w X_{5},} & {\left[X_{4}, X_{6}\right]^{\mathcal{H}}=-2 m y X_{4}+2 m w X_{6},} \\
{\left[X_{4}, X_{7}\right]^{\mathcal{H}}=-2 m z X_{4}+2 m w X_{7},} & {\left[X_{5}, X_{6}\right]^{\mathcal{H}}=-2 m y X_{5}+2 m x X_{6},}
\end{array}
$$

and so on.

It is easy to see that the torsion of the horizontal connection $\nabla^{\mathcal{H}}$ satisfies

$$
T^{\mathcal{H}}\left(X_{u}, X_{v}\right)=0, \quad u, v=4, \ldots, 7 .
$$

As for the curvature of the horizontal connection $\nabla^{\mathcal{H}}$ we remember that

$$
\begin{aligned}
R^{\mathcal{H}}(A, B, C, D) & =-A\left\langle\nabla_{B}^{\mathcal{H}} C, D\right\rangle+B\left\langle\nabla_{A}^{\mathcal{H}} C, D\right\rangle \\
& +\left\langle\nabla_{B}^{\mathcal{H}} C, \nabla_{A}^{\mathcal{H}} D\right\rangle-\left\langle\nabla_{A}^{\mathcal{H}} C, \nabla_{B}^{\mathcal{H}} D\right\rangle+\left\langle\nabla_{[A, B]^{\mathcal{H}}} C, D\right\rangle .
\end{aligned}
$$

Proposition 20 The only non-zero components of the curvature of the horizontal connection are

$$
R^{\mathcal{H}}\left(X_{u}, X_{v}, X_{u}, X_{v}\right)=-R^{\mathcal{H}}\left(X_{u}, X_{v}, X_{v}, X_{u}\right)=4 m, u, v=4, \ldots, 7, u<v .
$$

Corollary 21 The curvature $R^{\mathcal{H}}$ is parallel whit respect to $\nabla^{\mathcal{H}}$.

Remark: $\quad \nabla^{\mathcal{H}} g=0, \quad \nabla^{\mathcal{H}} T^{\mathcal{H}}=0, \quad \nabla^{\mathcal{H}} R^{\mathcal{H}}=0$.

Following [2], we denote by $\mathcal{H}$ and $\mathcal{V}$ the projection morphisms of $T(E B C V)$ on $\mathcal{H}(E B C V)$ and $\mathcal{V}(E B C V)$, respectively, and define the mapping

$$
\begin{gathered}
F: \Gamma\left(\mathcal{H}(E B C V)^{2} \times \Gamma(\mathcal{V}(E B C V) \rightarrow \mathcal{F}(E B C V)\right. \\
F(\mathcal{H} L, \mathcal{H} M, \mathcal{V} N)=\mathcal{V} N(g(\mathcal{H} L, \mathcal{H} M))-g(\mathcal{H}[\mathcal{V} N, \mathcal{H} L], \mathcal{H} M)-g(\mathcal{H}[\mathcal{V} N, \mathcal{H} M], \mathcal{H} L)
\end{gathered}
$$

for all $L, M, N \in T(E B C V)$. Then, it is easy to check that $F$ is an adapted tensor field on $E B C V$.

Definition 22 We say that the sub-Riemannian manifold $(\mathcal{M}, \mathcal{H} \mathcal{M}, g, \mathcal{V M})$ is a nearly Riemannian manifold if the adapted tensor field $F$ vanishes identically on $\mathcal{M}$.

Corollary 23 The subriemannian manifold ( $E B C V, \mathcal{H}(E B C V), g, \mathcal{V}(E B C V)$ is a nearly Riemannian manifold.

Indeed, for our manifold, $F(\mathcal{H} L, \mathcal{H} M, \mathcal{V} N)=0$, for all $L, M, N \in T(E B C V)$.

Acknowledgements. We wish to thank Professors F. J. Carreras, V. Miquel and U. Semmelmann for his valuable and enlightening comments on the subject of this talk. AF has been partially supported by MINECO/FEDER grant MTM2015-65430-P and Fundación Séneca project 19901/GERM/15. AMN has been partially supported by MINECO-FEDER grant MTM2016-77093-P and Generalitat Valenciana Project PROMETEOII/2014/064. ADT has been partially supported by Red IEMath-Galicia, reference CN 2012/077, Spain.

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