# On 7-dimensional Bianchi-Cartan-Vranceanu spaces 

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## 1 Introduction (see [8])

A subriemannian geometry is a manifold endowed with a distribution and a fiber inner product on that distribution.

A distribution here means a family of $k$-planes, that is, a linear sub-bundle of the tangent bundle of the manifold. We refer to the distribution as the horizontal space, and the objects tangent to it as "horizontal".

Given such geometry we can define the distance between two points just as in Riemannian geometry, except that we are only allowed to travel about horizontal curves when we joining two points.

The simplest non trivial subriemannian geometry is called Heisenberg group [10].
It is very interesting to study the relation between this subriemannian geometry and the classical isoperimetric problem, where the following 1-form has a important rolle:

$$
\omega=d z-\frac{1}{2}(x d y-y d x)
$$

we can write

$$
\begin{aligned}
\mathcal{H} & =\operatorname{Ker} \omega=\{\omega(x, y, z)=0\} \\
& =\left\{\left(v_{1}, v_{2}, v_{3}\right): v_{3}-\frac{1}{2}\left(x v_{2}-y v_{1}\right)=0\right\} \subset \mathbb{R}^{3} .
\end{aligned}
$$

This $\mathcal{H}$ is a field of two-planes in three-space.
The restriction of the usual metric to these two-planes defines a smoothly varying family of inner products
Definition $1 \mathbb{R}^{3}$ endowed with the structure of this distribution $\mathcal{H}$ and this family of inner products $\mathrm{ds}^{2}$ on $\mathcal{H}$ is called the Heisenberg group (complex contact group).

## The Bianchi-Cartan-Vranceanu (BCV) spaces (see [4])

For real numbers $l$ and $m$, with $l \geq 0$, consider the set

$$
B C V(l, m)=\left\{(x, y, z) \in \mathbb{R}^{3}: 1+m\left(x^{2}+y^{2}\right)>0\right\}
$$

equipped with the metric

$$
\mathrm{ds}_{l, m}^{2}=\frac{d x^{2}+d y^{2}}{\left(1+m\left(x^{2}+y^{2}\right)\right)^{2}}+\left(d z+\frac{l}{2} \frac{x d y-y d x}{1+m\left(x^{2}+y^{2}\right.}\right)^{2} .
$$

Observe that this metric is obtained as a conformal deformation of the planar Euclidean metric by adding the imaginary part of $z d \bar{z}$, for a complex number $z$.

Take the vector fields $E_{i}$ and its corresponding dual 1-forms $\omega^{j}$

$$
\begin{array}{ll}
E_{1}=\left(1+m\left(x^{2}+y^{2}\right)\right) \partial_{x}-\frac{l}{2} y \partial_{z} & \omega^{1}
\end{array}=\frac{d x}{1+m\left(x^{2}+y^{2}\right)}, ~ \omega^{2}=\frac{d y}{1+m\left(x^{2}+y^{2}\right)}, ~ \omega_{2}=\left(1+m\left(x^{2}+y^{2}\right)\right) \partial_{y}+\frac{l}{2} x \partial_{z} \quad=d z+\frac{l}{2} \frac{y d x-x d y}{\left(1+m\left(x^{2}+y^{2}\right)\right.}
$$

Let $\mathcal{D}$ be the distribution generated by $\left\{E_{1}, E_{2}\right\}$. The manifold $\left(B C V(l, m), \mathcal{D}, \mathrm{ds}_{l, m}^{2}\right)$ is called a Bianchi-Cartan-Vranceanu (BCV for short) space ( $[1,2,3,12]$ ), which is an example of sub-riemannian geometry (see $[4,10]$ ) and the horizontal distribution is a 2 -step breaking-generating distribution everywhere.

## 2 Extended Bianchi-Cartan-Vranceanu spaces

Observe that letting $z=x+i y$, we see that $\operatorname{Im}(z d \bar{z})=y d x-x d y$, which reminds us the map $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R} \times \mathbb{C}$ given by $\left(z_{1}, z_{2}\right) \mapsto\left(\left|z_{1}\right|^{2}-\left|z_{2}\right|^{2}, 2\left(z_{1} \bar{z}_{2}\right)\right)$, that easily leads to the classical Hopf fibration $\mathbb{S}^{1} \hookrightarrow \mathbb{S}^{3} \rightarrow \mathbb{S}^{2}$, where coordinates in $\mathbb{S}^{2}$ are given by $\left(\left|z_{1}\right|^{2}-\left|z_{2}\right|^{2}, 2 \operatorname{Re}\left(z_{1} z_{2}\right), 2 \operatorname{Im}\left(z_{1} z_{2}\right)\right)$.

In the same line we get the fibration $\mathbb{S}^{3} \hookrightarrow \mathbb{S}^{7} \rightarrow \mathbb{S}^{4}$, but using quaternions $\mathbb{H}$ instead of complex numbers. Quaternions are usually presented with the imaginary units $i, j, k$ in the form $q=x_{0}+x_{1} i+x_{2} j+x_{3} k, x_{0}, x_{1}, x_{2}, x_{3} \in \mathbb{R}$ with $i^{2}=j^{2}=k^{2}=i j k=-1$. They can also be defined equivalently, using the complex numbers $c_{1}=x_{0}+x_{1} i$ and $c_{2}=x_{2}+x_{3} i$, in the form $q=c_{1}+c_{2} j$. Then for a point $\left(q_{1}=\alpha+\beta j, q_{2}=\gamma+\delta j\right) \in \mathbb{S}^{7}$, we get the following coordinate expressions $\left(\left|q_{1}\right|^{2}-\left|q_{2}\right|^{2}, 2 \operatorname{Re}(\bar{\alpha} \gamma+\bar{\beta} \delta), 2 \operatorname{Im}(\bar{\alpha} \gamma+\bar{\beta} \delta), 2 \operatorname{Re}(\alpha \delta-\right.$ $\beta \gamma), 2 \operatorname{Im}(\alpha \delta-\beta \gamma))$.

For any $q=w+x i+y j+z k \in \mathbb{H}$ we find that $q d \bar{q}=w d w+x d x+y d y+z d z+(x d w-$ $w d x+z d y-y d z) i+(y d w-w d y+x d z-z d x) j+(z d w-w d z+y d x-x d y) k$. As the
quaternionic contact group $\mathbb{H} \times \operatorname{Im} \mathbb{H}$, with coordinates $(w, x, y, z, r, s, t)$ can be equipped with the metric

$$
\begin{aligned}
\mathrm{ds}^{2}= & \left(d w^{2}+d x^{2}+d y^{2}+d z^{2}\right)+\left(d r+\frac{1}{2}(x d w-w d x+z d y-y d z)\right)^{2} \\
& +\left(d s+\frac{1}{2}(y d w-w d y+x d z-z d x)\right)^{2}+\left(d t+\frac{1}{2}(z d w-w d z+y d x-x d y)\right)^{2}
\end{aligned}
$$

Then, by extending this metric, it seems natural to find a 7 -dimensional generalization of the 3 -dimensional $B C V$ spaces endowed with the two-parameter family of metrics

$$
\begin{aligned}
\mathrm{ds}_{l, m}^{2}= & \frac{d w^{2}+d x^{2}+d y^{2}+d z^{2}}{K^{2}}+\left(d r+\frac{l}{2} \frac{w d x-x d w+y d z-z d y}{K}\right)^{2} \\
& +\left(d s+\frac{l}{2} \frac{w d y-y d w+z d x-x d z}{K}\right)^{2}+\left(d t+\frac{l}{2} \frac{w d z-z d w+x d y-y d x}{K}\right)^{2},
\end{aligned}
$$

where $l, m$ are real numbers and $K=1+m\left(w^{2}+x^{2}+y^{2}+z^{2}\right)$.
That metric is obtained as a conformal deformation of the Euclidean metric of $\mathbb{R}^{4}$ by adding three suitable terms which depend on $l$ and $m$ concerning the imaginary part of $q d \bar{q}$, for a quaternion $q$. When $m=0$ we get a one-parameter of Riemannian metrics depending on $l$. Furthermore, if $l=1$, we find the 7 -dimensional quaternionic Heisenberg group (see [7] and [13]).

That metric can also be written as

$$
\mathrm{ds}_{l, m}^{2}=\sum_{i=1}^{7} \omega^{i} \otimes \omega^{i}
$$

where

$$
\begin{array}{rlrl}
\omega^{1}=d r+\frac{l}{2 K}(w d x-x d w+y d z-z d y), & & \omega^{4}=d w / K, \\
\omega^{2}=d s+\frac{l}{2 K}(w d y-y d w+z d x-x d z), & \omega^{5}=d x / K, \\
\omega^{3}=d t+\frac{l}{2 K}(w d z-z d w+x d y-y d x), & \omega^{6}=d y / K, \\
\omega^{7}=d z / K, \\
X_{1}=\partial_{r}, \quad X_{2}=\partial_{s}, & X_{3}=\partial_{t}, & \\
X_{4}=K \partial_{w}+\frac{l x}{2} \partial_{r}+\frac{l y}{2} \partial_{s}+\frac{l z}{2} \partial_{t}, \quad X_{5}=K \partial_{x}-\frac{l w}{2} \partial_{r}-\frac{l z}{2} \partial_{s}+\frac{l y}{2} \partial_{t}, \\
X_{6}=K \partial_{y}+\frac{l z}{2} \partial_{r}-\frac{l w}{2} \partial_{s}-\frac{l x}{2} \partial_{t}, \quad X_{7}=K \partial_{z}-\frac{l y}{2} \partial_{r}+\frac{l x}{2} \partial_{s}-\frac{l w}{2} \partial_{t} .
\end{array}
$$

Then we find that

Lemma $2\left\{X_{1}, X_{2}, \cdots, X_{7}\right\}$ is an orthonormal basis of vector fields whit respect to the given metric $\mathrm{ds}_{l, m}^{2}:=\langle$,$\rangle .$

Let $\mathcal{D}$ be the distribution generated by $\left\{X_{4}, X_{5}, X_{6}, X_{7}\right\}$.
The manifold $\left(E B C V(l, m), \mathcal{D}, \mathrm{ds}_{l, m}^{2}\right)$ will be called Extended Bianchi-Cartan-Vranceanu space (EBCV for short) and provides a new example of sub-riemannian geometry, $\mathcal{D}$, the horizontal distribution, is a 2-step breaking-generating distribution everywhere.

For the Lie brackets writing $1 \leq a, b \leq 3 ; 4 \leq u, v \leq 7$, we find that

$$
\left[X_{a}, X_{b}\right]=0 ; \quad\left[X_{a}, X_{u}\right]=0
$$

as well as

| [, ] | $X_{4}$ | $X_{5}$ | $X_{6}$ | $X_{7}$ |
| :---: | :---: | :---: | :---: | :---: |
| $X_{4}$ | 0 | $\begin{aligned} & -\operatorname{lm}\left(x^{2}+y^{2}+\frac{1}{m}\right) X_{1}+ \\ & \operatorname{lm}(w z+x y) X_{2}- \\ & \operatorname{lm}(w y-x z) X_{3}- \\ & 2 m x X_{4}+2 m w X_{5} \end{aligned}$ | $\begin{aligned} & -\operatorname{lm}(w z-x y) X_{1}- \\ & \operatorname{lm}\left(x^{2}+z^{2}+\frac{1}{m}\right) X_{2}+ \\ & \operatorname{lm}(w x+y z) X_{3}- \\ & 2 m y X_{4}+2 m w X_{6} \end{aligned}$ | $\begin{aligned} & \operatorname{lm}(w y+x z) X_{1}- \\ & \operatorname{lm}(w x-y z) X_{2}- \\ & \operatorname{lm}\left(x^{2}+y^{2}+\frac{1}{m}\right) X_{3}- \\ & 2 m z X_{4}+2 m w X_{7} \end{aligned}$ |
| $X_{5}$ |  | 0 | $\begin{aligned} & -\operatorname{lm}(w y+x z) X_{1}+ \\ & \operatorname{lm}(w x-y z) X_{2}- \\ & \operatorname{lm}\left(\frac{1}{m}+w^{2}+z^{2}\right) X_{3}- \\ & 2 m y X_{5}+2 m x X_{6} \end{aligned}$ | $\begin{aligned} & \operatorname{lm}(x y-w z) X_{1}+ \\ & \operatorname{lm}\left(w^{2}+y^{2}+\frac{1}{m}\right) X_{2}+ \\ & \operatorname{lm}(w x+y z) X_{3}- \\ & 2 m z X_{5}+2 m x X_{7} \end{aligned}$ |
| $X_{6}$ |  |  | 0 | $\begin{aligned} & -\operatorname{lm}\left(w^{2}+x^{2}+\frac{1}{m}\right) X_{1}- \\ & \operatorname{lm}(w z+x y) X_{2}+ \\ & \operatorname{lm}(w y-x z) X_{3}- \\ & 2 m z X_{6}+2 m y X_{7} \end{aligned}$ |
| $X_{7}$ |  |  |  | 0 |

For later use, when $m=0$ brackets reduce to

| $[]$, | $X_{4}$ | $X_{5}$ | $X_{6}$ | $X_{7}$ |
| :---: | :---: | :---: | :---: | :---: |
| $X_{4}$ | 0 | $-l X_{1}$ | $-l X_{2}$ | $-l X_{3}$ |
| $X_{5}$ |  | 0 | $-l X_{3}$ | $l X_{2}$ |
| $X_{6}$ |  |  | 0 | $-l X_{1}$ |
| $X_{7}$ |  |  |  | 0 |

Remark 3 When $l=1$, we have the brackets of the quaternionic Heisenberg group.

Definition 4 ([4])Let $M$ be a differentiable manifold of dimension $m+p$. A subriemannian manifold $(M, \mathcal{F}, g)$ is called Heisenberg manifold if

0 . It is step 2 everywhere

1. There are $m$ locally defined vector fields $X_{1}, \cdots, X_{m}$ on $M$ such that

$$
\mathcal{F}=\operatorname{span}\left\{X_{1}, \cdots, X_{m}\right\}
$$

2. The vector fields of $\mathcal{F}$ are orthonormal.
3. There are $p$ locally defined one-forms $\omega^{\alpha}$ whit $\omega^{\alpha}\left(X_{i}\right)=0$ which satisfies the nonvanishing conditions

$$
\operatorname{det}\left(\omega^{\alpha}\left[X_{i}, X_{j}\right]\right) \neq 0 ; \quad 1 \leq i, j \leq m ; \quad 1 \leq \alpha \leq p
$$

4. If the vector fields $\left\{X_{i}\right\},\left\{Y_{j}\right\}, 1 \leq i, j \leq m$ are defined in the local charts $U$ and $U^{\prime}$ respectively, then the distributions match on the overlap

$$
\operatorname{span}\left\{X_{1}, \cdots, X_{m}\right\}_{q}=\operatorname{span}\left\{Y_{1}, \cdots, Y_{m}\right\}_{q}, \text { for } \quad q \in U \cap U^{\prime}
$$

Corollary $5\left(E B C V(l, m), \mathcal{D}, \mathrm{ds}_{l, m}^{2}\right)$ is a Heisenberg manifold whit $p=3$ and $\left\{\omega^{\alpha}\right\}=$ $\left\{\omega^{1}, \omega^{2}, \omega 3\right\}$.

Let's note that condition 3 is satisfied using previous table of Lie brackets and the definition of $\omega^{1}, \omega^{2}, \omega^{3}$, that is

$$
\operatorname{det}\left(\omega^{\alpha}\left[X_{i}, X_{j}\right]\right)=l^{2} K^{4} \neq 0,1 \leq i, j \leq m,=1,1 \leq \alpha \leq 3 ;
$$

In [4] we can see more examples of Heisenberg manifolds and interesting properties that our manifold satisfies, particularly we can define the volume element $d v$ for a Heisenberg manifold.

For the EBCV manifold it has:

$$
d v=f d \omega_{1} \wedge \cdots \wedge d \omega_{7}
$$

For the Levi-Civita connection in a Riemannian manifold ([6], p. 160) we find

$$
\nabla_{X_{a}} X_{b}=0 ; \quad \nabla_{X_{a}} X_{u}=\nabla_{X_{u}} X_{a}
$$

and

| $\nabla_{X_{i}} X_{j}$ | $X_{4}$ | $X_{5}$ | $X_{6}$ | $X_{7}$ |
| :---: | :---: | :---: | :---: | :---: |
| $X_{1}$ | $\begin{aligned} & \frac{m}{2}\left(y^{2}+z^{2}+\right. \\ & \left.\frac{1}{m}\right) X_{5}+\frac{l m}{2}(w z- \\ & x y) X_{6}-\frac{l m}{2}(w y+ \\ & x z) X_{7} \end{aligned}$ | $\begin{aligned} & -\frac{l m}{2}\left(y^{2}+z^{2}+\right. \\ & \left.\frac{1}{m}\right) X_{4}+\frac{l m}{2}(w y+ \\ & x z) X_{6}+\frac{l m}{2}(w z- \\ & x y) X_{7} \end{aligned}$ | $\begin{aligned} & -\frac{l m}{2}(w z \\ & x y) X_{4}-\frac{l m}{2}(w y+ \\ & x z) X_{5}+\frac{l m}{2}\left(w^{2}+\right. \\ & \left.x^{2}+\frac{1}{m}\right) X_{7} \end{aligned}$ | $\begin{aligned} & \frac{l m}{2}(w y+x z) X_{4}- \\ & \frac{l m}{2}(w z-x y) X_{5}- \\ & \frac{l m}{2}\left(w^{2}+x^{2}+\right. \\ & \left.\frac{1}{m}\right) X_{6} \end{aligned}$ |
| : | : |  | : |  |
| $X_{7}$ | $\begin{aligned} & -\frac{l m}{2}(w y \\ & x z) X_{1}+\frac{l m}{2}(w x- \\ & y z) X_{2}+\frac{l m}{2}\left(x^{2}+\right. \\ & \left.y^{2}+\frac{1}{m}\right) X_{3}- \\ & 2 m w X_{7} \\ & \hline \end{aligned}$ | $\begin{aligned} & \frac{l m}{2}(w z-x y) X_{1}- \\ & \frac{l m}{2}\left(w^{2}+y^{2}+\right. \\ & \left.\frac{1}{m}\right) X_{2}-\frac{l m}{2}(w x+ \\ & y z) X_{3}-2 m x X_{7} \end{aligned}$ | $\begin{aligned} & \frac{l m}{2}\left(w^{2}+x^{2}+\right. \\ & \left.\frac{1}{m}\right) X_{1}+\frac{l m}{2}(w z+ \\ & x y) X_{2}-\frac{l m}{2}(w y- \\ & x z) X_{3}-2 m y X_{7} \end{aligned}$ | $\begin{aligned} & 2 m\left(w X_{4}+x X_{5}+\right. \\ & \left.y X_{6}\right) \end{aligned}$ |

When $m=0$, the Levi-Civita connection reduces to

| $\nabla_{X_{i}} X_{j}$ | $X_{4}$ | $X_{5}$ | $X_{6}$ | $X_{7}$ |
| :---: | :---: | :---: | :---: | :---: |
| $X_{1}$ | $\frac{l}{2} X_{5}$ | $-\frac{l}{2} X_{4}$ | $\frac{l}{2} X_{7}$ | $-\frac{l}{2} X_{6}$ |
| $X_{2}$ | $\frac{l}{2} X_{6}$ | $-\frac{l}{2} X_{7}$ | $-\frac{l}{2} X_{4}$ | $\frac{l}{2} X_{5}$ |
| $X_{3}$ | $\frac{l}{2} X_{7}$ | $\frac{l}{2} X_{6}$ | $-\frac{l}{2} X_{5}$ | $-\frac{l}{2} X_{4}$ |
| $X_{4}$ | 0 | $-\frac{l}{2} X_{1}$ | $-\frac{l}{2} X_{2}$ | $-\frac{l}{2} X_{3}$ |
| $X_{5}$ | $\frac{l}{2} X_{1}$ | 0 | $-\frac{l}{2} X_{3}$ | $\frac{l}{2} X_{2}$ |
| $X_{6}$ | $\frac{l}{2} X_{2}$ | $\frac{l}{2} X_{3}$ | 0 | $-\frac{l}{2} X_{1}$ |
| $X_{7}$ | $\frac{l}{2} X_{3}$ | $-\frac{l}{2} X_{2}$ | $\frac{l}{2} X_{1}$ | 0 |

Remark 6 When $l=1$, we find the Levi-Civita connection of the quaternionic Heisenberg group.

If $R$ denotes the curvature tensor we can prove that

| $\frac{4}{l^{2}} R$ | $\left(X_{1}, X_{4}\right)$ | $\vdots$ | $\left(X_{6}, X_{7}\right)$ |
| :---: | :--- | :--- | :--- |
| $\left(X_{1}, X_{4}\right)$ | $m^{2}\left[\left(y^{2}+z^{2}+1 / m\right)^{2}+\right.$ <br> $\left.(w z-x y)^{2}+(w y+x z)^{2}\right]$ | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\left(X_{6}, X_{7}\right)$ | $\vdots$ | $\vdots$ | $\frac{16 m}{l^{2}}-3 m^{2}\left[\left(w^{2}+x^{2}+\right.\right.$ <br> $\left.x z)^{2}\right]$ |

Remark 7 When $m=0$, the curvature of the quaternionic Heisenberg group reduces to

| $R$ | $\left(X_{1}, X_{4}\right)$ | $\vdots$ | $\left(X_{6}, X_{7}\right)$ |
| :---: | :--- | :--- | :--- |
| $\left(X_{1}, X_{4}\right)$ | $\frac{l^{2}}{4}$ | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\left(X_{6}, X_{7}\right)$ | $\vdots$ | $\vdots$ | $-\frac{3 l^{2}}{4}$ |

## 3 The curvature of the horizontal distribution in EBCV (see [8])

The Riemannian curvature tensor dominates discussions of Riemannian geometry. It is the local invariant of a Riemannian metric. What is the analogue of the Riemannian curvature tensor for subriemannian geometry? There is no good analogue. What we will call the curvature depends only on the distribution, and not at all on the metric.

Definition 8 The curvature of a distribution $\mathcal{H}$ is the linear bundle map

$$
F: \wedge^{2} \mathcal{H} \rightarrow T Q / \mathcal{H}
$$

defined by $F(X, Y)=-[X, Y] \bmod \mathcal{H}, \quad X, Y \in \mathcal{H}$.
This map is tensorial, that is, $F(X, Y)(q)$ depends only on the vectors $X(q), Y(q) \in \mathcal{H}$ and not on how they are extended to form horizontal vector fields $X, Y$.

Thurston [11] calls the curvature "twistedness" of the distribution. Other authors call it the "nonintegrability" tensor. We call $F$ the curvature of the distribution $\mathcal{H}$.

We can see ([8]) what is the value of $F$ in some interesting examples such as:
-Principal bundle connections

- Involutive distributions
- Contact distributions

Remark 9 Let $\mathcal{H}$ be the distribution spanned by $\left\{X_{4}, X_{5}, X_{6}, X_{7}\right\}$ in $E B C V$. Then it is of codimension 3, that is, $T(E B C V) / \mathcal{H}$ is a real rank 3 bundle.

Remark 10 Using Lema 2, the curvature of the distribution $\mathcal{H}=\left\{X_{4}, X_{5}, X_{6}, X_{7}\right\}$ in the $E B C V$ manifold is given by

| $F\left(X_{u}, X_{v}\right)$ | $X_{4}$ | $X_{5}$ | $X_{6}$ | $X_{7}$ |
| :---: | :---: | :--- | :--- | :--- |
| $X_{4}$ | 0 | $\operatorname{lm}\left(x^{2}+y^{2}+\frac{1}{m}\right) X_{1}-$ <br> $\operatorname{lm}(w z+x y) X_{2}+$ <br> $l m(w y-x z) X_{3}$ | $\operatorname{lm}(w z-x y) X_{1}+$ <br> $\operatorname{lm}\left(x^{2}+z^{2}+\frac{1}{m}\right) X_{2}-$ <br> $\operatorname{lm}(w x+y z) X_{3}$ | $-\operatorname{lm}(w y+x z) X_{1}+$ <br> $\operatorname{lm}(w x-y z) X_{2}+$ <br> $\operatorname{lm}\left(x^{2}+y^{2}+\frac{1}{m}\right) X_{3}$ |
| $X_{5}$ |  | 0 | $\operatorname{lm}(w y+x z) X_{1}-$ <br> $\operatorname{lm}(w x-y z) X_{2}+$ <br> $l m\left(\frac{l m}{}(x y-w z) X_{1}-\right.$ <br> $\operatorname{lm}\left(\frac{1}{m}+w^{2}+z^{2}\right) X_{3}$ | $\operatorname{lm}\left(w^{2}+y^{2}+\right.$ <br> $\left.\frac{1}{m}\right) X_{2}-\operatorname{lm}(w x+$ <br> $y z) X_{3}$ |
| $X_{6}$ |  |  | 0 | $\operatorname{lm}\left(w^{2}+x^{2}+\right.$ <br> $\left.\frac{1}{m}\right) X_{1}+\operatorname{lm}(w z+$ <br> $x y) X_{2}-\operatorname{lm}(w y-$ <br> $x z) X_{3}$ |
| $X_{7}$ |  |  |  | 0 |

For later use, when $m=0$ the curvature of the distribution $\mathcal{H}$ reduces to

| $F()$, | $X_{4}$ | $X_{5}$ | $X_{6}$ | $X_{7}$ |
| :---: | :---: | :---: | :---: | :---: |
| $X_{4}$ | 0 | $l X_{1}$ | $l X_{2}$ | $l X_{3}$ |
| $X_{5}$ |  | 0 | $l X_{3}$ | $-l X_{2}$ |
| $X_{6}$ |  |  | 0 | $l X_{1}$ |
| $X_{7}$ |  |  |  | 0 |

## 4 The Ricci tensor in EBCV

Proposition 11 The matrix representing the Ricci tensor is given by

$$
\left[\begin{array}{lllll}
l^{2} / 2\left(K^{2}+1\right) & 0 & 0 & \vdots & \\
0 & l^{2} / 2\left(K^{2}+1\right) & 0 & \vdots & \\
0 & 0 & l^{2} / 2\left(K^{2}+1\right) & \vdots & \\
-l m x(K+2) & -l m y(K+2) & -l m z(K+2) & \vdots & \\
l m w(K+2) & l m z(K+2) & -l m y(K+2) & \vdots & \\
-l m z(K+2) & l m w(K+2) & l m x(K+2) & \vdots & \\
l m y(K+2) & -l m x(K+2) & l m w(K+2) & \vdots & \\
\vdots & -l m x(K+2) & l m w(K+2) & -l m z(K+2) & l m y(K+2) \\
\vdots & -l m y(K+2) & l m z(K+2) & l m w(K+2) & -l m x(K+2) \\
\vdots & -l m z(K+2) & -l m y(K+2) & l m x(K+2) & l m w(K+2) \\
\vdots & A\left(K-1-m w^{2}\right)+B & l^{2} m(K+1) w x & l^{2} m(K+1) w y & l^{2} m(K+1) w z \\
\vdots & l^{2} m(K+1) w x & A\left(K-1-m x^{2}\right)+B & l^{2} m(K+1) x y & l^{2} m(K+1) x z \\
\vdots & l^{2} m(K+1) w y & l^{2} m(K+1) x y & A\left(K-1-m y^{2}\right)+B & l^{2} m(K+1) y z \\
\vdots & l^{2} m(K+1) w z & l^{2} m(K+1) x z & l^{2} m(K+1) y z & A\left(K-1-m z^{2}\right)+B
\end{array}\right]
$$

where $A=-l^{2}(K+1)$ and $B=12 m-3 / 2 l^{2}$.

Some particular cases could be interesting, for instance we get the following Ricci matrix when $K=1$ (or $m=0$ )

$$
\operatorname{Ric}_{1}=\left(\begin{array}{ccccccc}
l^{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & l^{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & l^{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 / 2 l^{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 / 2 l^{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 / 2 l^{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 / 2 l^{2}
\end{array}\right)
$$

Remark 12 When $l=1$, we find the Ricci curvature of the quaternionic Heisenberg group.

An easy computation leads to
Corollary 13 The EBCV manifold has constant scalar curvature $S=48 \mathrm{~m}$.

## 5 Killing vector fields in $E B C V$

Remember that a Killing vector field is a vector field on a Riemannian manifold that preserves the metric. Killing vector fields are the infinitesimal generators of isometries, that is, flows generated by Killing fields are continuous isometries of the manifold. Specifically, a vector field $X$ is a Killing vector field if the Lie derivative with respect to $X$ of the metric g vanishes: $\mathcal{L}_{X} g=0$ or equivalently

$$
\begin{equation*}
\mathcal{L}_{X} \mathrm{ds}_{l, m}^{2}=\left(\mathcal{L}_{X} \omega^{i}\right) \otimes \omega^{i}=0, \tag{1}
\end{equation*}
$$

where

$$
\mathcal{L}_{X} \omega^{i}=\iota_{X} d \omega^{i}+d\left(\iota_{X} \omega^{i}\right)
$$

In terms of the Levi-Civita connection, Killing's condition is equivalent to

$$
\begin{equation*}
g\left(\nabla_{Y} X, Z\right)+g\left(Y, \nabla_{Z} X\right)=0 \tag{2}
\end{equation*}
$$

It is easy to prove that
Proposition $14 \mathcal{L}_{X} g(Y, Z)=0$ if and only if $\mathcal{L}_{X} g\left(X_{i}, X_{j}\right)=0$ for basic vector fields $X_{i}, X_{j}$.

We know that the dimension of the Lie algebra of the Killing vector fields is

$$
m \leq n(n+1) / 2,
$$

and the maximum is reached on constant curvature manifolds ([6], p. 238, Vol. II), then for our manifold $m<28$. Then obviously

Proposition 15 The basic vertical vector fields $X_{1}, X_{2}, X_{3}$ are Killing fields.

From (2) it is easy to prove that the horizontal basic vector fields $X_{4}, \cdots, X_{7}$ are not Killing vector fields.

In her thesis, Profir [9] proved that the Lie algebra of Killing vector fields is 4dimensional. Our problem now is to determine the space of Killing vector fields in $E B C V$. By using (1) and the values of $\omega^{i}$ and $d \omega^{i}$ we obtain that the Killing vector fields are characterized by the following system of partial differential equations (28 equations).

## The Killing equations

In the usual coordinate system ( $r, s, t, w, x, y, z$ ) on $E B C V$, a vector field $X=\sum_{i=1}^{7} f_{i} X_{i}$ will be a Killing field if and only if the real functions $f_{i}$ satisfy the following system of 28-partial differential equations:

$$
\begin{array}{c|l}
01 & \partial_{r}\left(f_{1}\right)=0 \\
\vdots & \vdots \\
04 & \partial_{r}\left(f_{2}\right)+\partial_{s}\left(f_{1}\right)=0 \\
\vdots & \vdots \\
07 & \partial_{r}\left(f_{4}\right)+K \partial_{w}\left(f_{1}\right)+\frac{l y}{2} \partial_{s}\left(f_{1}\right)+\frac{l z}{2} \partial_{t}\left(f_{1}\right)-\operatorname{lm}\left\{\frac{1}{m}+\left(y^{2}+z^{2}\right)\right\} f_{5}-\operatorname{lm}(w z-x y) f_{6}+\operatorname{lm}(w y+x z) f_{7}=0 \\
\vdots & \vdots \\
19 & K \partial_{w}\left(f_{4}\right)+\frac{l x}{2} \partial_{r}\left(f_{4}\right)+\frac{l y}{2} \partial_{s}\left(f_{4}\right)+\frac{l z}{2} \partial_{t}\left(f_{4}\right)-2 m x f_{5}-2 m y f_{6}-2 m z f_{7}=0 \\
\vdots & \vdots \\
28 & \partial_{z}\left(f_{7}\right)-\frac{l y}{2} \partial_{r}\left(f_{7}\right)+\frac{l x}{2} \partial_{s}\left(f_{7}\right)-\frac{l w}{2} \partial_{t}\left(f_{7}\right)=0
\end{array}
$$

It seems that the solution of the system is very difficult, so that we focus on solving the system for $m=0$, that is:

$$
\begin{array}{c|l}
01 & \partial_{r}\left(f_{1}\right)=0 \\
\vdots & \vdots \\
04 & \partial_{r}\left(f_{2}\right)+\partial_{s}\left(f_{1}\right)=0 \\
\vdots & \vdots \\
07 & \partial_{r}\left(f_{4}\right)+\partial_{w}\left(f_{1}\right)+\frac{l y}{2} \partial_{s}\left(f_{1}\right)+\frac{l z}{2} \partial_{t}\left(f_{1}\right)-l f_{5}=0 \\
\vdots & \vdots \\
19 & \partial_{w}\left(f_{4}\right)+\frac{l x}{2} \partial_{r}\left(f_{4}\right)+\frac{l y}{2} \partial_{s}\left(f_{4}\right)+\frac{l z}{2} \partial_{t}\left(f_{4}\right)=0 \\
\vdots & \vdots \\
28 & \partial_{z}\left(f_{7}\right)-\frac{l y}{2} \partial_{r}\left(f_{7}\right)+\frac{l x}{2} \partial_{s}\left(f_{7}\right)-\frac{l w}{2} \partial_{t}\left(f_{7}\right)=0
\end{array}
$$

Proceeding as in [9], Profir considered a harder condition $L_{X} \omega^{i}=0$, then we find the following result

Proposition 16 When $m=0$, the following vector fields

$$
\begin{aligned}
& K_{1}=X_{1} \\
& K_{2}=X_{2} \\
& K_{3}=X_{3} \\
& K_{4}=-l x X_{1}-l y X_{2}-l z X_{3}+X_{4} \\
& K_{5}=l w X_{1}+l z X_{2}-l y X_{3}+X_{5} \\
& K_{6}=-l z X_{1}+l w X_{2}+l x X_{3}+X_{6} \\
& K_{7}=l y X_{1}-l x X_{2}+l w X_{3}+X_{7}
\end{aligned}
$$

are Killing ones.

Remark 17 (1) If $l=1$, we obtain Killing fields for the quaternionic Heisenberg group.
(2) We have just known that $K_{i}=X_{i}, i=1,2,3$ are Killing vector fields, however the Lie brackets of $K_{i}$ do not produce new Killing fields.

## $6 \quad B C V$ as a submanifold of $E B C V$

We define a basis of vector fields in $B C V$, seen as a submanifold of $E B C V$, adapted to the coordinates $(r, s, t, w, x, y, z)$ as follows:

$$
M_{1}=\partial_{r}, \quad M_{2}=L \partial_{w}+\frac{l x}{2} \partial_{r}, \quad M_{3}=L \partial_{x}-\frac{l w}{2} \partial_{r},
$$

where $L=1+m\left(w^{2}+x^{2}\right)$
We complete this basis to obtain a new one $\mathcal{B}=\left\{M_{1}, M_{2}, M_{3}, M_{4}, M_{5}, M_{6}, M_{7}\right\}$ for $E B C V$, given by:

$$
\begin{aligned}
& M_{1}=\partial_{r}, \\
& M_{2}=L \partial_{w}+\frac{l x}{2} \partial_{r}, \\
& M_{3}=L \partial_{x}-\frac{l w}{2} \partial_{r}, \\
& M_{4}=\partial_{s}, \\
& M_{5}=\partial_{t}, \\
& M_{6}=K \partial_{y}+\frac{l z}{2} \partial_{r}-\frac{l w}{2} \partial_{s}-\frac{l x}{2} \partial_{t}, \\
& M_{7}=K \partial_{z}-\frac{l y}{2} \partial_{r}+\frac{l x}{2} \partial_{s}-\frac{l w}{2} \partial_{t} .
\end{aligned}
$$

This basis is well defined and $\left\{M_{a}, a=1,2,3\right\}$ span the tangent space of the submanifold and $\left\{M_{u}, u=4,5,6,7\right\}$ span an orthonormal basis of the normal space.

We define the metric

$$
\mathrm{ds}_{l,\left.m\right|_{B C V} ^{2}}=\frac{d w^{2}+d x^{2}}{L^{2}}+\left(d r+\frac{l}{2 L}(w d x-x d w)\right)^{2} .
$$

We can also consider $\mathcal{B}$ as an orthonormal basis of $E B C V$. Then we are going to get the induced Levi-Civita connection to study the geometry of $B C V$ as a submanifold of $E B C V$.

The only non null brackets are
$\left[M_{2}, M_{3}\right]=-l M_{1}-2 m x M_{2}+2 m w M_{3}$,
$\left[M_{2}, M_{6}\right]=-\frac{L}{2 K}\left\{2 l m w z M_{1}+\left(K l-2 l m w^{2}\right) M_{4}-2 l m w x M_{5}-4 m w M_{6}\right\}$,
$\left[M_{2}, M_{7}\right]=\frac{L}{2 K}\left\{2 l m w y M_{1}-2 l m w x M_{4}-\left(K l-2 l m w^{2}\right) M_{5}+4 m w M_{7}\right\}$,
$\left[M_{3}, M_{6}\right]=-\frac{L}{2 K}\left\{2 l m x z M_{1}-2 l m w x M_{4}+\left(K l-2 l m x^{2}\right) M_{5}-4 m x M_{6}\right\}$,
$\left[M_{3}, M_{7}\right]=\frac{L}{2 K}\left\{2 l m x y M_{1}+\left(K l-2 l m x^{2}\right) M_{4}+2 l m w x M_{5}+4 m x M_{7}\right\}$,
$\left[M_{6}, M_{7}\right]=-L l M_{1}-\operatorname{lm}(x y+w z) M_{4}+\operatorname{lm}(w y-x z) M_{5}-2 m z M_{6}+2 m y M_{7}$.

## Gauss and Weingarten formulas

Let us write the Gauss and Weingarten formulas (see [6])

$$
\begin{aligned}
\nabla_{X}^{\prime} Y & =\nabla_{X} Y+\alpha(X, Y) \\
\nabla_{X}^{\prime} \xi & =-A_{\xi} X+D_{X} \xi
\end{aligned}
$$

where $X, Y \in \mathfrak{X}(B C V), \xi \in \mathfrak{X}^{\perp}(B C V), \nabla^{\prime}, \nabla$ are the Levi-Civita connections on $E B C V$ and $B C V$, respectively, and $D$ is the normal connection. Then, for instance, we find that

$$
\begin{aligned}
& \nabla_{M_{1}}^{\prime} M_{1}=0 \\
& \nabla_{M_{1}}^{\prime} M_{3}=-\frac{l}{2} M_{2}+\frac{L l m x}{2 K}\left(z M_{6}-y M_{7}\right) \\
& \nabla_{M_{3}}^{\prime} M_{1}=-\frac{l}{2} M_{2}+\frac{L l m x}{2 K}\left(z M_{6}-y M_{7}\right) \\
& \nabla_{M_{2}}^{\prime} M_{2}=2 m x M_{3} \\
& \nabla_{M_{3}}^{\prime} M_{3}=2 m w M_{2}
\end{aligned}
$$

so that

$$
\begin{array}{ll}
\nabla_{M_{1}} M_{1}=0, & \alpha\left(M_{1}, M_{1}\right)=0, \\
\nabla_{M_{1}} M_{2}=\frac{l}{2} M_{3}, & \alpha\left(M_{1}, M_{2}\right)=\frac{L l m w}{2 K}\left(z M_{6}-y M_{7}\right), \\
\nabla_{M_{1}} M_{3}=-\frac{l}{2} M_{2}, & \alpha\left(M_{1}, M_{3}\right)=\frac{L l m x}{2 K}\left(z M_{6}-y M_{7}\right), \\
\nabla_{M_{2}} M_{3}=-\frac{l}{2} M_{1}-m x M_{2}, & \alpha\left(M_{2}, M_{3}\right)=0
\end{array}
$$

we get
Corollary 18 Only when $m=0, B C V$ is a totally geodesic submanifold of $E B C V$.

Using the theory of submanifolds of a Riemannian manifold, we can now study a lot of problems such that the equations of Gauss, Codazzi and Ricci and their consequences.

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