New classical string solutions in $\mathbf{AdS}_3$ through null scrolls

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Abstract

We introduce a new way to study null scrolls in $\mathbf{AdS}_3$. They are timelike surfaces generated by the evolution of a curve through a transversal lightlike geodesic flow. This new approach deals with $\mathbf{AdS}_3$ as a quadric in $\mathbb{R}^{2,2}$ and that allows us to obtain an algorithm to construct null scrolls explicitly.

We see that those surfaces are strongly related with the solutions of generalized Liouville equations. In fact, under the Virasoro constraints, we show that there exists a one-to-one correspondence between null scrolls and solutions of these equations. In particular, those with constant mean curvature are modeled by Liouville equations. That also holds for stationary null scrolls (zero mean curvature), which provide classical string solutions. As a consequence, we get that the classical string solutions modeled by stationary null scrolls appear, in the Pohlmeyer reduced theory, as wave solutions of a Liouville equation. Even more, we exploit the new approach to determine the moduli space of classical string solutions modeled by null scrolls. This space can be identified with that of parameterized timelike curves in a Lorentz plane, modulo affine parameterizations. In addition, we obtain a simple algorithm to explicitly construct those classical string solutions which can be considered as an alternative to the own Pohlmeyer reduced mechanism for classical string solutions.

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1 Introduction and motivation

The theory of soliton equations, integrable systems, has had, and still has, a huge impact not only in applied mathematics, but also in a wide variety of nonlinear phenomena in physics. An important aspect of these equations is that a single example can be of interest in many different and apparently unrelated contexts. This kind of universality is strongly related to the fact that such equations frequently have an underlying geometric meaning. A well known example in this direction is provided by the string theories.

In the development of the **AdS/CFT** correspondence, classical string solutions in **AdS** spaces have played an important role. The discussion of strings historically began associated with the dynamics governed by the Nambu-Goto action that measures the area, the simplest geometrical invariant,

$$\mathcal{NG}(\phi) = c_o \int_S dA_\phi,$$

where the elementary fields, ϕ, are immersions of a surface S in **AdS** fixing the boundary ∂S. Now strings are curves that evolve in the background generating surfaces that provide extremals of the above energy. This topic, from a geometric point of view, is well understood for a long time and the classical string solutions correspond to those surfaces with zero mean curvature ($H = 0$). In a Riemannian setting they are called minimal surfaces. This term, also used for spacelike surfaces in a Lorentzian context, seems not to be appropriate for timelike surfaces. As usual, timelike surfaces in $\mathbf{AdS}_3$ are nondegenerate ones containing timelike tangent vectors. Since the induced metric on that class of surfaces is Lorentzian, sometimes they are also called Lorentzian surfaces. Furthermore, timelike surfaces with $H = 0$ will be called *stationary surfaces*.

The Nambu-Goto action presents problems if one wishes to quantize the string using a path-integral approach. In this respect, in 1981, A. M. Polyakov, [13], proposed to replace the area action by an equivalent string action that involves an intrinsic metric besides the induced one from the ambient spacetime metric. Both theories provide the so called classical string solutions that correspond to stationary surfaces.

In particular, it is important to build classical string solutions in $\mathbf{AdS}_5 \times \mathbb{S}^5$. However, people construct classical string solutions in $\mathbf{AdS}_3 \times \mathbb{S}^3$ and then, after embedding, in the bigger space (see for example [5, 8, 14] and references therein). The Pohlmeyer reduction provides a powerful and elegant tool in this process, because it makes equivalent the related sigma models in both factors to the sinh-Gordon and sin-Gordon equations, respectively. To apply this construction in the anti de Sitter factor several methods are used. For example, the dressing method is based on the choice of a vacuum solution of the string equation. People usually pick a minimal (spacelike) or stationary (timelike) surface that corresponds to a Hopf surface obtained when lifting a geodesic either in $\mathbf{AdS}_2$ or in $\mathbb{H}^2$ (the hyperbolic plane), respectively, via the corresponding Hopf mapping from $\mathbf{AdS}_3$ to each surface. The chosen vacuum solutions play the same role as the Clifford torus in the 3-sphere. These surfaces are precisely the only Hopf tubes providing classical string solutions (see [3, 10, 11])

The first main aim of this paper is to provide a new way to construct stationary surfaces in $\mathbf{AdS}_3$ and so new classical string solutions. To do that, we consider curves, no matter their causal characters, in $\mathbf{AdS}_3$ that evolve through lightlike (or null) geodesic flows to generate timelike surfaces which are called null scrolls (see [6, 7], as well as [1, 2] for the Lorentz-Minkowski three space version). Then we will study the beautiful geometry of null scrolls from two viewpoints. First, as classical parameterized surfaces in $\mathbf{AdS}_3$. Secondly, we give a new approach which takes advantage of the quadric model of $\mathbf{AdS}_3$ in $\mathbb{R}^{2,2}$. This new way gives us, for instance, a powerful algorithm to explicitly construct null scrolls. We then study, as a test, flat (zero Gaussian curvature) null scrolls as the simplest examples.

The new classical string solutions are provided by stationary null scrolls. As a first attempt to classify, up to motions in $\mathbf{AdS}_3$, the new family of classical string solutions, we show that it can be geometrically identified, up to similarities, with that of unit speed timelike curves in the light cone $\Lambda \subset \mathbb{R}^{2,2}$. However, Λ is viewed as a cone built over a Lorentzian squared torus (the Lorentzian product of two unit circles) which allows us to see, up to radial functions, the timelike curves in Λ as the projection, via the natural Lorentzian covering map, of the timelike curves in a Lorentzian plane, $\mathbb{L}^2$. We conclude that the moduli space of the new classical string solutions is identified with that of parameterized timelike curves in a Lorentzian plane, modulo affine parameterizations. Then we exhibit a simple algorithm to construct explicitly the whole family of classical string solutions in $\mathbf{AdS}_3$, which consists of stationary null scrolls. Furthermore, we give explicit parameterizations of several new classical string solutions in $\mathbf{AdS}_3$.

This new way to study null scrolls in $\mathbf{AdS}_3$ can be considered as an alternative to the Pohlmeyer reduced mechanism. Then, we describe how to translate these new solutions to the language of Pohlmeyer theory. Let us briefly explain how to do that.

(i) The starting point is to solve the equation of motion for strings in $\mathbf{AdS}_3$ subject to the Virasoro constraints. From a geometric point of view, this is equivalent to solve the Gauss-Codazzi equations for timelike surfaces in $\mathbf{AdS}_3$ in terms of null coordinates. In this setting, the geometry, intrinsic and extrinsic, of a timelike surface is codified in a wave function ϕ (which determines the intrinsic geometry), the mean curvature H, and a pair of differentials (the Hopf differentials with coefficients P and Q). Now, all these ingredients must satisfy the following generalized sinh-Gordon equation

$$\phi_{z\bar{z}} + \frac{1}{2}(H^2 - 1)e^{\phi} - 2PQe^{-\phi} = 0. \qquad \text{(GSG)}$$

(ii) This equation, when H is constant, in particular for stationary timelike surfaces, or classical string solutions, by a suitable Bäcklund transform (see for example [8]), becomes the classical sinh-Gordon equation

$$\hat{\phi}_{z\bar{z}} - 2\sinh\hat{\phi} = 0.$$

(iii) If the timelike surface is a null scroll in $\mathbf{AdS}_3$, then it is foliated by null geodesics in the anti de Sitter three space, which implies that either $P = 0$ or $Q = 0$. Therefore, the

equation (GSG) for null scrolls turns into the following generalized Liouville equation

$$\phi_{z\bar{z}} + \frac{1}{2}(H^2 - 1)e^{\phi} = 0. \qquad \text{(GL)}$$

In addition, we solve the inverse scattering problem for this model to obtain one of the main statement of the paper. Actually, up to a one variable function determining the non trivial Hopf differential, we show a one-to-one correspondence between the class of null scrolls in $\mathbf{AdS}_3$ and the space of solutions of (GL).

(iv) In particular, the subfamily of stationary null scrolls (which provides classical string solutions) can be identified with the space of solutions of the Liouville equation

$$\phi_{z\bar{z}} - \frac{1}{2}e^{\phi} = 0. \qquad \text{(L)}$$

Summing up, we obtain that the new space of classical string solutions can be identified with the following models:

- The space of unit speed timelike curves in the light cone in $\mathbb{R}^{2,2}$, modulo similarities.
- The space of parameterized timelike curves in a Lorentzian plane, up to affine parameterizations.
- The Liouville model (L), modulo one variable functions.

2 The geometry of null scrolls in $\mathbf{AdS}_3$

In a differentiable manifold M endowed with a linear connection ∇, one can study the evolution of curves by transversal geodesic flows. More precisely, if $\gamma(s)$, $s \in I \subset \mathbb{R}$, is a regular curve immersed in M and $B(s)$ is a vector field along $\gamma(s)$ which is everywhere transversal to the curve, then we can construct the surface, $S(\gamma, B)$, in M just writing

$$X : I \times (-\epsilon, \epsilon) \to M, \qquad X(s,t) = \exp_{\gamma(s)} tB(s).$$

It should be noted that each coordinate curve $s =$ constant is the geodesic, $\gamma_s(t)$, in M uniquely determined by the initial conditions $\gamma_s(0) = \gamma(s)$ and $\gamma_s'(0) = B(s)$. This idea extends that of a ruled surface in the Euclidean three space and, obviously, it works when the connection is the Riemannian one associated with either a Riemannian or a Lorentzian space. Unlike Riemannian spaces, in Lorentzian backgrounds there are, a priori, four possibilities for getting this kind of ruled surfaces, or *scrolls*, according to the causal characters of both the base curve $\gamma(s)$ and the ruling flow $B(s)$. However, when $B(s)$ is non-null, it is not difficult to see that we can change, if necessary, the base curve to get a non-null curve. Similarly, if the ruling flow is null (or lightlike), then the base curve can be chosen to be also lightlike. Consequently, we have two kind of ruled surfaces, those with non-null ruling flow

and those with lightlike ruling flow. The latter are called *null scrolls*. Certainly, every null scroll is a timelike surface. In this paper, we focus on the study of null scrolls in the anti de Sitter three space $\mathbf{AdS}_3$ (see [6] for more details). If we denote by $g = \langle\,,\rangle$ the metric of $\mathbb{R}^{2,2}$, it is not difficult to see that the null scroll data can be normalized to have

$$\langle \gamma'(s), B(s)\rangle = -1.$$

The way to study null scrolls in $\mathbf{AdS}_3$ is different from that used in [6]. In fact, while an intrinsic study was given there, in this paper we will see $\mathbf{AdS}_3$, the anti de Sitter three space with curvature -1, as the following quadric in $\mathbb{R}^{2,2}$

$$\mathbf{AdS}_3 = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1|^2 - |z_2|^2 = -1\},$$

endowed with the induced metric from that in $\mathbb{R}^{2,2}$. The use of this picture has an important advantage, because it allows us to see the lightlike geodesics of $\mathbf{AdS}_3$ as straight lines in $\mathbb{R}^{2,2}$ (the lines obtained when cutting the quadric by degenerate planes). Consequently, the null scroll $S(\gamma, B)$ in $\mathbb{R}^{2,2}$ is parameterized by

$$X(s,t) = \gamma(s) + t\, B(s) \tag{1}$$

and the induced metric writes as follows

$$\begin{pmatrix} \langle X_s, X_s\rangle & \langle X_s, X_t\rangle \\ \langle X_s, X_t\rangle & \langle X_t, X_t\rangle \end{pmatrix} = \begin{pmatrix} 2t\langle \gamma', B'\rangle(s) + t^2\langle B', B'\rangle(s) & -1 \\ -1 & 0 \end{pmatrix}.$$

To study the extrinsic geometry of null scrolls, in particular the Gauss map $N(s,t)$, we consider an orientation on $\mathbf{AdS}_3$ and define a volume element, Ω, by $\Omega(V,W,Z) = -1$ for any positively oriented orthonormal frame $\{V, W, Z\}$. It allows us to define the cross product $V \times W$ of two tangent vectors by $\langle V \times W, Z\rangle = \Omega(V,W,Z)$ for any tangent vector Z. It is not difficult to see that $\langle V \times W, V \times W\rangle = \langle V, W\rangle^2 - \langle V, V\rangle\langle W, W\rangle$.

The core of a null scroll in $\mathbf{AdS}_3$ consists of a lightlike curve $\gamma(s) \subset \mathbf{AdS}_3$, and a frame $\{\gamma(s), A(s) = \gamma'(s), B(s), C(s) = A(s)\times B(s)\}$ in $\mathbb{R}^{2,2}$ along the curve satisfying the following two conditions

(i) The $\mathbb{R}^{2,2}$-metric in this frame is given by the following matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) The frame evolves along the curve according to the following linear differential equation

$$\frac{d}{ds}\begin{pmatrix} \gamma(s) \\ A(s) \\ B(s) \\ C(s) \end{pmatrix} = \begin{pmatrix} \gamma'(s) \\ A'(s) \\ B'(s) \\ C'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \sigma(s) & 0 & \kappa(s) \\ -1 & 0 & -\sigma(s) & \tau(s) \\ 0 & \tau(s) & \kappa(s) & 0 \end{pmatrix}\begin{pmatrix} \gamma(s) \\ A(s) \\ B(s) \\ C(s) \end{pmatrix}. \tag{2}$$

Therefore, from (2), the core of a null scroll in $\mathbf{AdS}_3$ is encoded in three functions:

(f1) $\sigma(s) = \langle \gamma'(s), B'(s) \rangle$, which provides an obstruction to the geodesibility of γ in the null scroll;

(f2) $\kappa(s) = \langle \gamma''(s), C(s) \rangle$, which, in some sense, could be viewed as the curvature function of γ in $\mathbf{AdS}_3$; and

(f3) $\tau(s) = \langle B'(s), C(s) \rangle$, whose geometrical meaning will be given a little further in this section.

They all provide geometrical invariants of the null scroll, even more, one can obtain the null scroll from these three functions, just solving (2). The null scroll is completely determined, up to motions in $\mathbf{AdS}_3$, by its core. Moreover, using the Gauss equation of $\mathbf{AdS}_3$ in $\mathbb{R}^{2,2}$, we have

$$A'(s) = D_s A(s), \quad B'(s) = -\gamma(s) + D_s B(s), \quad C'(s) = D_s C(s), \tag{3}$$

where D_s denotes the covariant derivative, in $\mathbf{AdS}_3$, along the curve $\gamma(s)$.

The Gauss map $N(s,t)$ is a spacelike unit vector field, along the null scroll, satisfying that

$$\langle N(s,t), X(s,t) \rangle = \langle N(s,t), X_s(s,t) \rangle = \langle N(s,t), X_t(s,t) \rangle = 0.$$

Then a direct computation yields

$$N(s,t) = X_s \times X_t = C(s) + t\,\tau(s)\,B(s). \tag{4}$$

As for the shape operator we have

$$\begin{aligned} \frac{\partial N}{\partial s} &= D_s N = \tau(s)\, X_s + (\kappa(s) + t\tau'(s))\, X_t, \\ \frac{\partial N}{\partial t} &= D_t N = \tau(s)\, X_t, \end{aligned}$$

and then we get

$$dN(s,t) \equiv \begin{pmatrix} \tau(s) & 0 \\ \kappa(s) + t\tau'(s) & \tau(s) \end{pmatrix}.$$

Now, the mean curvature, H, and the Gaussian curvature, K, of a null scroll in $\mathbf{AdS}_3$ are given by

$$H(s,t) = \tau(s) = \|D_s B\|(s), \tag{5}$$

$$K(s,t) = -1 + \det(dN) = -1 + \tau^2(s) = -1 + H^2(s,t) = \langle B'(s), B'(s) \rangle. \tag{6}$$

In these equations are encoded deep properties on the geometry of null scrolls, a few of them are listed below:

(a) Mean and Gaussian curvatures are invariant along the ruling flow.

(b) The equation $H^2 = 1 + K$ provides a nice relation between extrinsic and intrinsic geometries of a null scroll. In particular, the mean curvature can be measured, via the egregium theorem of Gauss, by the people living in the null scroll.

(c) It is well known that the values of the Nambu-Goto energy, in particular extremal ones (those reached by the string solutions), have an intrinsic nature. In other words, they can be computed inside the worldsheets, so people living in a string solution are able to measure the Nambu-Goto tension that its world receives from the surrounding spacetime. Quite the contrary, those people have no idea of the extremal nature of the world where they are living, because the mean curvature has an extrinsic nature. Thus, though the nature of the Nambu-Goto is intrinsic, it provides string solutions whose critical nature can not be intrinsically valued. If we restrict ourselves to null scrolls, this unsatisfactory intrinsic-extrinsic disagreement, between action and solutions, does not make sense, because the solutions ($H = 0$) correspond to null scrolls with Gaussian curvature $K = -1$.

3 A new way to view the null scrolls of $\mathbf{AdS}_3$

As $\mathbf{AdS}_3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 - |z_2|^2 = -1\}$, any curve $\gamma(s)$ in $\mathbf{AdS}_3$ can be seen in $\mathbb{C}^2$ as

$$\gamma(s) = \left(\rho(s)\, e^{i\theta(s)}, \sqrt{1+\rho^2(s)}\, e^{i\eta(s)}\right), \tag{7}$$

for certain functions ρ, θ, η. In particular, we wish to know the way to express, in $\mathbb{C}^2$, the lightlike curves of $\mathbf{AdS}_3$. Then we have

$$\gamma'(s) = \left((\rho'(s) + i\rho(s)\theta'(s))\, e^{i\theta(s)}, \left(\frac{\rho(s)\rho'(s)}{\sqrt{1+\rho^2(s)}} + i\sqrt{1+\rho^2(s)}\eta'(s) \right) e^{i\eta(s)} \right). \tag{8}$$

Now, the condition $\langle \gamma'(s), \gamma'(s) \rangle = 0$ allows one to determine, for example, the function $\eta(s)$ in terms of the functions $\rho(s)$ and $\theta(s)$, namely we have

$$\eta(s) = \int \sqrt{\frac{(\rho'(s))^2}{(1+\rho^2(s))^2} + \frac{\rho^2(s)}{1+\rho^2(s)}(\theta'(s))^2}\, ds. \tag{9}$$

Therefore, to build in $\mathbb{C}^2$ a lightlike curve in $\mathbf{AdS}_3$ we use (7) and the constraint (9). In particular, we need a pair of functions $\rho(s)$ and $\theta(s)$ to determine any lightlike curve. Let us give some simple examples.

Example 3.1 *It is clear, from* (9)*, that there exists no lightlike curve with $\eta(s)$ being a constant function. However, we can determine the lightlike curves with $\rho(s) = \rho_o =$ constant. In fact, from* (9)*, we have*

$$\eta(s) = \frac{\rho_o}{\sqrt{1+\rho_o^2}}\, \theta(s) + m, \qquad m \in \mathbb{R}.$$

Then we get the following one-parameter family of lightlike curves

$$\gamma(s) = \left(\rho_o\, e^{i\theta(s)}, \sqrt{1+\rho_o^2}\, e^{\,i\left(\frac{\rho_o}{\sqrt{1+\rho_o^2}}\,\theta(s)+m\right)} \right), \qquad m \in \mathbb{R}. \tag{10}$$

Example 3.2 *We can also determine the lightlike curves with $\theta(s) = \theta_o =$ constant. In this case, from (9), we obtain $\eta(s) = \arctan \rho(s) + m$, where $m \in \mathbb{R}$. Therefore, we get the following one-parameter class of lightlike curves*

$$\gamma(s) = \left(\rho(s)\, e^{i\theta_o}, \sqrt{1+\rho^2(s)}\, e^{i(\arctan \rho(s)+m)} \right), \qquad m \in \mathbb{R}. \tag{11}$$

It should be remembered that to construct a null scroll, in particular in $\mathbf{AdS}_3$, we need a lightlike curve, the base curve $\gamma(s)$, and a transversal lightlike vector field, the ruling flow $B(s)$, along $\gamma(s)$. Since we already know how to build the base curve, we have to look for suitable ruling flows. To do it, we first consider the light cone in $\mathbb{R}^{2,2}$

$$\Lambda = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1|^2 = |z_2|^2\}, \tag{12}$$

which geometrically corresponds to a cone, in $\mathbb{C}^2$, built over a torus $\mathrm{T}^2 = \mathbb{S}^1(r) \times (-\mathbb{S}^1(r))$. Therefore, those ruling flows, along $\gamma(s)$, will be determined by three functions $(r(s),\, \varphi(s),\, \psi(s))$, that we will call *radial*, *plus-rotation* and *minus-rotation*, respectively, as follows

$$B(s) = r(s) \left(e^{i\varphi(s)}, e^{i\psi(s)} \right). \tag{13}$$

However, they can not be arbitrarily chosen. In fact, the admissible ruling flows must be tangent to $\mathbf{AdS}_3$ at $\gamma(s)$, that is, $\langle \gamma(s), B(s) \rangle = 0$. In our framework, this condition says that

$$\rho(s)\, \cos\left[\theta(s) - \varphi(s)\right] = \sqrt{1+\rho^2(s)}\, \cos\left[\eta(s) - \psi(s)\right]. \tag{14}$$

It should be noted that (14) does not involve the function $r(s)$, however it allows to determine any of the two angular functions by

$$\psi(s) = \eta(s) + \arccos\left(\frac{\rho(s)}{\sqrt{1+\rho^2(s)}}\, \cos\left[\theta(s) - \varphi(s)\right] \right).$$

Obviously the above constraint is not enough. To generate a null scroll the ruling flow should be everywhere transversal to the base curve. In other words, the function $\langle \gamma'(s), B(s) \rangle$ should never vanish. To analyze this constraint, we observe that $\gamma'(s)$ lies on the light cone and, consequently, we can write

$$\gamma'(s) = c(s) \left(e^{i\omega(s)}, e^{i\lambda(s)} \right), \tag{15}$$

which, jointly with (13), yields

$$\langle \gamma'(s), B(s) \rangle = c(s)\, r(s) \left(\cos\left[\omega(s) - \varphi(s)\right] - \cos\left[\lambda(s) - \psi(s)\right] \right). \tag{16}$$

Now we have to ensure that the function $G(s) = \cos[\omega(s) - \varphi(s)] - \cos[\lambda(s) - \psi(s)]$ does not vanish everywhere. To see that we consider, in the light cone Λ, the Lorentzian squared torus

$$\mathrm{T}^2 = \mathbb{S}^1(1) \times (-\mathbb{S}^1(1)) = \{(z_1, z_2) \in \Lambda \,:\, |z_1|^2 = |z_2|^2 = 1\} = \{(e^{iu}, e^{iv}) \,:\, u, v \in \mathbb{R}\},$$

and then the Lorentzian covering map

$$\Phi : \mathbb{L}^2 \to \mathrm{T}^2, \quad \Phi(u, v) = (e^{iu}, e^{iv}).$$

It is clear that $G(s) = 0$ if and only if $\omega(s) - \varphi(s) = \pm[\lambda(s) - \psi(s)]$. Thus, we define the following two curves in the Lorentzian plane $\mathbb{L}^2$

$$\alpha(s) = (\omega(s), \lambda(s)), \qquad \beta(s) = (\varphi(s), \psi(s)).$$

Then, $G(s) = 0$ if and only if $\alpha(s) - \beta(s)$ is orthogonal, in $\mathbb{L}^2$, to either $(1, 1)$ or $(1, -1)$. In other words, G vanishes if and only if the curve $\alpha(s) - \beta(s)$ lies on the light cone of $\mathbb{L}^2$. This confirms the naive suspect that the transversal constraint holds almost everywhere.

Finally, we wish to normalize null scrolls by imposing that $\langle \gamma'(s), B(s) \rangle = -1$. To do it, we combine (8) with (13) to obtain the following equation (compare with (16))

$$\langle \gamma'(s), B(s) \rangle = c(s)\, r(s)\, G(s) = r(s)\, F(s),$$

where

$$\begin{aligned} F(s) \;=\; & \rho'(s)\, \cos[\theta(s) - \varphi(s)] - \rho(s)\theta'(s)\, \sin[\theta(s) - \varphi(s)] - \\ & \frac{\rho(s)\rho'(s)}{\sqrt{1 + \rho^2(s)}}\, \cos[\eta(s) - \psi(s)] + \sqrt{1 + \rho^2(s)}\, \eta'(s)\, \sin[\eta(s) - \psi(s)]. \end{aligned}$$

Once we know that $G(s)$ never vanishes, we choose

$$r(s) = \frac{-1}{F(s)}, \tag{17}$$

to obtain the asked normalization $\langle \gamma'(s), B(s) \rangle = -1$.

All this process can be summarized as follows.

Algorithm to construct null scrolls in $\mathbf{AdS}_3$. We propose the following three steps:

(1) The base curve. Choose any pair of functions, $\rho(s)$ and $\theta(s)$, and let

$$\gamma(s) = \left(\rho(s)\, e^{i\theta(s)}, \sqrt{1 + \rho^2(s)}\, e^{i\eta(s)}\right)$$

be the lightlike curve with

$$\eta(s) = \int \sqrt{\frac{(\rho'(s))^2}{(1 + \rho^2(s))^2} + \frac{\rho^2(s)}{1 + \rho^2(s)}(\theta'(s))^2}\, ds.$$

(2) The ruling flow. Set

$$B(s) = r(s)\left(e^{i\varphi(s)}, e^{i\psi(s)}\right),$$

for functions satisfying the following conditions

(2.1) *Tangency*: $\psi(s) = \eta(s) + \arccos\left(\frac{\rho(s)}{\sqrt{1+\rho^2(s)}}\cos\left(\theta(s) - \varphi(s)\right)\right)$.

(2.2) *Transversality*: The function $G(s)$, or equivalently $F(s)$, never vanishes.

(2.3) *Normalization*: The radial function of the ruling flow is normalized by $r(s) = -\dfrac{1}{F(s)}$.

(3) The construction. Once we have the above data, we build the null scroll $S(\gamma, B)$ by

$$X(s,t) = \exp_{\gamma(s)} tB(s).$$

However, the approach we are using allows us to see the lightlike geodesics of $\mathbf{AdS}_3$ as straight lines in $\mathbb{R}^{2,2}$ (the lines that we obtain when cutting the quadric by degenerate planes). Consequently, we can see the above null scroll $S(\gamma, B)$ in $\mathbb{R}^{2,2}$ parameterized by

$$X(s,t) = \gamma(s) + t\,B(s). \tag{18}$$

Next, we apply the algorithm to give examples of null scrolls in $\mathbf{AdS}_3$.

Example 3.3 *By choosing the constant radial function $\rho(s) = 1$ and the angular function $\theta(s) = s$, we determine the second angular function by $\eta(s) = \frac{\sqrt{2}}{2}\,s + m_1$. Now, the base curve is given by*

$$\gamma(s) = \left(e^{is}, \sqrt{2}\,e^{\frac{\sqrt{2}}{2}\,s + m_1}\right).$$

To find the ruling flow we choose $\varphi(s) = s$ to get the second angular function as $\psi(s) = \eta(s) + \frac{\pi}{4}$. Then we write

$$F(s) = \sqrt{2}\,\eta'(s)\,\sin\left[\eta(s) - \psi(s)\right] = -\frac{\sqrt{2}}{2},$$

which allows us to determine the radial function of the ruling flow by $r(s) = \sqrt{2}$. Finally, we use (18) to give the explicit parameterization of the corresponding null scroll

$$X(s,t) = \left(e^{is}, \sqrt{2}\,e^{\frac{\sqrt{2}}{2}\,s + m_1}\right) + \sqrt{2}\,t\left(e^{is}, \sqrt{2}\,e^{\frac{\sqrt{2}}{2}\,s + \frac{\pi}{4} + m_1}\right).$$

4 Flat null scrolls

We will consider now flat null scrolls, i. e., null scrolls having zero Gaussian curvature. They also have mean curvature $H = \pm 1$ in $\mathbf{AdS}_3$. However, the simplest extrinsic curvature

behavior correspond to the stationary ($H = 0$) null scrolls which provided classical string configurations. This class will be treated in next sections.

Therefore, we start by computing the geometrical invariants, mean and Gaussian curvatures, of a null scroll in terms of their data associated with the immersion in $\mathbb{R}^{2,2}$. To do that we use (5) and (6), to obtain

$$H^2(s,t) = r^2(s)\left[\varphi'(s)^2 - \psi'(s)^2\right] + 1, \tag{19}$$

$$K(s,t) = r^2(s)\left[\varphi'(s)^2 - \psi'(s)^2\right]. \tag{20}$$

Remark 4.1 *We already know several important properties of the geometrical invariants, in particular the mean curvature function, of null scrolls. The new approach has allowed us to obtain the above formulas* (19) *and* (20)*, which provide another fundamental deep property: mean and Gaussian curvature functions of a null scroll are both encoded in its ruling flow.*

Then a null scroll is flat, i. e., $K = 0$ if and only if both rotational functions agree up to both orientation and constants. Note that they correspond to those null scrolls with constant mean curvature $H^2(s,t) = 1$.

To build flat null scrolls we may assume, without loss of generality, that $\varphi(s) = \psi(s)$. Then we consider the base curve

$$\gamma(s) = \left(\rho_o\, e^{i\theta(s)}, \sqrt{1+\rho_o^2}\, e^{i\eta(s)}\right), \qquad \text{where} \quad \eta(s) = \frac{\rho_o}{\sqrt{1+\rho_o^2}}\,\theta(s).$$

As the ruling flows should be tangent to $\mathbf{AdS}_3$, they will satisfy the constraint (14) with $\varphi(s) = \psi(s)$, which allows us to compute this rotational function as

$$\varphi(s) = \arctan\left(\frac{\rho_o\cos\theta(s) - \sqrt{1+\rho_o^2}\,\cos\eta(s)}{\sqrt{1+\rho_o^2}\,\sin\eta(s) - \rho_o\sin\theta(s)}\right).$$

Finally, to check transversality and normalization conditions we compare equations (8) and 15) to find that

$$-\rho_o\theta'(s)\sin\theta(s) = c(s)\cos\omega(s), \qquad \rho_o\theta'(s)\cos\theta(s) = c(s)\sin\omega(s),$$

$$-\sqrt{1+\rho_o^2}\,\eta'(s)\sin\eta(s) = c(s)\cos\lambda(s), \qquad \sqrt{1+\rho_o^2}\,\eta'(s)\cos\eta(s) = c(s)\sin\lambda(s),$$

which give

$$\tan\omega(s) = -\cot\theta(s), \qquad \tan\lambda(s) = -\cot\eta(s). \tag{21}$$

On the other hand, the curve $\alpha(s)-\beta(s) = (\omega(s)-\varphi(s), \lambda(s)-\varphi(s))$ is lightlike in the Lorentz plane $\mathbb{L}^2$ if and only if $\omega(s) = \pm\lambda(s)$, which combined with (21) yields $\cot\theta(s) = \pm\cot\eta(s)$. Now, we use that $\eta(s) = \frac{\rho_o}{\sqrt{1+\rho_o^2}}\,\theta(s)$ to conclude that $\theta(s)$ should be constant, which can not hold. Therefore, $\alpha(s) - \beta(s)$ is non-null everywhere, and that ensures the transversality.

To normalize we compute the function $F(s)$ given by

$$F(s) = \rho_o\,\theta'(s)\,\left(\sin\left[\eta(s) - \varphi(s)\right] - \sin\left[\theta(s) - \varphi(s)\right]\right),$$

which never vanishes, because (14). We then consider

$$r(s) = \frac{1}{\sin\left[\theta(s) - \varphi(s)\right] - \sin\left[\eta(s) - \varphi(s)\right]}.$$

Therefore, we obtain the following class of flat null scrolls in $\mathbf{AdS}_3$

$$X(s,t) = \left(\rho_o e^{i\theta(s)}, \sqrt{1+\rho_o^2}e^{i\eta(s)}\right) + t\, r(s)\left(e^{i\varphi(s)}, e^{i\varphi(s)}\right),$$

where

$$\begin{aligned}
\eta(s) &= \frac{\rho_o}{\sqrt{1+\rho_o^2}}\,\theta(s), \\
\varphi(s) &= \arctan\left(\frac{\rho_o\cos\theta(s) - \sqrt{1+\rho_o^2}\,\cos\eta(s)}{\sqrt{1+\rho_o^2}\,\sin\eta(s) - \rho_o\sin\theta(s)}\right), \\
r(s) &= \frac{1}{\sin\left(\theta(s) - \varphi(s) - \sin\left(\eta(s) - \varphi(s)\right)\right.},
\end{aligned}$$

$\theta(s)$ being an arbitrary function and $\rho_o \in \mathbb{R}$.

The lightlike curves in $\mathbf{AdS}_3$, with $\theta(s) = \theta_o = $ constant, given in Example 3.2, can be expressed in terms of the function $\eta(s)$ as follows

$$\gamma(s) = \left(\tan\eta(s)\, e^{i\theta_o}, \sec\eta(s)\, e^{i\eta(s)}\right).$$

Then we can construct, up to circle orientations and constants, the whole class of flat null scrolls admitting $\gamma(s)$ as base curve. Then, we have the following parameterization of these null scrolls

$$X(s,t) = \left(\tan\eta(s)\, e^{i\theta_o} + t\, r(s)\, e^{i\varphi(s)}, \sec\eta(s)\, e^{i\eta(s)} + t\, r(s)\, e^{i\varphi(s)}\right),$$

where

$$\begin{aligned}
\varphi(s) &= \arctan\left(\frac{\cos\theta_o - \cot\eta(s)}{1 - \sin\theta_o}\right), \\
r(s) &= \frac{\cos\eta(s)(\cos\eta(s) - \sin\eta(s)\cos\theta_o)}{\eta'(s)\sin\eta(s)(1-\sin\theta_o)}.
\end{aligned}$$

This family is obviously determined by two moduli: a constant, $\theta_o \in \mathbb{R}$, and a real valued function, $\eta(s)$, defined on a suitable real interval.

Remark 4.2 *It should be observed that flat null scrolls, in* $\mathbf{AdS}_3$*, are important not only because they present the simplest scroll geometry, but also by their applications to the study of degenerate helices (see to this respect the seminal paper* [4]*). In a forthcoming paper, the authors will use flat null scrolls to complete the study of degenerate helices in* $\mathbf{AdS}_3$*, even showing the corresponding moduli space.*

5 Stationary null scrolls or classical string solutions

We will explicitly describe the stationary null scrolls providing new classical string solutions. First of all, it is worth pointing out that while the core of a null scroll is made up of three geometrical invariant, actually it can be reduced to a pair of them. In fact, the invariant σ measures the obstruction of the base curve $\gamma(s)$ to be a geodesic of the null scroll. Therefore, it vanishes if and only if $\gamma(s)$ is a geodesic of $S(\gamma, B)$. However, $\gamma(s)$ is always a pre-geodesic of $S(\gamma, B)$, that is, it is a geodesic up to reparameterization. To check this claim, we start with the natural parameterization of a null scroll $X(s,t) = \gamma(s) + t\, B(s)$, so we have $\sigma(s) = \langle \gamma', B' \rangle(s) = -\langle \gamma'', B \rangle(s)$. Now, we determine a function $s(u)$ as a solution of the following differential equation

$$\frac{d^2 s}{du^2} = \left(\frac{ds}{du}\right)^2 \sigma(s(u)),$$

to get

$$\langle \frac{d^2 \gamma}{du^2}, B \rangle = 0.$$

Although the essential core of a null scroll is determined by the functions κ, doing the role of the acceleration (the curvature) of the base curve, and τ, measuring the mean curvature function of the null scroll, sometimes, for certain computations, it is convenient to treat the base curve as a pre-geodesic, so σ does not vanishes identically.

Therefore, if we parameterize a null scroll $S(\gamma, B)$ viewing the base curve, $\gamma(s)$, as a geodesic, then from (2) we get $B'(s) = -\gamma(s) + \tau(s)\, C(s)$, where we recall that the function $\tau(s)$ is nothing but the mean curvature function H of the null scroll. Since this function is completely codified in the ruling flow $B(s)$, which can be regarded as a curve in the light cone $\Lambda \subset \mathbb{R}^{2,2}$, it seems natural to state the following question:

How to choose the ruling flow in order the null scroll to be stationary?

To get a reasonable answer, we start with a stationary null scroll. As $H = \tau = 0$, we get $B'(s) = -\gamma(s)$, which can be read as follows: *The flow of a stationary null scroll is a unit speed timelike curve in the lightcone* $\Lambda \subset \mathbb{R}^{2,2}$. Conversely, let $L(s)$ be a unit speed timelike curve in Λ, so $\langle L(s), L(s) \rangle = 0$ and $\langle L'(s), L'(s) \rangle = -1$. Then define the curve $\beta(s) = -L'(s)$ in $\mathbf{AdS}_3$ which need not be lightlike. Then, compare $\beta(s)$ and $L(s)$ to see that $L(s)$ is both

(i) Tangent to $\mathbf{AdS}_3$ along $\beta(s)$. In fact, we have

$$\langle \beta(s), L(s) \rangle = \langle -L'(s), L(s) \rangle = 0;$$

and

(ii) Transversal everywhere to $\beta(s)$, because

$$\langle \beta'(s), L(s) \rangle = -\langle \beta(s), L'(s) \rangle = \langle \beta(s), \beta(s) \rangle = -1.$$

Therefore, we can consider the null scroll $X(s,t) = \beta(s) + t\,L(s)$ and then change the base curve, if necessary, $\gamma(s) = \beta(s) + t(s)\,L(s)$, to obtain a lightlike base curve. Now, the mean curvature of this null scroll is

$$H(s) = \Omega(\gamma', L, D_s L)(s) = \Omega(\beta', L, D_s L)(s) = 0,$$

which shows that the null scroll is stationary. Therefore, we have the following

Proposition 5.1 *The ruling flow of a stationary null scroll is a unit speed timelike curve in $\Lambda \subset \mathbb{R}^{2,2}$. Conversely, every unit speed timelike curve in $\Lambda \subset \mathbb{R}^{2,2}$ can be regarded as the ruling flow of a stationary null scroll.*

As a consequence, we know how to get classical string solutions in $\mathbf{AdS}_3$.

Corollary 5.2 *Whenever the tangent indicatrix of a unit speed timelike curve in $\Lambda \subset \mathbb{R}^{2,2}$ evolves, in $\mathbf{AdS}_3$, along the geodesic flow associated to the own curve, it is generating a classical string solution.*

Let $\mathcal{M}_1$ be the space of arclength parameterized timelike curves in $\Lambda \subset \mathbb{R}^{2,2}$ and let $\mathcal{M}_2$ be the space of stationary null scrolls (the space of classical string solutions modeled by null scrolls) in $\mathbf{AdS}_3$. The above proposition can be described in terms of the surjective map $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$, where $\Psi(L)$ is parameterized by $X(s,t) = -L'(s) + t\,L(s)$. In the Lorentz plane $\mathbb{L}^2$, equipped with the metric $(+,-)$, we consider the space $\mathcal{TC}(\mathbb{L}^2)$ of timelike curves in $\mathbb{L}^2$ and the Lorentzian covering map $\Phi : \mathbb{L}^2 \to \mathbf{T}^2 \subset \Lambda$ defined by $\Phi(x,y) = (e^{ix}, e^{iy})$. Now let $F : \mathcal{TC}(\mathbb{L}^2) \to \mathcal{M}_1$ be the map defined by

$$\mathbf{x}(s) = (x(s), y(s)) \mapsto L(s) = r(s)(e^{ix(s)}, e^{iy(s)}), \quad \text{where} \quad r(s)^2 = -\frac{1}{\langle \mathbf{x}'(s), \mathbf{x}'(s)\rangle}.$$

It is obvious that F is surjective. Now, from $F(\mathbf{x}(s)) = F(\mathbf{y}(s))$, we see that

$$L(s) = r(s)(e^{ix(s)}, e^{iy(s)}) = m(u)(e^{ia(u)}, e^{ib(u)}) = \tilde{L}(u),$$

which is equivalent to $r(s) = m(u)$ and $\Phi(\mathbf{x}(s)) = \Phi(\mathbf{y}(u))$. In other words, $\mathbf{y}(u(s)) = \mathbf{x}(s) +$ constant. Therefore, $F^{-1}(L(s)) = \{\mathbf{x}(s) + q \,:\, q \in \mathbb{R}\}$, said otherwise, $F : \mathcal{TC}(\mathbb{L}^2) \to \mathcal{M}_1$ is a line bundle.

The above two maps can be used to build classical string solutions in $\mathbf{AdS}_3$ from timelike curves in a Lorentzian plane. In fact, just consider the composition map

$$\mathcal{TC}(\mathbb{L}^2) \quad \xrightarrow{F} \quad \mathcal{M}_1 \quad \xrightarrow{\Psi} \quad \mathcal{M}_2,$$

$$\mathbf{x}(s) = (x(s), y(s)) \mapsto L(s) = r(s)\Phi(\mathbf{x}(s)) \mapsto -L'(s) + t\,L(s),$$

where $r(s)^2 = -\frac{1}{\langle \mathbf{x}'(s), \mathbf{x}'(s)\rangle}$.

Looking for the size of the new family of classical string solutions, it should be interesting to identify this family with another one of geometric objects whose size could be determined. In other words, we are interested in the following problem

Determine the moduli space of classical string solutions which are modeled by stationary null scrolls.

As the map $\Psi \circ F$ is surjective, to answer this problem it seems natural to analyze its injectivity degree. Therefore, we are going to study the relation between two timelike curves in $\mathbb{L}^2$, say $\mathbf{x}(s)$ and $\mathbf{y}(u)$, providing the same null scroll in $\mathbf{AdS}_3$. Let $\Psi(F(\mathbf{x}(s))) = \Psi(F(\mathbf{y}(u)))$ and write

$$L(s) = F(\mathbf{x}(s)) = r(s)\Phi(\mathbf{x}(s)), \qquad \tilde{L}(u) = F(\mathbf{y}(u)) = m(u)\Phi(\mathbf{y}(u)),$$

with

$$r^2(s) = -\frac{1}{\langle \frac{d\mathbf{x}}{ds}, \frac{d\mathbf{x}}{ds} \rangle}, \qquad m^2(u) = -\frac{1}{\langle \frac{d\mathbf{y}}{du}, \frac{d\mathbf{y}}{du} \rangle}.$$

Then it is clear that $\tilde{L}(u) = f(s(u))L(s(u))$, where f satisfies $m(u) = f(s(u))r(s(u))$. Moreover, $\Phi(\mathbf{y}(u)) = \Phi(\mathbf{x}(s(u))$, which allows us to conclude that $\mathbf{y}(u)$ must be, up to a constant, a reparameterization of $\mathbf{x}(s)$, that is $\mathbf{y}(u) = \mathbf{x}(s(u))$. On the other hand, an easy computation yields

$$f(s(u))\,\frac{ds}{du} = \pm 1. \tag{22}$$

Now, both null scrolls agree if and only if

$$-\frac{d^2\tilde{L}}{du^2} + t\frac{d\tilde{L}}{du} \in \mathrm{Span}\left\{-\frac{d^2L}{ds^2} + t\frac{dL}{ds}, L\right\}.$$

A direct and long computation shows that this condition holds if and only if f is a constant function, which combined with (22) says that $\Psi(F(\mathbf{x}(s))) = \Psi(F(\mathbf{y}(u)))$ if and only if both curves agree, up to an affine reparameterization. So the space of stationary null scrolls can be identified with that of parameterized timelike curves, modulo affine parameterizations. Setting $\mathrm{Diff}(\mathbb{R})$ the space of diffeomorphisms of $\mathbb{R}$ (here $\mathbb{R}$ could be substituted by some real interval), we mean by $\mathcal{A}(\mathbb{R}) \equiv (\mathbb{R} - \{0\}) \times \mathbb{R}$ the space of affine diffeomorphisms. Then, the above result can be summarized as follows, which completely determines the whole moduli space of classical string solutions modeled by null scrolls.

Theorem 5.3 *The moduli space of classical string solutions in* $\mathbf{AdS}_3$*, which are modeled by null scrolls, is identified with*

$$\mathcal{TC}(\mathbb{L}^2) \times (\mathrm{Diff}(\mathbb{R})/\mathcal{A}(\mathbb{R}))\,.$$

In addition, we provide a simple algorithm which allows one to build explicitly the whole moduli space of classical string solutions in $\mathbf{AdS}_3$ which are configured by null scrolls. This algorithm works according to the following steps:

1. Choose any time like curve, say $\mathrm{x}(s) = (x(s), y(s))$, no matter its parameterization, in the Lorentz plane $\mathbb{L}^2$ with metric $(+,-)$.

2. Consider the natural Lorentzian covering map, $\Phi : \mathbb{L}^2 \to \mathrm{T}^2 \subset \Lambda$, given by

$$\Phi(x,y) = \left(e^{ix}, e^{iy}\right),$$

and then the image of the previous curve to obtain a timelike curve in $\mathrm{T}^2 \subset \Lambda$

$$\Phi(\mathrm{x}(s)) = \left(e^{ix(s)}, e^{iy(s)}\right).$$

3. Since the previous curve is arbitrarily parameterized, we introduce a radial function,

$$r(s) = \sqrt{-\frac{1}{\langle \mathrm{x}'(s), \mathrm{x}'(s)\rangle}},$$

to ensure that $B(s) = r(s)\,\Phi(\mathrm{x}(s))$ defines a stationary flow.

4. Then, we construct the classical string solutions modeled by the stationary null scroll

$$X(s,t) = -B'(s) + t\,B(s) = (-r'(s) + t\,r(s))\,\Phi(\mathrm{x}(s)) - r(s)\,\frac{d}{ds}(\Phi(\mathrm{x}(s))).$$

5. Moreover, all classical string solutions modeled by null scrolls are obtained in this way.

To illustrate the algorithm, let us exhibit some explicit parameterizations of classical string solutions modeled on null scrolls.

5.1 Some examples of classical string solutions

(**1**) The simplest example appears when choosing the timelike curve $\mathrm{x}(s) = (\cosh s, \sinh s)$ in $\mathbb{L}^2$. Since it is arclength parameterized, the radial function is identically one. Therefore, we get the stationary flow $B(s) = \left(e^{i\cosh s}, e^{i\sinh s}\right)$, yielding the following classical string solution

$$X(s,t) = \left(-\sinh(s)\,ie^{i\cosh s} + t\,e^{i\cosh s}, -\cosh(s)\,ie^{i\sinh s} + t\,e^{i\sinh s}\right).$$

(**2**) Let us consider, in $\mathbb{L}^2$, the timelike curve $\mathrm{x}(s) = (\sin s, \sqrt{2}\,s)$, which provides the stationary flow

$$B(s) = \frac{1}{(1+\sin^2 s)^{1/2}}\,\left(e^{i\sin s}, e^{i\sqrt{2}s}\right).$$

Now, it gives a classical string solution, in $\mathbf{AdS}_3$, that can be parameterized by

$$\begin{aligned} X(s,t) &= \left(-\frac{\sin s\cos s}{(1+\sin^2 s)^{3/2}} + \frac{t}{(1+\sin^2 s)^{1/2}}\right)\,\left(e^{i\sin s}, e^{i\sqrt{2}s}\right) \\ &- \frac{i}{(1+\sin^2 s)^{1/2}}\,\left(\cos s\,e^{i\sin s}, \sqrt{2}\,e^{i\sqrt{2}s}\right). \end{aligned}$$

(**3**) In $\mathbb{L}^2$, we consider the timelike curve $\mathrm{x}(s) = (\ln s, s)$, where $s > 1$. It generates the following stationary flow

$$B(s) = \frac{s}{(s^2-1)^{1/2}} \left(e^{i \ln s}, e^{is}\right),$$

yielding a classical string solution which can be explicitly defined by

$$\begin{aligned} X(s,t) &= \left(-\frac{1}{(s^2-1)^{3/2}} + \frac{t\,s}{(s^2-1)^{1/2}}\right) \left(e^{i \ln s}, e^{is}\right) \\ &- \frac{i\,s}{(s^2-1)^{1/2}} \left(\frac{1}{s}\, e^{i \ln s}, e^{is}\right). \end{aligned}$$

(**4**) The timelike curve $\mathrm{x}(s) = (\sin s, \sqrt{2}\, e^s)$ in $\mathbb{L}^2$ defines the following stationary flow in the light cone $\Lambda \subset \mathbb{R}^{2,2}$

$$B(s) = \frac{1}{(2e^{2s} - \cos^2 s)^{1/2}} \left(e^{i \sin s}, e^{i\sqrt{2}e^s}\right),$$

which provides the following classical string solution

$$\begin{aligned} X(s,t) &= \left(-\frac{2e^{2s} + \cos s\, \sin s}{(2e^{2s} - \cos^2 s)^{3/2}} + \frac{t}{(2e^{2s} - \cos^2 s)^{1/2}}\right) \left(e^{i \sin s}, e^{i\sqrt{2}e^2}\right) \\ &- \frac{i}{(2e^{2s} - \cos^2 s)^{1/2}} \left(\cos s\, e^{i \sin s}, \sqrt{2}\, e^s\, e^{i\sqrt{2}e^s}\right). \end{aligned}$$

(**5**) For any positive real number, $\varepsilon > 0$, we consider in $\mathbb{L}^2$ the following timelike curve $\mathrm{x}_\varepsilon(s) = (\sinh s, \varepsilon s + \sinh s)$ which, by the algorithm, defines the following stationary flow

$$B(s) = \frac{1}{(\varepsilon(\varepsilon + 2\cosh s))^{1/2}} \left(e^{i \sinh s}, e^{i(\varepsilon s + \sinh s)}\right).$$

Then we get the following classical string solution

$$\begin{aligned} X(s,t) &= \left(-\frac{\varepsilon \sinh s}{(\varepsilon(\varepsilon + 2\cosh s))^{3/2}} + \frac{t}{(\varepsilon(\varepsilon + 2\cosh s))^{1/2}}\right) \left(e^{i \sinh s}, e^{i(\varepsilon s + \sinh s)}\right) \\ &- \frac{i}{(\varepsilon(\varepsilon + 2\cosh s))^{1/2}} \left(\cosh s\, e^{i \sinh s}, (\varepsilon + \cosh s)\, e^{i(\varepsilon s + \sinh s)}\right). \end{aligned}$$

6 The new classical string solutions via the Pohlmeyer mechanism

It is well known that the classical string theory in the anti de Sitter 3-space is equivalent to the sinh-Gordon theory via the Pohlmeyer reduction (see [9, 12]). Therefore, each classical

string solution in $\mathbf{AdS}_3$ can be written, at least theoretically, in terms of a wavefunction (a solution of the sinh-Gordon equation). However, finding explicit solutions via this inverse Pohlmeyer mechanism, in general, involves formidable computations (see [8] and references therein). In this section, we describe how to translate the new classical string solutions to the language of Pohlmeyer reduced theory. In this sense, we show that every null scroll in $\mathbf{AdS}_3$, no matter the value of its mean curvature, can be viewed as a solution of the Liouville equation, which as it is well known defines a submodel of that associated with the sinh-Gordon equation. However, the converse of this fact also works and then we obtain our main result:

Theorem 6.1 *Every null scroll provides a solution of the generalized Liouville model* (GL). *Conversely, every solution of the generalized Liouville model* (GL) *provides a class, labeled in the space of one variable functions, of non congruent null scrolls. Therefore, up to one variable functions, there exists a one-to-one correspondence between the class of null scrolls in* $\mathbf{AdS}_3$ *and the solutions of the generalized Liouville model* (GL).

To prove this result, it is worth remembering several important facts about the geometry of timelike surfaces in $\mathbf{AdS}_3$. It is well known that *every timelike surface in* $\mathbf{AdS}_3$*, can be parameterized by two families of lightlike curves.* In other words, we can use *null coordinates* to parameterize timelike surfaces. This is a chief point that provides the geometric support to the Virasoro constraints. To make it clear, let us use the following better known notation in this context. The choice of a null coordinate system $Y(z,\bar{z})$ on a certain timelike surface, say $S \subset \mathbf{AdS}_3$, allows one to write the induced metric as $e^{\phi}\, dz\, d\bar{z}$ for a certain function $\phi(z,\bar{z})$. If we write $\partial Y = Y_z$ and $\bar{\partial} Y = Y_{\bar{z}}$, then we automatically obtain the Virasoro constraints

$$\partial Y \cdot \partial Y = \langle Y_z, Y_z \rangle = 0, \qquad \bar{\partial} Y \cdot \bar{\partial} Y = \langle Y_{\bar{z}}, Y_{\bar{z}} \rangle = 0,$$

so they are equivalent to the use of null coordinates. Then we see that the Virasoro constraints are invariant under conformal changes in the surface. In particular, they are also invariants under conformal changes in the anti de Sitter metric and so they are actually established in the conformal class of the target spacetime metric.

Now, the Gauss-Codazzi equations for $Y(z,\bar{z})$ in $\mathbf{AdS}_3$ have the following form

$$\phi_{z\bar{z}} + \frac{1}{2}(H^2 - 1)e^{\phi} - 2PQe^{-\phi} = 0, \tag{23}$$

$$P_{\bar{z}} = \frac{1}{2}e^{\phi} H_z, \tag{24}$$

$$Q_z = \frac{1}{2}e^{\phi} H_{\bar{z}}, \tag{25}$$

where H is the mean curvature function and $P\, dz^2$, $Q\, d\bar{z}^2$ are the Hopf differentials (all these data being invariant under the choice of null coordinates) and $P = \langle Y_{zz}, N \rangle$ and $Q = \langle Y_{\bar{z}\bar{z}}, N \rangle$, N being the Gauss map. Note that (23) is a generalized sinh-Gordon equation that can be

transformed, via a suitable change, in the sinh-Gordon equation (see [8]). In this setting, the Gaussian curvature is given by

$$K = H^2 - 1 - 4PQe^{-2\phi},$$

which is combined with (23) to obtain the following generalized Liouville equation

$$\phi_{z\bar{z}} + \frac{1}{2}\, Ke^{\phi} = 0. \tag{26}$$

If $Y(z,\bar{z})$ denotes a null coordinate system on a null scroll, $S(\gamma, B)$ in $\mathbf{AdS}_3$, then it admits a foliation by null straight lines in $\mathbf{AdS}_3$ and consequently a Hopf differential vanishes identically. Without loss of generality, we may assume that $Q = 0$, that is, we assume that the $\bar{z}$-curves correspond to the null scroll rulings. Hence, both mean and Gaussian curvatures are functions of one variable and then (23) and (26) are equivalent to the Liouville equation

$$\phi_{z\bar{z}} + \frac{1}{2}(H^2 - 1)e^{\phi} = 0, \quad H^2 = K + 1. \tag{27}$$

As a consequence, the wave function, $\phi(z,\bar{z})$, and the Hopf differential, $P(z,\bar{z})dz^2$, of a null scroll are solutions of (24) and (27), respectively. These equations can be completely integrated. The general solution of the Liouville equation (27) can be expressed in terms of a pair of one variable functions $R(z)$ and $\tilde{R}(\bar{z})$, as follows (see the Appendix for more details)

$$\phi(z,\bar{z}) = \ln\left[\frac{4R_z}{(H^2-1)}\left(\frac{1}{R+\tilde{R}}\right)_{\bar{z}}\right], \qquad \frac{R_z\tilde{R}_{\bar{z}}}{H^2-1} < 0. \tag{28}$$

Once the general solution of the Liouville equation is known, we can express the general solution of (24), using an additional one variable function $m(z)$, as follows

$$P(z,\bar{z}) = -\frac{2H_z(R+\tilde{R})}{H^2-1}\left(\frac{1}{R+\tilde{R}}\right)_z + m. \tag{29}$$

Therefore, we have proved that given a null scroll, $S(\gamma, B)$, with mean curvature function H in $\mathbf{AdS}_3$, then its wave function ϕ and its non trivial Hopf differential $P\,dz^2$ are given, respectively, by (28) and (29).

Now the natural problem is a sort of converse. *Given a solution of* (24) *and* (27)*, we ask for a null scroll in* $\mathbf{AdS}_3$ *whose wave function and non trivial Hopf differential are given, respectively, by* (28) *and* (29). In other words, we wish to recover, explicitly, the null scroll from the data (ϕ, P). To do that we propose a new algorithm:

(1) To determine a null scroll, naturally parameterized by

$$X(s,t) = \gamma(s) + t\, B(s),$$

from the data (ϕ, P), we have to compute $s(z)$ and $t(z,\bar{z})$ in order to

$$Y(z,\bar{z}) = X(s(z), t(z,\bar{z})) = \gamma(s(z)) + t(z,\bar{z})\, B(s(z)),$$

provides a null coordinates parameterization associated with (ϕ, P).

(2) As $\langle Y_{\bar z}, Y_{\bar z}\rangle = 0$, we must to determine the above function as solutions of $\langle Y_z, Y_z\rangle = 0$ and $\langle Y_{\bar z}, Y_z\rangle = \frac{1}{2}\, e^{\phi(z,\bar z)}$. An easy computation gives us

$$s_z\, t_{\bar z} = \frac{2R_z\tilde R_{\bar z}}{(H^2-1)(R+\tilde R)}, \qquad 2\, t_z = t^2 s_z(H^2-1),$$

which yield

$$s(z) = 2\int \frac{R_z}{H^2-1}, \qquad t(z,\bar z) = -\frac{1}{R(z)+\tilde R(\bar z)}.$$

(3) We can now find the three functions, that control the core of the null scroll, in terms of the Liouville data. It is clear that $\sigma = 0$ and $\tau(s) = H(s)$. To compute the curvature of the base curve, we proceed as follows. First, we observe that

$$Y_{zz} = s_{zz}\, X_s + s_z^2\, X_{ss} + t_{zz}\, X_t + t_z^2\, X_{tt},$$

so that

$$P(z,\bar z) = \langle Y_{zz}, N\rangle = s_z^2\langle X_{ss}, N\rangle.$$

Therefore

$$\kappa(s) = \lim_{t\to 0}\left(\langle X_{ss}, N\rangle\right) = \lim_{t\to 0}\left(P(z,\bar z)\frac{(H^2-1)^2}{(2R_z)^2}\right) = \frac{m(H^2-1)^2}{(2R_z)^2}.$$

(4) The chief point is that this argument can be reversed. Once the data (ϕ, P) are given, we can construct the functions (σ,κ,τ), and then use them to obtain the null scroll. In other words, we can solve the first order linear differential equation (2).

This algorithm shows that the Liouville data, (ϕ, P), determine completely the congruence class of a null scroll in $\mathbf{AdS}_3$. However, each solution ϕ of the generalized Liouville equation (GL) determines a family of non congruent isometric null scrolls, labeled in the space of one variable functions $\{m(z) : z\in\mathbb{R}\}$. This completes the proof of the theorem.

In particular, we can consider the class of stationary ($H = 0$) null scrolls, which provides classical string solutions in $\mathbf{AdS}_3$. Then, as a consequence of the theorem, we show how the Pohlmeyer reduced mechanism works on this new class of string solutions.

Corollary 6.2 *There exists a one-to-one correspondence between the class of stationary null scrolls (classical string solutions) and the solution of the Liouville equation*

$$\phi_{z\bar z} = \frac{1}{2}\, e^{\phi}. \tag{30}$$

The solutions of (30) can be written, in terms of two one variable functions $R(z)$ and $\tilde R(\bar z)$, as follows

$$\phi(z,\bar z) = \ln\frac{4R_z\tilde R_{\bar z}}{(R+\tilde R)^2}, \qquad R_z\tilde R_{\bar z} > 0. \tag{31}$$

Now, the inverse scattering problem can be solved according to the following algorithm. We start with a wave function, a solution of (31), and then we construct a classical string solution in $\mathbf{AdS}_3$ as a stationary null scroll as follows:

(1) Choose an additional one variable function, $m(z)$, which will take control on the non trivial Hopf differential of the solution.

(2) Define the function $\kappa(z) = \dfrac{-m(z)}{(2R_z)^2}$, and solve the linear system of differential equations

$$\gamma' = A, \quad A' = \kappa\, C, \quad B' = -\gamma, \quad C' = \kappa\, B,$$

to obtain the core $\{\gamma, A, B, C\}$ of the solution.

(3) Finally, the null scroll parameterized by

$$Y(z,\bar{z}) = \gamma\left(-2\int R(z)\,dz\right) - \frac{1}{R(z)+\tilde{R}(\bar{z}}\,B\left(-2\int R(z)\,dz\right),$$

provides the classical string solution associated with the given wave function.

Appendix: On the solutions of the Liouville equation

In 1853, J. Liouville studied the second order hyperbolic nonlinear partial differential equation (27) and obtained the general solution (28). However his proof was incomplete in the sense that it did not address uniqueness considerations for initial conditions value problem associated with the equation. Several simple approaches to obtain the general solution of the Liouville equation are well known (see for example ([15])). Perhaps, the most typical one uses a Bäcklund transform to turn the Liouville equation $\phi_{z\bar{z}} = e^{\phi}$, which is obviously nonlinear, into the linear wave equation $\tilde{\phi}_{z\bar{z}} = 0$. Then they consider the well known general solution of this equation to obtain, via that Bäcklund transformation, the general solution of the Liouville equation.

Below, we use a different approach to briefly revise the complete proof to obtain the general solution of this equation. For simplicity, we will consider that H is constant, which does not involve any constraint. We are indeed interested in the case of constant mean curvature and particularly in stationary null scrolls ($H = 0$), because they provide the classical string configurations of the model. We start with a solution $\phi(z,\bar{z})$ of (27) and write $\mu(z,\bar{z}) = \phi_z(z,\bar{z})$. Then

$$\phi(z,\bar{z}) = \int_0^z \mu(t,\bar{z})\,dt + g(\bar{z}), \qquad g(\bar{z}) = \phi(0,\bar{z}),$$

and $\mu_{\bar{z}} = \phi_{z\bar{z}} = -\frac{1}{2}(H^2-1)e^{\phi}$. Therefore, as

$$\left[\mu_z - \frac{\mu^2}{2}\right]_{\bar{z}} = \left[\phi(z,\bar{z}) + \frac{1}{2}(H^2-1)e^{\phi}\right]_z = 0,$$

there exists a second one variable function $f(z) = \mu_z(z,0) - \frac{\mu^2(z,0)}{2}$. This equation, for any $\bar{z}$, has a unique solution, $\mu(z,\bar{z})$, provided we specify an initial condition at $z = 0$. When

moving $\bar{z}$, all those conditions are encoded in the function $\delta(\bar{z}) = \mu(0,\bar{z}) = \phi_z(0,\bar{z})$, which satisfies

$$\delta_{\bar{z}} = \phi_{z\bar{z}|_{z=0}} = -\frac{1}{2}(H^2-1)e^{g(\bar{z})}.$$

To know $\delta(\bar{z})$, we only need $\xi = \mu(0,0)$.

The above argument can be summarized as follows: A solution $\phi(z,\bar{z})$ is uniquely determined provided the following initial data are specified

$$f(z) = \phi_{zz}(z,0) - \frac{\phi_z^2(z,0)}{2}, \quad g(\bar{z}) = \phi(0,\bar{z}), \quad \xi = \phi_z(0,0). \tag{32}$$

Finally, to show that all solutions of the Liouville equation, (27), are considered in (28), we can proceed as follows. First, it is not difficult to check that every function of the type (28) is automatically a solution of (27). As for the converse, we first compute the initial conditions, (32), associated with a solution of the type (28). Then, given any solution $\phi(z,\bar{z})$ of (27), we turn the above argument back to construct a second solution, say $\tilde{\phi}(z,\bar{z})$, of the type (28), satisfying the same initial conditions as $\phi(z,\bar{z})$, and so they should agree.

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