How big is the family of stationary null scrolls?

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Abstract

Every minimal ruled surface in the Euclidean three space, $\mathbb{E}^3$, is congruent to either a plane or a helicoid. In particular, the plane is the only minimal cylinder in $\mathbb{E}^3$. An analogous result is known in the Lorentz-Minkowski three space, $\mathbb{L}^3$, if we restrict ourselves to those ruled surfaces where the ruling flow is non-null. In fact, the moduli space of stationary Lorentzian ruled surfaces with non-null ruling flow is formed by five congruence classes. However, when the ruled surfaces are generated by lightlike (null) ruling flows in $\mathbb{L}^3$, then a deep difference, in the study of cylinders seen as null scrolls, is obtained. We show that the moduli space of stationary null scrolls in $\mathbb{L}^3$ can be regarded as a kind of circle bundle on the moduli space of null curves in that background. This completes the classification of stationary ruled surfaces in $\mathbb{L}^3$.

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1 Motivation, examples and main result

Let us begin with the following elemental experiment. Let $\delta(s)$ be an arclength regular curve in the Euclidean three space $\mathbb{E}^3$. Then for any unit vector $\vec{x} \in \mathbb{E}^3$, everywhere

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transversal to $\delta(s)$, we can build the corresponding cylinder $\mathbf{C}(\delta, \vec{x})$ with base curve $\delta(s)$ and whose generatrices are parallel to $\vec{x}$ parameterized by

$$\mathbf{X}(s,t) = \delta(s) + t\,\vec{x}.$$

A classical manipulation allows one to view the cylinder as the right one constructed on the planar curve

$$\alpha(s) = \delta(s) - \langle \delta(s), \vec{x} \rangle\, \vec{x}$$

obtained when projecting the base curve on the plane orthogonal to $\vec{x}$. Now, an elementary computation shows that the mean curvature function of $\mathbf{C}(\delta, \vec{x})$ is, up to a constant, the curvature function of that new planar base curve

$$H = \frac{1}{2}\,\kappa_\alpha.$$

Therefore, the plane turns out to be the only minimal (mean curvature vanishing identically) cylinder in $\mathbb{E}^3$.

Stationary surfaces, those Lorentzian (also called timelike) ones vanishing identically their mean curvature function in $\mathbb{L}^3$, play an important role not only in Mathematics but also in Physics, especially in problems related to General Relativity. Formally, they could be considered as the Lorentzian counterpart of minimal surfaces in $\mathbb{E}^3$. In particular, stationary ruled surfaces in $\mathbb{L}^3$ have been widely considered in the literature (see for example [1, 2, 3, 4, 5, 6] and references therein), and several properties of minimal surfaces have well-known Lorentzian counterparts. For example, we can consider right cylinders built on non null planar curves and then verify that both the Euclidean and the Lorentzian planes are the only stationary cylinders with non-null ruling flow.

However, important and deep differences are observed when one studies stationary surfaces in $\mathbb{L}^3$. For example, a classical exercise shows that, besides the plane, the helicoid is the only ruled minimal surface in the Euclidean three space. The corresponding problem for stationary ruled surfaces in $\mathbb{L}^3$ needs, obviously, a previous consideration derived from the causal character of the so-called ruling flow which, at first, could be either null (or lightlike) or not null (spacelike or timelike).

We will see $\mathbb{L}^3$ as $\mathbb{R}^3$ endowed with the inner product

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1\, y_1 + x_2\, y_2 - x_3\, y_3.$$

Let $\gamma : I \subset \mathbb{R} \to \mathbb{L}^3$ be a curve in $\mathbb{L}^3$ and $B(s)$, $s \in I$, a vector field along that curve which is transversal to $\gamma(s)$ everywhere. Let $X : I \times \mathbb{R} \to \mathbb{L}^3$ be the immersion defined by

$$X(s,t) = \gamma(s) + t\, B(s).$$

As for the first fundamental form we have $X_s = \gamma'(s) + t\, B'(s)$ and $X_t = B(s)$; so $X(s,t)$ parameterizes a Lorentzian surface, $S(\gamma, B)$, on the domain $\{(s,t) \in I \times \mathbb{R} : \mathbf{g}(s,t) = \langle X_s, X_s \rangle \langle X_t, X_t \rangle - \langle X_x, X_t \rangle^2 < 0\}$, which we will call a ruled surface with base curve

$\gamma(s)$ and ruling flow $B(s)$. From now on, both the base curve and the ruling flow will be referred to as the fundamental data of a ruled surface. In this paper we deal with the following question

Q1. *What is the size of the class of stationary Lorentzian ruled surfaces*?

We can, a priori, consider three subfamilies of ruled surfaces according to the causal character (spacelike, timelike or lightlike) of the ruling flow. Nevertheless, it will be sufficient to consider only two classes. In fact, we can collect the first two cases in the class of ruled surfaces with non-null ruling flow, while the third one corresponds to the class of ruled surfaces with null ruling flow, the so-called null scrolls. The following surfaces, defined on suitable open intervals, belong to the class of ruled surfaces with non-null ruling flow (see [7] and references therein).

(1) *Lorentzian helicoid of the first kind.* Defined by

$$\mathcal{H}_1: \ \ \mathbf{X}(s,t) = \gamma(s) + t\,B(s) = (0,s,0) + \frac{t}{\sqrt{1-c^2\,s^2}}\,(cs,0,1).$$

(2) *Lorentzian helicoid of the second kind.* Defined by

$$\mathcal{H}_2: \ \ \mathbf{X}(s,t) = \gamma(s) + t\,B(s) = (0,s,0) + \frac{t}{\sqrt{c^2\,s^2-1}}\,(1,0,cs).$$

(3) *Lorentzian helicoid of the third kind.* Defined by

$$\mathcal{H}_3: \ \ \mathbf{X}(s,t) = \gamma(s) + t\,B(s) = (0,0,s) + \frac{t}{\sqrt{1+c^2\,s^2}}\,(1,-cs,0).$$

(4) *Conjugate of Enneper's Lorentzian surface of the second kind.* Defined by

$$\mathcal{E}: \ \ \mathbf{X}(s,t) = \gamma(s) + t\,B(s) = (0,s,0) + \frac{t}{\sqrt{1-2cs}}\,(cs,0,1-cs).$$

It should be noted that while the third helicoid has spacelike ruling flow, the other three ruled surfaces have timelike ruling flow. A nice well-known result (see for example [7]) guarantees that, besides the Lorentzian plane, the above ruled surfaces define the moduli space of stationary ruled surfaces with non-null ruling flow. Therefore, this space is made up of five congruence classes of surfaces.

Consequently, to obtain the complete answer to the stated question, we will focus on the class of stationary ruled surfaces with lightlike ruling flow, that is, the family of stationary null scrolls. In this paper we will completely solve the following problem

P1. *Determine the moduli space of stationary null scrolls.*

In particular, we will exhibit the exceptionally big size of the space of congruence classes of stationary null scrolls, which will allow us to obtain the complete classification of the family of stationary ruled surfaces in $\mathbb{L}^3$ according to the following result

Main theorem. *Every stationary ruled surface in* $\mathbb{L}^3$ *is congruent to one of the following surfaces*

1. *The Lorentzian plane.*
2. *The helicoid of the first kind.*
3. *The helicoid of the second kind.*
4. *The helicoid of the third kind.*
5. *The conjugate surface of Enneper of the second kind.*
6. *A null scroll* $\mathbf{S}(c,\omega,\varphi)$ *parameterized by* $\mathbf{X}(\gamma_{c\omega}, B_\varphi)$, *with*

$$\gamma_{c\omega}(s) = \int_0^s c(t)\,(\cos\omega(t), \sin\omega(t), 1)\,dt$$

and

$$B_\varphi(s) = \frac{\csc^2\left(\frac{\omega(s)-\varphi}{2}\right)}{2\,c(s)}\,(\cos\varphi, \sin\varphi, 1),$$

where $c(s)$ *and* $\omega(s)$ *are real functions,* $c(s) > 0$ *everywhere, both defined on a certain real interval, and* $\varphi \in \mathbb{R}$.

Moreover, in the above list, every two of them can not be congruent. So the moduli space of stationary ruled surfaces is that of congruence classes of these surfaces.

2 About the geometry of null scrolls

We wish to know in what measure the geometry of null scrolls is encoded in its fundamental data. Let $\mathbf{S}(\gamma, B)$ be a null scroll parameterized by $\mathbf{X}(s,t) = \gamma(s) + t\,B(s)$. Then the ruling flow satisfies that $\langle B(s), B(s)\rangle = 0$ and it can be chosen so that

$$\langle \gamma'(s), B(s)\rangle = -1 \qquad \textit{normalization condition.}$$

Moreover, if the base curve is non-null then, without loss of generality, it can be parameterized by its arclength and so $\langle \gamma'(s), \gamma'(s)\rangle = \epsilon$, where $\epsilon = \pm 1$ according to the curve being spacelike or timelike. However, if necessary, we can change, the base curve to get a lightlike one (see [8]). Indeed, when $\langle \gamma'(s), \gamma'(s)\rangle = \epsilon$, we look for a curve $\beta(s) = \gamma(s) + t(s)\,B(s)$ satisfying $\langle \beta'(s), \beta'(s)\rangle = 0$ and $\langle \beta'(s), B(s)\rangle \neq 0$. From $\beta'(s) = \gamma'(s) + t'(s)\,B(s) + t(s)\,B'(s)$, we determine the function $t(s)$ to be a solution of the following Ricatti differential equation

$$2\,t'(s) = \langle B'(s), B'(s)\rangle\, t^2(s) + 2\langle \gamma'(s), B'(s)\rangle\, t(s) + \epsilon, \tag{1}$$

where it should be noted that $\langle \beta'(s), B(s)\rangle = \langle \gamma'(s), B(s)\rangle = -1$. Consequently, any null scroll $\mathbf{S}(\gamma, B)$ can be parameterized by $\mathbf{X}(s,t) = \gamma(s) + t\,B(s)$ with null base curve such that

$$\langle \gamma'(s), \gamma'(s)\rangle = \langle B(s), B(s)\rangle = 0, \qquad \langle \gamma'(s), B(s)\rangle = -1.$$

Example 2.1 *Consider the null scroll built on the base curve* $\gamma(s) = (\cos s, \sin s, 1)$ *with lightlike ruling flow* $B(s) = (h(s), 0, h(s))$, *where* $h(s) = \csc s$, *given by*

$$\mathbf{X}(s,t) = (\cos s + t\,h(s), \sin s, 1 + t\,h(s)).$$

To parameterize this null scroll on a null base curve, we need to obtain a solution of the Ricatti differential equation (1) which, in this case, becomes a linear differential equation,

$$t'(s) = \frac{\epsilon}{2} - t(s)\,\cot s,$$

because $B'(s)$ *still is lightlike. The general solution is*

$$t(s) = \sin s \left[\frac{\epsilon}{2}\int \frac{ds}{\sin s} + C\right], \qquad C = \textit{constant}.$$

This provides a one-parameter family of lightlike curves that work as base curves for the given null scroll.

In that setting, the induced metric on the null scroll $\mathbf{S}(\gamma, B)$ is given by the following matrix

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 2t\,\langle \gamma'(s), B'(s)\rangle + t^2\,\langle B'(s), B'(s)\rangle & -1 \\ -1 & 0 \end{pmatrix}.$$

To compute the Gauss map and the shape operator of $\mathbf{S}(\gamma, B)$, we define $C(s) = \gamma'(s) \times B(s)$ (see [8]). Then it is a unit spacelike vector field along $\gamma(s)$, which is anywhere orthogonal to $\mathbf{S}(\gamma, B)$ and so it defines the Gauss map of the null scroll along its directrix. The Gauss map is given by

$$N(s,t) = \frac{X_s(s,t) \times X_t(s,t)}{|g_{11}g_{22} - g_{12}^2|} = C(s) + t\,B'(s) \times B(s).$$

Now, an easy computation allows us to see that

$$B'(s) \times B(s) = -f(s)\,B(s),$$

where $f(s) = \langle \gamma'(s), B'(s) \times B(s)\rangle = \det[\gamma'(s), B'(s), B(s)]$, and then the Gauss map of the null scroll is given by

$$N(s,t) = -t\,f(s)\,B(s) + C(s).$$

Sometimes the function $-f(s)$ is called the *parameter of distribution* of the ruled surface $\mathbf{S}(\gamma, B)$.

A straightforward computation yields

$$N_s = -f\,X_s - \langle \gamma', \gamma'' \times B\rangle + t\,f')\,X_t$$

and

$$N_s = -f\,X_t,$$

so the matrix of the shape operator dN is

$$dN \equiv \begin{pmatrix} f & t\,f' + \langle \gamma', \gamma'' \times B \rangle \\ 0 & f \end{pmatrix}.$$

Therefore, the mean and Gauss curvature functions of the null scroll $\mathbf{S}(\gamma, B)$ are given by

$$H(s,t) = f(s) \qquad \text{and} \qquad K(s,t) = f(s)^2.$$

The mean curvature function of a null scroll is, up to the sign, the parameter of distribution.

3 Constructing null scrolls

It is clear that a null scroll in $\mathbb{L}^3$ is determined when giving both a lightlike curve, making the role of base curve, and a lightlike ruling flow. Furthermore, these data must satisfy a normalized constraint of transversality. The tangent vector field to the base curve lies in the light cone and we recuperate the curve by quadratures, up to congruences in $\mathbb{L}^3$, from its tangent vector field. Consequently, a null scroll, $\mathbf{S}(\gamma, B)$, is completely determined up to motions when we know $\gamma'(s)$ and $B(s)$ lying in the light cone and satisfying

$$\langle \gamma'(s), B(s) \rangle = -1 \qquad \textit{normalization condition.}$$

As usual, choose an orthonormal frame in $\mathbb{L}^3$, where the Lorentz-Minkowski metric is written as $dx^2 + dy^2 - dz^2$ and the light cone $\mathbf{C}$ by

$$x^2 + y^2 = z^2.$$

In this framework any vector $\vec{u} \in \mathbf{C}$ may be written as

$$\vec{u} = (x, y, z) = z\,(\cos\alpha, \sin\alpha, 1).$$

Therefore, (z, α), which we will call *elliptic polar coordinates*, parameterize $\mathbf{C}$. Observe that the corresponding coordinate curves are straight lines and circles, respectively. We have that

$$\gamma'(s) = c(s)\,(\cos\omega(s), \sin\omega(s), 1), \qquad B(s) = r(s)\,(\cos\varphi(s), \sin\varphi(s), 1),$$

and the normalization condition yields

$$\cos(\omega - \varphi) = 1 - \frac{1}{c(s)\,r(s)}.$$

A straightforward computation allows us to obtain the mean curvature of a null scroll

$$H(s,t) = f(s) = -\det[\mathbf{x}'(s), B(s), B'(s)] =$$

$$= -c(s)\, r(s)\, r'(s) \det \begin{pmatrix} \cos\omega(s) & \sin\omega(s) & 1 \\ \cos\varphi(s) & \sin\varphi(s) & 1 \\ \cos\varphi(s) & \sin\varphi(s) & 1 \end{pmatrix} -$$

$$-c(s)\,(r(s))^2 \det \begin{pmatrix} \cos\omega(s) & \sin\omega(s) & 1 \\ \cos\varphi(s) & \sin\varphi(s) & 1 \\ -\varphi'(s)\,\sin\varphi(s) & \varphi'(s)\,\cos\varphi(s) & 0 \end{pmatrix}$$

and then

$$H(s,t) = -c(s)\,(r(s))^2\,[\varphi'(s) - \varphi'(s)\,\cos(\omega-\varphi)] = -c(s)\,(r(s))^2 \left[\varphi'(s) - \varphi'(s) + \frac{\varphi'(s)}{c(s)\,r(s)}\right].$$

Consequently, we find that

$$H(s,t) = f(s) = -r(s)\,\varphi'(s).$$

Remark 3.1 *The mean curvature of a null scroll only depends on the lightlike ruling flow.*

The stationary surfaces in $\mathbb{L}^3$ are the critical points of the area, and they correspond to surfaces with zero mean curvature function. In particular, the mean curvature function of a null scroll vanishes identically if and only if the lightlike ruling flow is made up of parallel rulings. In other words, the angular function associated with this flow is constant $\varphi(s) = \varphi_o \in \mathbb{R}$. As a consequence, we obtain the following algorithm to construct stationary null scrolls.

1. We first need a lightlike base curve, $\gamma(s)$, defined on a certain interval $I \subset \mathbb{R}$. To get it, we build its tangent vector field in the light cone and recuperate the curve, up to congruences, by quadratures. However,

$$\gamma'(s) = c(s)\,(\cos\omega(s), \sin\omega(s), 1), \qquad s \in I,$$

therefore, to find the base curve we need two functions $c, \omega : I \subset \mathbb{R} \to \mathbb{R}$, defined in a suitable interval, one of them positive, $c(s) > 0$ anywhere.

2. To construct the ruling flow we need, a priori, a real number, say $\varphi_o \in \mathbb{R}$, and a function $r : I \subset \mathbb{R} \to \mathbb{R}$, $r(s) > 0$ anywhere. Then write

$$B(s) = r(s)\,(\cos\varphi_o, \sin\varphi_o, 1).$$

However, these data should obey the normalization condition, which allows one to determine the function $r(s)$ in terms of the remaining data by

$$r(s) = \frac{\csc^2\left(\frac{\omega(s)-\varphi_o}{2}\right)}{2\,c(s)}.$$

3. Therefore, to determine a stationary null scroll we need three parameters: a positive function $c \in \mathcal{F}^+(I)$, a function $\omega \in \mathcal{F}(I)$ and real number $\varphi_o \in \mathbb{R}$. With these data, the stationary null scroll $\mathbf{S}(c, \omega, \varphi_o)$ will be parameterized by $\mathbf{X}(\gamma_{c\omega}, B_{\varphi_o})$, where

$$\gamma_{c\omega}(s) = \int_0^s c(t)\,(\cos\omega(t), \sin\omega(t), 1)\,dt$$

and

$$B_{\varphi_o}(s) = \frac{\csc^2\left(\frac{\omega(s)-\varphi_o}{2}\right)}{2\,c(s)}\,(\cos\varphi_o, \sin\varphi_o, 1).$$

Remark 3.2 *As one will see later, this algorithm allows one to construct the complete moduli space of stationary null scrolls built on a given lightlike curve as base curve. Moreover, it will allow us to identify the complete moduli space, or the space of congruence classes, of stationary null scrolls in* $\mathbb{L}^3$.

For example, identify $\mathbb{L}^3$ with $\mathbb{C} \times \mathbb{R}$ in an obvious way, and choose the lightlike helix

$$\gamma(s) = (-i\,e^{i\,s}, s), \qquad s \in \mathbb{R}.$$

Then, the space of stationary null scrolls built on this helix as a base curve is given by

$$\left\{ \mathbf{X}(s,t) = (-i\,e^{i\,s}, s) + \frac{t}{1-\cos(s-\varphi_o)}\,(e^{i\varphi_o}, 1) \,:\, e^{\varphi_o} \in \mathbb{S}^1 \right\},$$

and, as we will see, this set identifies the moduli space of stationary null scrolls built on this helix in the sense that these null scrolls determine different congruence classes in $\mathbb{L}^3$ providing that they proceed from different points in the unit circle.

4 Congruence classes of stationary null scrolls

We have just seen that the construction of a stationary null scroll needs a couple of functions (one of them positive) and a real number. Now we wish to know the congruence classes of stationary null scrolls; said otherwise, we want to find out how those classes of $\mathbf{S}(c, \omega, \varphi_o)$ evolve when changing each one of the corresponding parameters.

An obvious comment, for instance, is that $\mathbf{S}(c, \omega, \varphi_o)$ and $\mathbf{S}(c, \omega, \varphi_o + 2\pi n)$, with $n \in \mathbb{Z}$, define the same congruence class, for any integer, because they coincide. Therefore, a previous partial question is the following

Q2. *Determine the moduli space of stationary null scrolls built on the same base curve.*

To give an appropriate answer, we have to parameterize a stationary null scroll by a double family of lightlike curves. Let $\mathbf{S}(\gamma, \varphi_o) = \mathbf{S}(c, \omega, \varphi_o)$ be a stationary null scroll and write

$$B_{\varphi_o}(s) = r(s)\,(\cos\varphi_o, \sin\varphi_o, 1) = r(s)\,\vec{u}_o,$$

$\vec{u}_o$ being a fixed lightlike vector. Then, we can define the following parameterization

$$\mathbf{Z}(s,t) = \gamma(s) + t\,\vec{u}_o.$$

The induced metric, in this setting, is given by the following matrix:

$$\begin{pmatrix} 0 & h(s) \\ h(s) & 0 \end{pmatrix},$$

where the function h is given by

$$h(s) = \langle \gamma'(s), \frac{1}{r(s)}\,B_{\varphi_o}(s)\rangle = -\frac{1}{r(s)}.$$

Let $\mathbf{S}_1(\gamma_1,\varphi_o) = \mathbf{S}_1(c_1,\omega_1,\varphi_o)$ and $\mathbf{S}_2(\gamma_2,\psi_o) = \mathbf{S}_2(c_2,\omega_2,\psi_o)$ be two null scrolls which are congruent through a rigid motion, $\mathbf{M} : \mathbb{L}^3 \to \mathbb{L}^3$, defined by

$$\mathbf{M}(p) = \mathbf{A}(p) + \vec{x}.$$

Parameterizing both null scrolls as above, it is clear that this motion preserves them and so $\mathbf{M}(\gamma_1(s))$ is a coordinate curve in the second null scroll

$$\mathbf{M}(\gamma_1(s)) = \gamma_2(s) + t_o\,\vec{v}_o.$$

In particular, we have

$$\mathbf{A}(\gamma_1'(s)) = \gamma_2'(s).$$

Define a new rigid motion in $\mathbb{L}^3$ by $\bar{\mathbf{M}} = \mathbf{T} \circ \mathbf{M}$, where $\mathbf{T}$ stands for the translation associated with the lightlike vector $-t_o\,\vec{v}_o$. Then it is clear that

$$\bar{\mathbf{M}}(\mathbf{S}_1) = \mathbf{S}_2, \qquad \bar{\mathbf{M}}(\gamma_1(s)) = \gamma_2(s).$$

Thus, with the obvious meaning, we have

$$\bar{\mathbf{M}}\,(\mathbf{Z}_1(s,t)) = \mathbf{Z}_2(s,t),$$

and

$$\bar{\mathbf{M}}\,(\mathbf{Z}_1(s,t)) = \bar{\mathbf{M}}\,(\gamma_1(s) + t\,\vec{u}_o) = \bar{\mathbf{M}}(\gamma_1(s)) + t\,\mathbf{A}(\vec{u}_o) = \gamma_2(s) + t\,\mathbf{A}(\vec{u}_o).$$

Finally, we get

$$\gamma_2(s) + t\,\mathbf{A}(\vec{u}_o) = \gamma_2(s) + t\,\vec{v}_o.$$

to conclude that

$$\mathbf{A}(\vec{u}_o) = \vec{v}_o. \tag{2}$$

Now, the motion restricted to $\mathbf{S}_1(\gamma_1,\varphi_o)$ provides an isometry between both null scrolls, so that $r_1(s) = r_2(s)$, which implies that

$$c_1(s)\,(1 - \cos(\omega_1(s) - \varphi_o)) = c_2(s)\,(1 - \cos(\omega_2(s) - \psi_o)).$$

As a first consequence, if both stationary null scrolls are built on the same base curve, then $\psi_o = \varphi_o + 2\pi n$, with $n \in \mathbb{Z}$. Therefore, we have answered the previous problem according to the following statement

Theorem 4.1 *The moduli space of stationary null scrolls built on the same base curve is identified with a circle.*

Furthermore, we obtain the complete moduli space of stationary null scrolls as a consequence of the following result

Theorem 4.2 *Two stationary null scrolls are congruent if and only if their base curves are congruent through a rigid motion whose linear part preserves the ruling flows. Moreover, the rigid motion that realizes the congruence between both null scrolls coincides, up to a translation, with that performing the congruence between their base curves.*

Proof. The necessary condition is clear from (2). Conversely, by assuming that $\mathbf{M}$ is a rigid motion, with linear part $\mathbf{A}$, satisfying (2), then it is obvious that $\mathbf{M}\left(\mathbf{Z}_1(s,t)\right) = \mathbf{Z}_2(s,t)$ and consequently it realizes the congruence between both null scrolls.

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