# Variational problems on curves 

Angel Ferrández<br>Departamento de Matemáticas, Universidad de Murcia<br>Campus de Espinardo, 30100 Murcia<br>E-mail: aferr@um.es<br>URL: webs.um.es/aferr

These notes are part of the book Topics in modern differential geometry. Simon Stevin Institute lectures 2008-2009 Series: Simon Stevin Transactions on Geometry, Volume 1, eds.: S. Haesen and L. Verstraelen. Published by Simon Stevin Institute for Geometry, Tilburg (The Netherlands), 2010. ISBN: 978-90-815185-1-2.


#### Abstract

In this two hours talk we will survey on some variational problems concerning curves as well as their significance in the interplay between Geometry and Physics.


## 1 What's a variational problem?

It's a pair:
(1) Something we wish to study how it is changing; and
(2) An "admissible" action, i. e., a way to do the change. By "admissible" we mean "isometry invariant".

Example 1 Consider the set

$$
\mathcal{C}=\{\gamma:[a, b] \rightarrow M, \gamma(a)=p, \gamma(b)=q\}
$$

of nailed curves in a manifold $M$, i. e., curves having the same endpoints. Then consider the functional $E$ defined by

$$
E(\gamma)=\int_{a}^{b}\left|\gamma^{\prime}(t)\right|^{2} d t
$$



Figure 1: A curve variation

By asking ourselves for the critical points of $E$, it is well known that they are the geodesics.

In the following we wish to explain why we are interested in curves.

## 2 Willmore and Willmore-Chen functionals

Willmore functional (see [W1] and [W2]). Let $I\left(M^{2}, \mathbb{R}^{3}\right)=\{\phi$ : $\left.M^{2} \rightarrow \mathbb{R}^{3}\right\}$ be the set of immersions of a surface $M^{2}$ in the Euclidean 3-space. The Willmore functional is defined on $I\left(M^{2}, \mathbb{R}^{3}\right)$ by

$$
\mathcal{W}(\phi)=\int_{M} H^{2} d A
$$

where $H^{2}$ stands for the mean curvature of the surface in the 3 -space.
Then the extremals of $\mathcal{W}(\phi)$ are called Willmore surfaces.
In order to classify Willmore surfaces in $\mathbb{R}^{3}$ we have a first interesting result.
Theorem 1 (Willmore's theorem) Let $S \subset \mathbb{R}^{3}$ be a compact surface. Then we have
(i) $\mathcal{W}(S) \geq 4 \pi$; and
(ii) $\mathcal{W}(S)=4 \pi$ if and only if $S=\mathbb{S}^{2}(r)$.

To leave $\mathbb{R}^{3}$, let $S$ be a surface in a Riemannian manifold $(M,\langle\rangle$,$) . Now$ the Willmore functional writes down as

$$
\mathcal{W}(S)=\int_{S}\left(H^{2}+R\right) d A
$$

where $H^{2}$ stands for the mean curvature of $S$ in $M$ and $R$ is the sectional curvature of $M$ along the tangent bundle $T S$.

A crucial fact is that the Willmore functional is conformally invariant. Then $\mathbb{R}^{3}$ and $\mathbb{S}^{3}$ have the same Willmore surfaces, being minimal surfaces the trivial ones.

A test surface. It is well known that the Clifford torus $T_{C}$ is minimal, then Willmore, in $\mathbb{S}^{3}$. Furthermore, it is easy to see that

$$
\mathcal{W}\left(T_{C}\right)=2 \pi^{2}
$$

Let $I\left(T, \mathbb{R}^{3}\right)$ be the set of immersions of a torus $T$ in the Euclidean 3-space. The Willmore functional is now defined by

$$
\mathcal{W}(\phi)=\int_{T} H_{\phi}^{2} d A_{\phi}
$$

We then have
The Willmore conjecture which states that $\mathcal{W}(\phi) \geq 2 \pi^{2}$, equality holding if and only if $\phi(T)$ is conformal to the Clifford torus.

Now we get a second key point
Pinkall's theorem (see [P]). Willmore tori in $\mathbb{S}^{3}$ are obtained by lifting, via the Hopf mapping, closed elastic curves in $\mathbb{S}^{2}$.

Up to now we have worked in codimension one, but this restriction can be dropped as follows.

Willmore-Chen functional (see [Ch]). Let $I\left(\left(M^{m}, g\right),(\bar{M}, \bar{g})\right)$ be the set of immersions between two Riemannian manifolds. The Willmore-Chen functional is defined on $I\left(\left(M^{m}, g\right),(\bar{M}, \bar{g})\right)$ by

$$
\mathcal{W C}(\phi)=\int_{M}\left(H^{2}-\tau_{e}\right)^{\frac{m}{2}} d A
$$

where $H^{2}$ stands for the mean curvature of $M$ in $\bar{M}$ and $\tau_{e}$ is the extrinsic scalar curvature.

Then the extremals of $\mathcal{W C}(\phi)$ are called Willmore-Chen submanifolds.
Problem. Look for Willmore surfaces and Willmore-Chen submanifolds in pseudo-Riemannian space forms.

The solution of this problem will show us a natural and nice connection between Geometry and Physics.

## 3 Interplay between Geometry and Physics (see [B2])

We first note that the Hopf map is more than a simple tool. It provides excellent applications in different contexts in Physics, as we can see in the nice paper [U] by H. K. Urbantke.

Secondly, the Willmore functional has a certain universality regarding its physical applications: from strings and branes to membranes and vesicles, because all of them are extremals of a certain action. We are interested in knowing what kind of action.

In that line, Poisson and Sophie Germaine proposed that action should be an even and symmetric function of the principal curvatures of the surface.

In the seventies, thinking about membranes, Canham and Helfrich proposed a new model, now based on a quadratic function of the principal curvatures, given by

$$
\mathcal{C H}(S)=\int_{S}\left(a+b H^{2}+c G\right) d A
$$

$H^{2}$ and $G$ standing for the mean and Gauss curvatures, respectively, of the surface, and $a, b, c \in \mathbb{R}$.

As the topology of membranes does not change by fluctuations, then the Gauss-Bonnet theorem reduces the Canham-Helfrich functional to

$$
\mathcal{C H}(S)=\int_{S}\left(a+b H^{2}\right) d A .
$$

Furthermore, as minimality and compacity are not good mates, we can assume that $b \neq 0$. Then, by taking $b=1$, the Canham-Helfrich functional is nothing but a modified Willmore funcional.

Thirdly, it seems that strings theories will play a key role to understand physical world. A string theory is carry out in a non-flat spacetime, where strings (curves) evolve generating surfaces (worldsheets). The problem now is looking for the action describing the dynamics.

The most widely accepted nowadays is that of Kleiner and Polyakov given by

$$
\mathcal{K P}(S)=a \int_{S} d A+b \int_{S} H^{2} d A
$$

which strongly sounds Willmore again.
Finally, we have to talk about elastic curves. In 1691 James Bernouilli looked at the shape of a beam under a load, where we assume that the beam will recover its size and shape when the load is removed. Let $\gamma(s)=(x(s), y(s))$ be a parameterization of the centerline of the beam (see [St]).


Figure 2: A beam under a load

Three years later, James Bernouilli announced his solution as the following system of differential equations

$$
\begin{aligned}
d y & =\frac{x^{2}}{\sqrt{\left(1-x^{4}\right)}} d x \\
d s & =\frac{1}{\sqrt{\left(1-x^{4}\right)}} d x
\end{aligned}
$$

with an extra hypothesis: the bending moment is directly proportional to some constant related to the composition of the bar and inversely proportional to the radius of curvature. The elliptic functions were born (see Appendix).

The problem was then taken up again forty years later by Daniel Bernouilli and L. Euler. In 1742 the former suggested to Euler that
"The way to determine the shape of an elastic rod subject to the pressure at both ends was to minimize

$$
\int_{0}^{L} \frac{1}{R^{2}} d s
$$

where $s$ is the arc length, $R$ the radius of curvature and $L$ is the length of the elastic rod".

Definition 1 An elastica, or elastic curve, is a regular curve $\gamma$, with fixed endpoints and fixed tangent vectors at endpoints, which is critical of the functional

$$
\mathcal{F}^{\lambda}(\gamma)=\int_{0}^{L}\left(k^{2}+\lambda\right) d s
$$

where $L$ is the length of $\gamma, k^{2}=\|\ddot{\gamma}\|^{2}$ and $\lambda$ is an arbitrary constant.
When $\lambda=0 \gamma$ will be called free elastica.
Remark 1 The parameter $\lambda$ is viewed as a length penalty.
L. Euler was writing his book on the calculus of variations when received D. Bernouilli's suggestion. Then he treated the elastica problem in the first appendix of the book. After a qualitative analysis of the differential equations, Euler gave (presumably based only on experiments) a complete description of all possible planar elastic curves (besides circle and right line), which are sketched as follows (See [T])


Figure 3: Euler elastic classes

## 4 Langer and Singer approach for elastic curves

To look for elastic curves in a Riemannian manifold ( $M,\langle$,$\rangle ), we follow$ the nice works by Joel Langer and David Singer [LS] and $[\mathrm{Si}]$.

Let $\gamma:[a, b] \rightarrow M$ be an immersed curve and write $\|\gamma(t)\|=\frac{d s}{d t}:=v \neq 0, s$ standing for the arc length parameter. Let $k$ be the curvature function, which can be assumed to be $k \neq 0, \tau$ the torsion and $\{T, N, B\}$ the Frenet frame along $\gamma$. Then

$$
\gamma^{\prime}=\frac{d \gamma}{d t}=\frac{d \gamma}{d s} \frac{d s}{d t}=v T
$$

and the Frenet equations are

$$
\begin{aligned}
\frac{d T}{d s} & =k N \\
\frac{d N}{d s} & =-k T+\tau B \\
\frac{d B}{d s} & =-\tau N
\end{aligned}
$$

As elastica are extremals of a functional, we have to pave the way to define the set where the functional will apply. To do that, let $\mathcal{C}=\{\gamma:[a, b] \rightarrow M\}$ be a set of nailed curves in $M$ and let $\mathcal{F}: \mathcal{C} \rightarrow \mathbb{R}$ be the functional. To find the extremals of $\mathcal{F}$ we have to compute its first variation, i. e., we must know $\delta \mathcal{F}_{\gamma}: T_{\gamma} \mathcal{C} \rightarrow \mathbb{R}$. To determine $T_{\gamma} \mathcal{C}$ we follow [B1].

As for a curve in $\mathcal{C}$ let us consider a map

$$
\begin{aligned}
\Gamma:(\varepsilon, \varepsilon) & \rightarrow \mathcal{C} \\
z & \mapsto \Gamma(z)
\end{aligned}
$$

so that

$$
\begin{aligned}
\Gamma(z):[a, b] & \rightarrow M \\
t & \mapsto \Gamma(z)(t) \in M .
\end{aligned}
$$

Said otherwise

$$
\begin{aligned}
\Gamma:(\varepsilon, \varepsilon) \times[a, b] & \rightarrow M \\
(z, t) & \mapsto \Gamma(z, t) .
\end{aligned}
$$

The curve $\Gamma$ goes throughout $\gamma \in \mathcal{C}$ provided that $\Gamma(0)=\gamma$, i. e., $\Gamma(0, t)=$ $\gamma(t)$. Then $\Gamma(z, t)$ is known as a variation of $\gamma(t)$. We then have defined two vector fields

$$
V(z, t)=\frac{\partial \Gamma}{\partial t}(z, t)
$$

and

$$
W(z, t)=\frac{\partial \Gamma}{\partial z}(z, t)
$$

usually known as "longitudinal" and "transversal", respectively, such that $V(0, t)=\gamma^{\prime}(t)$. Furthermore, $W(t):=W(0, t)$ is a vector field along $\gamma(t)$ so that

$$
\Gamma^{\prime}(0)=\left.\frac{\partial \Gamma}{\partial z}\right|_{z=0}=\frac{\partial \Gamma(z, t)}{\partial z}_{\mid z=0}=W(t) .
$$

Then $T_{\gamma} \mathcal{C}$ is a subset of the set of vector fields on $M$ along $\gamma$.
Conversely, let $W(t)$ be a vector field along $\gamma$ and define

$$
\Gamma:(-\varepsilon, \varepsilon) \times[a, b] \rightarrow M
$$

by

$$
\Gamma(z, t)=\exp _{\gamma(t)} z W(t)
$$

Then

$$
\begin{aligned}
\Gamma(0, t) & =\gamma(t) \\
\left.\frac{\partial}{\partial z}\right|_{z=0} & \Gamma(z, t)
\end{aligned}=W(t) .
$$

As a consequence, $T_{\gamma} \mathcal{C}=\{$ vector fields along $\gamma\}$.

Therefore

$$
\begin{aligned}
\delta \mathcal{F}_{\gamma}: T_{\gamma} \mathcal{C} & \rightarrow \mathbb{R} \\
W & \mapsto \delta \mathcal{F}_{\gamma}(W) .
\end{aligned}
$$

Take

$$
\Gamma:(-\varepsilon, \varepsilon) \rightarrow \mathcal{C}
$$

with $\Gamma(0)=\gamma$ and $\Gamma^{\prime}(0)=W$. Then

$$
\delta \mathcal{F}_{\gamma}(W)=\frac{d}{d z}{ }_{\mid z=0} F(\Gamma(z))=\frac{\partial}{\partial z}{ }_{\mid z=0} F(\Gamma(z, t)) .
$$

We write down, for short, $W=\frac{\partial \Gamma}{\partial z}, V=\frac{\partial \Gamma}{\partial t}$ and $V=v T$, so that $\langle V, V\rangle=$ $v^{2}$. Then we get

$$
0=[W, V]=[W, v T]=W(v) T+v[W, T] .
$$

So $[W, T]=-\frac{W(v)}{v} T=g T$, where $g:=-\frac{W(v)}{v}$.
Now

$$
\begin{aligned}
2 v W(v) & =W\left(v^{2}\right)=W\langle V, V\rangle \\
& =2\left\langle\nabla_{W} V, V\right\rangle=2\left\langle\nabla_{V} W, V\right\rangle \\
& =2\left\langle\nabla_{v T} W, v T\right\rangle=2 v^{2}\left\langle\nabla_{T} W, T\right\rangle .
\end{aligned}
$$

So $g=-\left\langle\nabla_{T} W, T\right\rangle$.
On the other hand, as $k^{2}=\left\langle\nabla_{T} T, \nabla_{T} T\right\rangle$, we find that

$$
\begin{aligned}
\frac{\partial k^{2}}{\partial z} & =W\left(k^{2}\right)=2\left\langle\nabla_{W} \nabla_{T} T, \nabla_{T} T\right\rangle \\
& =2\left\langle\nabla_{T} \nabla_{W} T+\nabla_{[W, T]} T+R(W, T) T, \nabla_{T} T\right\rangle \\
& =2\left\langle\nabla_{T}\left(\nabla_{T} W+[W, T]\right)+\nabla_{[W, T]} T+R(W, T) T, \nabla_{T} T\right\rangle \\
& =2\left\langle\nabla_{T}^{2} W+\nabla_{T}(g T)+g \nabla_{T} T+R(W, T) T, \nabla_{T} T\right\rangle \\
& =2\left\langle\nabla_{T}^{2} W, \nabla_{T} T\right\rangle+2 g\left\langle\nabla_{T} T, \nabla_{T} T\right\rangle+2\left\langle R(W, T) T, \nabla_{T} T\right\rangle+2 g\left\langle\nabla_{T} T, \nabla_{T} T\right\rangle \\
& =2\left\langle\nabla_{T}^{2} W, \nabla_{T} T\right\rangle+2\left\langle R(W, T) T, \nabla_{T} T\right\rangle+4 g\left\langle\nabla_{T} T, \nabla_{T} T\right\rangle .
\end{aligned}
$$

Remark 2 In the above computations we have used
(i) $R(W, T) T=\nabla_{W} \nabla_{T} T-\nabla_{T} \nabla_{W} T-\nabla_{[W, T]} T$;
(ii) $\nabla_{W} T=\nabla_{T} W+[W, T]$; and
(iii) $[W, T]=g T$.

As $\nabla_{T} T=k N$, we get

$$
\frac{\partial k^{2}}{\partial z}=2 k\left\langle\nabla_{T}^{2} W, N\right\rangle+4 g k^{2}+2 k\langle R(W, T) T, N\rangle .
$$

Now

$$
\frac{\partial k^{2}}{\partial z}=2 k \frac{\partial k}{\partial z}
$$

so that

$$
\frac{\partial k}{\partial z}=W(k)=\left\langle\nabla_{T}^{2} W, N\right\rangle+2 g k^{2}+\langle R(W, T) T, N\rangle .
$$

In what follows $\gamma:[0,1] \rightarrow M$ will be a curve of length $L$. Now for a fixed constant $\lambda$ let us consider the functional

$$
\mathcal{F}^{\lambda}(\gamma)=\frac{1}{2} \int_{0}^{L}\left(k^{2}+\lambda\right) d s
$$

Problem. Find the critical values of $\mathcal{F}^{\lambda}$.
To do that we first observe, as $v(t)=\frac{d s}{d t}$, that

$$
\mathcal{F}^{\lambda}(\gamma)=\frac{1}{2} \int_{0}^{1}\left(k^{2}+\lambda\right) v(t) d t .
$$

Then, take a variation $\Gamma(z, t)$ of $\gamma(t)$ as above, with variational vector field $W$ and compute

$$
\begin{aligned}
\frac{d}{d z} \mathcal{F}^{\lambda}(\Gamma(z, t)) & =\frac{1}{2} \int_{0}^{1} W\left[\left(k^{2}+\lambda\right) v\right] d t \\
& =\frac{1}{2} \int_{0}^{1}\left\{W\left(k^{2}\right) v+k^{2} W(v)+\lambda W(v)\right\} d t \\
& =\frac{1}{2} \int_{0}^{1}\left\{W\left(k^{2}\right)-\left(k^{2}+\lambda\right) g\right\} v d t \\
& =\frac{1}{2} \int_{0}^{L}\left\{W\left(k^{2}\right)-\left(k^{2}+\lambda\right) g\right\} d s \\
& =\int_{0}^{L}\left\{k\left\langle\nabla_{T}^{2} W, N\right\rangle+2 g k^{2}+k\langle R(W, T) T, N\rangle-\frac{1}{2}\left(k^{2}+\lambda\right) g\right\} d s \\
& =\int_{0}^{L}\left\{k\left\langle\nabla_{T}^{2} W, N\right\rangle+2 g k^{2}+k\langle R(N, T) T, W\rangle-\frac{1}{2}\left(k^{2}+\lambda\right) g\right\} d s
\end{aligned}
$$

Integrating by parts, and using that $g=-\left\langle\nabla_{T} W, T\right\rangle$, we get

$$
\begin{aligned}
\frac{d}{d z} \mathcal{F}^{\lambda}(\Gamma(z, t))= & \int_{0}^{L}\left\{\left\langle\nabla_{T}^{2} W, \nabla_{T} T\right\rangle-\left\langle\nabla_{T} W, 2 k^{2} T\right\rangle+\left\langle R\left(\nabla_{T} T, T\right) T, W\right\rangle\right. \\
& \left.+\frac{1}{2}\left\langle\nabla_{T} W,\left(k^{2}+\lambda\right) T\right\rangle\right\} d s \\
= & \int_{0}^{L}\left\{T\left\langle\nabla_{T} W, \nabla_{T} T\right\rangle-\left\langle\nabla_{T} W, \nabla_{T}^{2} T\right\rangle\right\} d s+\int_{0}^{L}\left\langle R\left(\nabla_{T} T, T\right) T, W\right\rangle d s \\
& +\int_{0}^{L}\left\langle\nabla_{T} W, \Lambda T\right\rangle d s \\
= & \left.\left\langle\nabla_{T} W, \nabla_{T} T\right\rangle\right|_{0} ^{L}-\int_{0}^{L} T\left\langle W, \nabla_{T}^{2} T\right\rangle d s+\int_{0}^{L}\left\langle W, \nabla_{T}^{3} T\right\rangle d s \\
& +\int_{0}^{L}\left\langle R\left(\nabla_{T} T, T\right) T, W\right\rangle d s+\int_{0}^{L} T\langle W, \Lambda T\rangle d s-\int_{0}^{L}\left\langle W, \nabla_{T} \Lambda T\right\rangle d s \\
= & {\left[\left\langle\nabla_{T} W, \nabla_{T} T\right\rangle+\left\langle W,-\nabla_{T}^{2} T+\Lambda T\right\rangle\right]_{0}^{L}+\int_{0}^{L}\langle E, W\rangle d s, }
\end{aligned}
$$

where $\Lambda=\frac{\lambda-3 k^{2}}{2}$ and

$$
E=\nabla_{T}^{3} T-\nabla_{T}(\Lambda T)+R\left(\nabla_{T} T, T\right) T
$$

is the Euler-Lagrange operator.
We assume that $(M,\langle\rangle$,$) is of constant curvature C$. Then

$$
R(X, Y) Z=C(\langle Y, Z\rangle X-\langle X, Z\rangle Y),
$$

so that $R\left(\nabla_{T} T, T\right) T=C \nabla_{T} T$ and the Euler-Lagrange operator writes down as follows

$$
E=\nabla_{T}^{3} T-\nabla_{T}\left(\Lambda_{C} T\right)
$$

where $\Lambda_{C}=\Lambda-C$.
To compute $E$ we use the Frenet equations. Then we find

$$
\begin{aligned}
E & =\nabla_{T}^{3} T-\nabla_{T}\left(\Lambda_{C} T\right) \\
& =\nabla_{T}\left(\nabla_{T}^{2} T-\Lambda_{C} T\right) \\
& =\nabla_{T}\left(\nabla_{T}(k N)-\Lambda_{C} T\right) \\
& =\nabla_{T}\left(k_{s} N+k \nabla_{T} N-\Lambda_{C} T\right) \\
& =\nabla_{T}\left(k_{s} N-k^{2} T+k \tau B-\frac{\lambda-2 C-3 k^{2}}{2} T\right) \\
& =\frac{2 k_{s s}+k^{3}-\lambda k+2 C k-k \tau^{2}}{2} N+\left(2 k_{s} \tau+k \tau_{s}\right) B .
\end{aligned}
$$

The curve $\gamma$ is an elastica provided that $E=0$, said otherwise, $\gamma$ is an elastica if and only if the following system of differential equations holds

$$
\begin{aligned}
2 k_{s s}+k^{3}-\lambda k+2 C k-k \tau^{2} & =0 \\
2 k_{s} \tau+k \tau_{s} & =0 .
\end{aligned}
$$

From the second equation we get

$$
k^{2} \tau=a \text { constant. }
$$

Eliminating $\tau$ from the first equation and integrating we have

$$
k_{s}^{2}+\frac{k^{4}}{4}+\left(C-\frac{\lambda}{2}\right) k^{2}+\frac{a^{2}}{k^{2}}=A .
$$

Letting $u=k^{2}$, this becomes

$$
u_{s}^{2}+u^{3}+4\left(C-\frac{\lambda}{2}\right) u^{2}-4 A u+4 A^{2}=0
$$

whose solutions are (see [Si])

1. $u=k^{2}=$ constant and $\tau=$ constant: helices and circles.
2. $k=k_{0} \operatorname{sech}\left(\frac{k_{0}}{2 w} s\right)$ and $\tau=0$ : borderline elastica.


Figure 4: Borderline elastica
3. $k=k_{0} \operatorname{dn}\left(\frac{k_{0}}{2 w} s, p\right)$ (see Appendix for elliptic functions) and $\tau=0$ : orbitlike elastica.


Figure 5: Orbitlike elastica
4. $k=k_{0} \mathrm{cn}\left(\frac{k_{0}}{2 w} s, p\right)$ (see Appendix for elliptic functions) and $\tau=0$ : wavelike elastica (see [Br]).


Figure 6: Wavelike elastica

## 5 The Plyushchay model

This note does not pretend teaching on elastic curves, however two facts will be pointed out:
(i) What's the relationship between critical points of Willmore and critical points of elastica functionals?; and
(ii) The integrand of the elastica functional

$$
\mathcal{F}(\gamma)=\int_{\gamma}\left(k^{2}+\lambda\right) d s
$$

can be modified to get a new functional

$$
\mathcal{P}(\gamma)=\int_{\gamma} f\left(k_{i}\right) d s
$$

depending on the curvatures of $\gamma$. Then could you find the critical points of $\mathcal{P}$ ? Furthermore, could you find the physical meaning of them?

We first answer to the question (i).
Let $\Pi: \mathbb{S}^{3}(1) \rightarrow \mathbb{S}^{2}\left(\frac{1}{2}\right)$ be the usual Hopf fibration. Let $\gamma: I \subset \mathbb{R} \rightarrow \mathbb{S}^{2}\left(\frac{1}{2}\right)$ be a unit speed curve and let $\bar{\gamma}$ be its horizontal lift. Then $M_{\gamma}=\Pi^{-1}(\gamma)$ is a flat surface, which we will call the Hopf tube over $\gamma$, parameterized by

$$
\phi(s, t)=e^{i t} \bar{\gamma}(s) .
$$

When $\gamma$ is closed in $\mathbb{S}^{2}\left(\frac{1}{2}\right)$ of length $L$, enclosing an area $A$, then the Hopf tube is a flat torus isometric to $\mathbb{R}^{2} / \Gamma, \Gamma$ being the lattice spanned by $\{(0,2 \pi),(L, 2 A)\}$.

The Euler-lagrange equation for Willmore tori in $\mathbb{S}^{3}$ is

$$
\Delta^{D} H=|A|^{2} H-2\langle H, H\rangle H,
$$

which becomes

$$
2 \bar{k}^{\prime \prime}+\bar{k}^{3}+4 \bar{k}=0,
$$

$\bar{k}$ standing for the curvature of the lifting $\bar{\gamma}$.

Therefore $M_{\gamma}$ is Willmore in $\mathbb{S}^{3}$ if and only if $\gamma$ is an elastica in $\mathbb{S}^{2}\left(\frac{1}{2}\right)$, (see $[\mathrm{P}]$ ).

This was the key point to design a programme to find Willmore surfaces in the anti de Sitter space $\mathbb{H}_{1}^{3}$, as well as Willmore-Chen submanifolds in pseudoRiemannian space forms.

As for (ii) remember that we wish to consider the general functional

$$
\mathcal{P}(\gamma)=\int_{\gamma} f\left(k_{i}\right) d s
$$

and look for its critical points.
Let $(M,\langle\rangle$,$) be an n$-dimensional Riemannian manifold. As above write

$$
\mathcal{C}=\{\gamma:[a, b] \rightarrow M\}
$$

and the simplest functional

$$
\mathcal{F}_{m}: \mathcal{C} \rightarrow \mathbb{R}
$$

for any $m \in \mathbb{R}$ defined by

$$
\mathcal{F}_{m}(\gamma)=\int_{\gamma}(k(s)+m) d s
$$

This is known as the Plyushchay functional (or Plyushchay model, see [Pl]), because he was the first to use it to study trajectories of relativistic particles.

Remark 3 (1) When $m=0$ then $\mathcal{F}_{0}(\gamma)$ is nothing but the total curvature of curves in $\mathcal{C}$. If $M=\mathbb{R}^{2}$ then $k=\theta^{\prime}$. Now, if $\gamma$ is closed then

$$
\mathcal{F}_{0}(\gamma)=\int_{\gamma} k(s) d s=2 \pi i(\gamma)
$$

$i(\gamma) \in \mathbb{Z}$ being the rotation index of $\gamma$ and $\mathcal{F}_{0}$ is constant on any homotopy class of curves.

If $\mathcal{C}$ is the space of clamped curves curves, i. e., $\gamma_{z}(a)=\gamma_{z}(b), \gamma_{z}{ }^{\prime}(a)=\vec{u}$ and $\gamma_{z}^{\prime}(b)=\vec{v}$, then

$$
\mathcal{F}_{0}(\gamma)=\varphi_{0}+2 \pi \# \text { (interior loops) }
$$

where $\varphi_{0}=\operatorname{angle}(\vec{u}, \vec{v})$. Therefore, $\mathcal{F}_{0}$ is also constant on any homotopy class of clamped curves.

Summarizing, the variational problem associated with $\mathcal{F}_{0}$ on $\mathbb{R}^{2}$ has no physical interest.
(2) What about $\mathcal{F}_{0}$ when $M^{2}$ is a surface in $\mathbb{R}^{3}$ ? Take now $\mathcal{C}$ the set of one-to-one closed curves in $M^{2}$ and let $D$ be a disc in $\mathbb{R}^{2}$. Consider the space of embeddings $\left\{\Phi: D \rightarrow M^{2}\right\}$


Figure 7: An embedding

Then we have

$$
\int_{\gamma} k(s) d s+\int_{\Phi(D)} K d A=2 \pi
$$

(3) As for $\mathcal{F}_{0}$ on $M=\mathbb{R}^{3}$, some classical results are known.
(3.1) If $\gamma \subset \mathbb{R}^{3}$ is one-to-one and closed, then

$$
\int_{\gamma} k(s) d s \geq 2 \pi
$$

equality holding if and only if $\gamma$ is planar and convex.
(3.2) If $\gamma \subset \mathbb{R}^{3}$ is one-to-one, closed and knotted, then

$$
\int_{\gamma} k(s) d s \geq 4 \pi
$$

To look for the critical points of $\mathcal{F}_{m}(\gamma)=\int_{\gamma}(k(s)+m) d s$ in the general background, let

$$
\Gamma:(-\varepsilon, \varepsilon) \times[a, b] \rightarrow M
$$

be a variation of $\gamma$ defined by

$$
\Gamma(z, t)=\exp _{\gamma(t)} z W(t)
$$

$W(t)$ being a vector field along $\gamma$. Then

$$
\begin{aligned}
\Gamma(0, t) & =\gamma(t) \\
\left.\frac{\partial}{\partial z}\right|_{z=0} & \Gamma(z, t)
\end{aligned}=W(t) .
$$

and

$$
\begin{aligned}
\delta \mathcal{F}_{m}(W) & =\frac{d}{d z} \mathcal{F}_{\mid z=0}(\Gamma(z)) \\
& =\frac{\partial}{\partial z} \mathcal{F}_{\mid z=0} \mathcal{F}_{m}(\Gamma(z, t)) \\
& =\frac{\partial}{\partial z} \int_{a}^{b} k(z, t) v(z, t) d t \\
& =\int_{a}^{b} W(k) v d t+\int_{a}^{b} k W(v) d t \\
& =\int_{\gamma}\langle\Omega(\gamma), W\rangle d s+[\mathcal{B}(\gamma, W)]_{0}^{L}
\end{aligned}
$$

where

$$
\Omega(\gamma)=\nabla_{T}^{2} N+\nabla_{T}((k-m) T)+R(N, T) T
$$

stands for the Euler-Lagrange operator and

$$
\mathcal{B}(\gamma, W)=\left\langle\nabla_{T} W, N\right\rangle+\langle W, m T+\tau B\rangle
$$

is the boundary operator.
Let $\mathcal{C}$ be the set of clamped curves defined by $\mathcal{C}=\{\gamma:[a, b] \rightarrow M / \gamma(a)=$ $\left.p, \gamma(b)=q, \gamma^{\prime}(a)=\vec{u}, \gamma^{\prime}(b)=\vec{v}\right\}$. Then $T_{\gamma} \mathcal{C}=\{W$ along $\gamma: W(a)=W(b)=$ $0\}$, so that

$$
[\mathcal{B}(\gamma, W)]_{0}^{L}=0
$$

Summarizing, $\gamma$ is a critical point of $\mathcal{F}_{m}$ if and only if $\Omega(\gamma)=0$.
The condition $\Omega(\gamma)=0$ is called the Euler-Lagrange equation of the variational problem.

By using the Frenet equations, the condition $\Omega(\gamma)=0$, in a space form $M^{n}(C)$, reads as follows

$$
\begin{aligned}
\tau^{2}+m k & =C, \\
\tau_{s}^{\prime} & =0, \\
\tau \eta & =0,
\end{aligned}
$$

where $\eta \perp\{T, N, B\}$.
As a first consequence we have that $\tau=$ constant, as well as $\eta=0$. Then the critical points of this model live in a 3-dimensional totally geodesic submanifold. Furthermore, when

- When $m \neq 0$, then the critical points form a 1-parameter family of helices $\left\{(k, \tau) \in \mathbb{R}^{2}: m k+\tau^{2}=C\right\}$.
- When $m=0$ we only know that $\tau^{2}=C$, i. e., the critical points are living in $\mathbb{S}^{3}(C)$.

Without loss of generality, we can take $C=1$ and state the following problem.

Problem. Look for $\tau^{2}=1$ curves in $\mathbb{S}^{3}(1)$.
To get an answer we recall the Hopf map to find that the lifting of any curve in $\mathbb{S}^{2}\left(\frac{1}{2}\right)$ provides a curve in $\mathbb{S}^{3}(1)$ with $\tau^{2}=1$.

Going for a walk in the realm of Lorentzian world, it is easy to see that the extremals of this variational problem is given by the one-parameter family $\left\{(k, \tau) \in \mathbb{R}^{2}: \varepsilon_{2} m k-\tau^{2}=C\right\}$, where $\varepsilon_{2}=\langle N, N\rangle$ is the causal character of $N$. So they are living in the anti de Sitter world $\mathbb{H}_{1}^{3}(-1)$. Then in the anti de Sitter world the dynamics of a system of particles governed by the action $\int_{\gamma}(k(s)+m) d s$ is also described by helices.

Final remark. The beauty of the model governed by actions of the form

$$
\int_{\gamma} f\left(k_{1}(s), \cdots, k_{n}(s)\right) d s
$$

lies in the fact that the degree of freedom that were added in the classical method is actually encoded in the geometry of the particle paths.

## 6 Appendix: elliptic functions

The formulas for elastic curves involve elliptic functions. The three basic elliptic ones are denoted $\operatorname{sn}(x, m), c n(x, m)$ and $d n(x, m)$. The so called modulus $m$ is a number between 0 and 1 . The elliptic functions generalize the trigonometric ones. For instance, $s n(x, 0)=\sin (x), \operatorname{cn}(x, 0)=\cos (x)$ and $d n(x, 0) \equiv 1$. They become hyperbolic functions when $m=1$. In this case $\operatorname{sn}(x, 1)=\operatorname{tgh}(x)$ and $\operatorname{cn}(x, 1)=d n(x, 1)=\operatorname{sech}(x)$.

The inverse $s n^{-1}$ is given by

$$
\operatorname{sn}^{-1}(y, m) \int_{0}^{y} \frac{d t}{\sqrt{1-t^{2}} \sqrt{1-m t^{2}}} .
$$

Analogous to the trigonometric case, $s n$ is a smooth periodic odd function defined for all real numbers. Once $s n$ is defined, the identities

$$
\begin{aligned}
c n^{2}+s n^{2} & =1, \\
d n^{2}+m s n^{2} & =1,
\end{aligned}
$$

define both $c n$ and $d n$ as smooth periodic even functions. When $m<1$, $c n$ and $d n$ differ qualitatively in that $d n$ is a positive function and its period is half the period of $c n$. The function $K(m)=s n^{-1}(1, m)$ generalizes $\frac{\pi}{2}$. The periods are $4 K(m)$ for $s n$ and $c n$, but $2 K(m)$ for $d n$.

As for derivatives we have $(s n)^{\prime}=c n d n,(c n)^{\prime}=-s n d n$ and $(d n)^{\prime}=$ -m sn cn.

Finally, some graphics are sketched as follows


Figure 8: Graphics of elliptic functions for $k=0.1$


Figure 9: Graphics of elliptic functions for $k=0.6$

## Acknowledgements

This work has been partially supported by MEC project MTM2007-64504, and Fundación Séneca project 04540/GERM/06, Spain. This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Regional Agency for Science and Technology (Regional Plan for Science and Technology 2007-2010).

I wish to thank to the organizers for the invitation to deliver this talk at the International Research School on Differential Geometry.

I'm strongly indebted to Professors M. Barros and M. H. Cifre. The former opened to me his personal notes, the latter was kind enough to enhance my handmade drawings.

## Bibliography

[B1] M. Barros, personal notes.
[B2] M. Barros, Simple Geometrical Models with Applications in Physics, Curvature and Variational Modeling in Physics and Biophysics, edited by O. J. Garay, E. García-Río and R. Vázquez-Lorenzo (2008) AIP, 71-113.
[Br] G. Brunnett, The curvature of plane elastic curves, Technical Report NPS-MA-93-013, Naval Postgraduate School.
[Ch] B.-Y. Chen, On a variational problem on hypersurfaces, J. London Math. 2 (1973), 321-325.
[LS] J. Langer and D. A. Singer, The total squared curvature of closed curves, J. Diff. Geom. 20 (1984), 512-520.
[P] U. Pinkall, Hopf tori in $\mathbb{S}^{3}$, Invent. Math. 81 (1985), 379-386.
[Pl] M. S. Plyushchay, Massless particle with rigidity as a model for the rescription of bosons and fermions, Phys. Lett. B 243 (1990), 383-388.
[Si] D. A. Singer, Lectures on Elastic curves and Rods, Curvature and Variational Modeling in Physics and Biophysics, edited by O. J. Garay, E. GarcíaRío and R. Vázquez-Lorenzo (2008) AIP, 3-32.
[St] D. H. Steinberg, Elastic curves in hyperbolic space, Thesis, Case Western Reserve University, 1995.
[T] C. Truesdell, The influence of elasticity on Abalysis: the classical heritage, Bull. A. M. S. 9, No. 3, Nov. 1983, 293-310.
[U] H. K. Urbantke, The Hopf fibration-seven times in Physics, J. of Geom. and Phys. 46 (2003) 125-150.
[W1] T. J. Willmore, Total curvature in Riemannian geometry, Ellis Horwood Ltd. 1982.
[W2] T. J. Willmore, Riemannian geometry, Oxford Science Publ. 1993.

