Some variational problems on curves and applications

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Abstract

Some variational problems are revisited showing elastic curves as a key tool to find solutions to some classical problems such as Willmore surfaces, Willmore-Chen submanifolds and 2-dimensional nonlinear sigma models. To deepen on the interplay between Geometry and Physics, some Plyushchay models have been considered.

1 Introduction

A variational problem is a pair formed by

(1) Something we wish to study how it is changing, and

(2) An "admissible" action, i. e., a way to do the change. By "admissible" we mean "isometry invariant".

Example 1 Consider the set

$$\mathcal{C} = \{\gamma : [a, b] \to M,\ \gamma(a) = p,\ \gamma(b) = q\}$$

of nailed curves in a manifold M, i. e., curves having the same endpoints.

Let E be the functional on $\mathcal{C}$ defined by

$$E(\gamma) = \int_a^b |\gamma'(t)|^2 \, dt.$$

By asking ourselves for the critical points of E, it is well known that they are the geodesics.

2 A little bit of history

Let us introduce elastic curves. In 1691 James Bernouilli looked at the shape of a beam under a load, where we assume that the beam will recover its size and shape when the load is removed. Let $\gamma(s) = (x(s), y(s))$ be a parameterization of the centerline of the beam (see [St]).

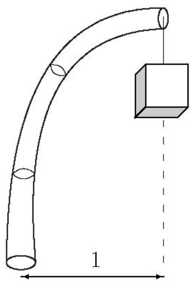

Figure 1: A beam under a load

Three years later, James Bernouilli announced his solution as the following system of differential equations

$$\begin{aligned} dy &= \frac{x^2}{\sqrt{(1-x^4)}}\,dx, \\ ds &= \frac{1}{\sqrt{(1-x^4)}}\,dx, \end{aligned}$$

with an extra hypothesis: the bending moment is directly proportional to some constant related to the composition of the bar and inversely proportional to the radius of curvature. The elliptic functions were born.

The problem was then taken up again forty years later by Daniel Bernouilli and L. Euler. In 1742 the former suggested to Euler that

"The way to determine the shape of an elastic rod subject to the pressure at both ends was to minimize

$$\int_0^L \frac{1}{R^2}\,ds,$$

where s is the arc length, R the radius of curvature and L is the length of the elastic rod".

Definition 1 An elastica, or elastic curve, is a regular curve γ, with fixed endpoints and fixed tangent vectors at endpoints, which is critical of the functional

$$\mathcal{F}^{\lambda}(\gamma) = \int_0^L (k^2 + \lambda)\, ds,$$

where L is the length of γ, $k^2 = ||\ddot{\gamma}||^2$ and λ is an arbitrary constant.

When $\lambda = 0$ γ will be called *free elastica.*

Remark 1 The parameter λ is viewed as a length penalty.

L. Euler was writing his book on the calculus of variations when received D. Bernouilli's suggestion. Then he treated the elastica problem in the first appendix of the book. After a qualitative analysis of the differential equations, Euler gave (presumably based only on experiments) a complete description of all possible planar elastic curves (besides circle and right line), which are sketched in Fig. 2 (See [T]).

3 Langer and Singer approach for elastic curves

To look for elastic curves in a Riemannian manifold $(M, \langle\ ,\ \rangle)$, we follow the nice works by Joel Langer and David Singer [LS] and [Si] .

Let $\gamma : [a, b] \to M$ be an immersed curve and write $||\gamma(t)|| = \frac{ds}{dt} := v \neq 0$, s standing for the arc length parameter. Let k be the curvature function, which can be assumed to be $k \neq 0$, τ the torsion and $\{T, N, B\}$ the Frenet frame along γ. Then

$$\gamma' = \frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt} = vT$$

and the Frenet equations are

$$\begin{aligned} \frac{dT}{ds} &= kN, \\ \frac{dN}{ds} &= -kT + \tau B, \\ \frac{dB}{ds} &= -\tau N. \end{aligned}$$

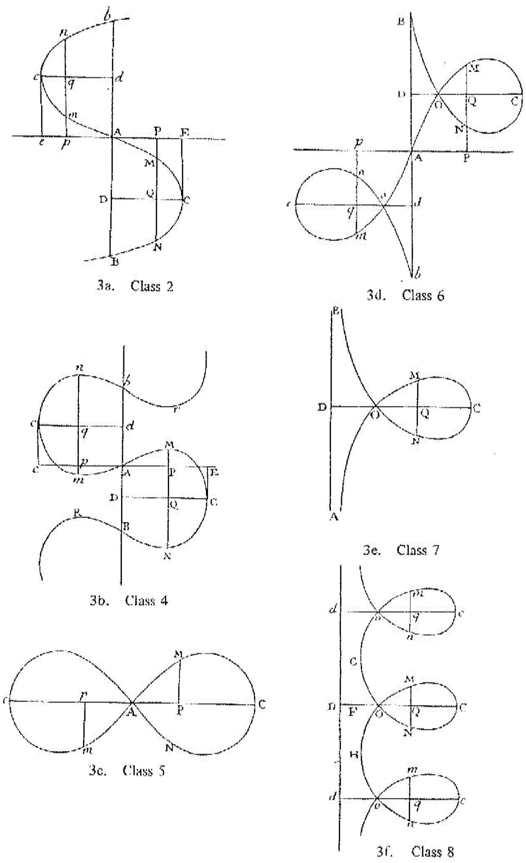

Figure 2: Euler elastic classes

As elastica are extremals of a functional, we have to pave the way to define the set where the functional will apply. To do that, let $\mathcal{C} = \{\gamma : [a, b] \to M\}$ be a set of nailed curves in M and let $\mathcal{F} : \mathcal{C} \to \mathbb{R}$ be the functional. To find the extremals of $\mathcal{F}$ we have to compute its first variation, i. e., we must know $\delta\mathcal{F}_\gamma : T_\gamma\mathcal{C} \to \mathbb{R}$. To determine $T_\gamma\mathcal{C}$ we follow [B1].

As for a curve in $\mathcal{C}$ let us consider a map

$$\begin{array}{rcl} \Gamma : (\varepsilon, \varepsilon) & \to & \mathcal{C} \\ z & \mapsto & \Gamma(z) \end{array}$$

so that

$$\begin{array}{rcl}\Gamma(z):[a,b] & \rightarrow & M\\ t & \mapsto & \Gamma(z)(t)\in M.\end{array}$$

Said otherwise

$$\begin{array}{rcl}\Gamma:(\varepsilon,\varepsilon)\times[a,b] & \rightarrow & M\\ (z,t) & \mapsto & \Gamma(z,t).\end{array}$$

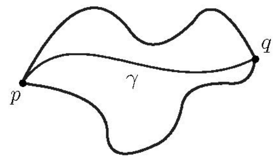

Figure 3: A curve variation

The curve Γ goes throughout $\gamma\in\mathcal{C}$ provided that $\Gamma(0)=\gamma$, i. e., $\Gamma(0,t)=\gamma(t)$. Then $\Gamma(z,t)$ is known as a variation of $\gamma(t)$. We then have defined two vector fields

$$V(z,t)=\frac{\partial\Gamma}{\partial t}(z,t)$$

and

$$W(z,t)=\frac{\partial\Gamma}{\partial z}(z,t),$$

usually known as "longitudinal" and "transversal", respectively, such that $V(0,t)=\gamma'(t)$. Furthermore, $W(t):=W(0,t)$ is a vector field along $\gamma(t)$ so that

$$\Gamma'(0)=\frac{\partial\Gamma}{\partial z}\Big|_{z=0}=\frac{\partial\Gamma(z,t)}{\partial z}\Big|_{z=0}=W(t).$$

Then $T_\gamma\mathcal{C}$ is a subset of the set of vector fields on M along γ.

Conversely, let $W(t)$ be a vector field along γ and define

$$\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$$

by

$$\Gamma(z, t) = \exp_{\gamma(t)} zW(t).$$

Then

$$\begin{aligned} \Gamma(0, t) &= \gamma(t), \\ \frac{\partial}{\partial z}\Big|_{z=0} \Gamma(z, t) &= W(t). \end{aligned}$$

As a consequence, $T_\gamma \mathcal{C} = \{\text{vector fields along } \gamma\}$.

Therefore

$$\begin{aligned} \delta \mathcal{F}_\gamma : T_\gamma \mathcal{C} &\to \mathbb{R} \\ W &\mapsto \delta \mathcal{F}_\gamma(W). \end{aligned}$$

Take

$$\Gamma : (-\varepsilon, \varepsilon) \to \mathcal{C}$$

with $\Gamma(0) = \gamma$ and $\Gamma'(0) = W$. Then

$$\delta \mathcal{F}_\gamma(W) = \frac{d}{dz}\Big|_{z=0} F(\Gamma(z)) = \frac{\partial}{\partial z}\Big|_{z=0} F(\Gamma(z, t)).$$

We write down, for short, $W = \frac{\partial \Gamma}{\partial z}$, $V = \frac{\partial \Gamma}{\partial t}$ and $V = vT$, so that $\langle V, V \rangle = v^2$. Then we get

$$0 = [W, V] = [W, vT] = W(v)T + v[W, T].$$

So $[W, T] = -\frac{W(v)}{v} T = gT$, where $g := -\frac{W(v)}{v}$.

Now

$$\begin{aligned} 2vW(v) &= W(v^2) = W \langle V, V \rangle \\ &= 2 \langle \nabla_W V, V \rangle = 2 \langle \nabla_V W, V \rangle \\ &= 2 \langle \nabla_{vT} W, vT \rangle = 2v^2 \langle \nabla_T W, T \rangle . \end{aligned}$$

So $g = -\langle \nabla_T W, T\rangle$.

On the other hand, as $k^2 = \langle \nabla_T T, \nabla_T T\rangle$, we find that

$$
\begin{aligned}
\frac{\partial k^2}{\partial z} &= W(k^2) = 2\langle \nabla_W \nabla_T T, \nabla_T T\rangle \\
&= 2\left\langle \nabla_T \nabla_W T + \nabla_{[W,T]} T + R(W,T)T, \nabla_T T\right\rangle \\
&= 2\left\langle \nabla_T (\nabla_T W + [W,T]) + \nabla_{[W,T]} T + R(W,T)T, \nabla_T T\right\rangle \\
&= 2\left\langle \nabla_T^2 W + \nabla_T (gT) + g\nabla_T T + R(W,T)T, \nabla_T T\right\rangle \\
&= 2\left\langle \nabla_T^2 W, \nabla_T T\right\rangle + 2g\langle \nabla_T T, \nabla_T T\rangle + 2\langle R(W,T)T, \nabla_T T\rangle + 2g\langle \nabla_T T, \nabla_T T\rangle \\
&= 2\left\langle \nabla_T^2 W, \nabla_T T\right\rangle + 2\langle R(W,T)T, \nabla_T T\rangle + 4g\langle \nabla_T T, \nabla_T T\rangle .
\end{aligned}
$$

Remark 2 In the above computations we have used

(i) $R(W,T)T = \nabla_W \nabla_T T - \nabla_T \nabla_W T - \nabla_{[W,T]} T$;

(ii) $\nabla_W T = \nabla_T W + [W,T]$; and

(iii) $[W,T] = gT$.

As $\nabla_T T = kN$, we get

$$
\frac{\partial k^2}{\partial z} = 2k\left\langle \nabla_T^2 W, N\right\rangle + 4gk^2 + 2k\langle R(W,T)T, N\rangle .
$$

Now

$$
\frac{\partial k^2}{\partial z} = 2k\frac{\partial k}{\partial z},
$$

so that

$$
\frac{\partial k}{\partial z} = W(k) = \left\langle \nabla_T^2 W, N\right\rangle + 2gk^2 + \langle R(W,T)T, N\rangle .
$$

In what follows $\gamma : [0,1] \to M$ will be a curve of length L. Now for a fixed constant λ let us consider the functional

$$
\mathcal{F}^\lambda(\gamma) = \frac{1}{2}\int_0^L (k^2 + \lambda)\, ds.
$$

Problem. Find the critical values of $\mathcal{F}^\lambda$.

To do that we first observe, as $v(t) = \frac{ds}{dt}$, that

$$\mathcal{F}^\lambda(\gamma) = \frac{1}{2}\int_0^1 (k^2+\lambda)\, v(t)\, dt.$$

Then, take a variation $\Gamma(z,t)$ of $\gamma(t)$ as above, with variational vector field W and compute

$$\begin{aligned}
\frac{d}{dz}\mathcal{F}^\lambda(\Gamma(z,t)) &= \frac{1}{2}\int_0^1 W[(k^2+\lambda)v]dt \\
&= \frac{1}{2}\int_0^1 \{W(k^2)v + k^2 W(v) + \lambda W(v)\}dt \\
&= \frac{1}{2}\int_0^1 \{W(k^2) - (k^2+\lambda)g\}vdt \\
&= \frac{1}{2}\int_0^L \{W(k^2) - (k^2+\lambda)g\}ds \\
&= \int_0^L \{k\left\langle \nabla_T^2 W, N\right\rangle + 2gk^2 + k\left\langle R(W,T)T,N\right\rangle - \frac{1}{2}(k^2+\lambda)g\}ds \\
&= \int_0^L \{k\left\langle \nabla_T^2 W, N\right\rangle + 2gk^2 + k\left\langle R(N,T)T,W\right\rangle - \frac{1}{2}(k^2+\lambda)g\}ds.
\end{aligned}$$

Integrating by parts, and using that $g = -\left\langle \nabla_T W, T\right\rangle$, we get

$$\begin{aligned}
\frac{d}{dz}\mathcal{F}^\lambda(\Gamma(z,t)) &= \int_0^L \{\left\langle \nabla_T^2 W, \nabla_T T\right\rangle - \left\langle \nabla_T W, 2k^2 T\right\rangle + \left\langle R(\nabla_T T, T)T, W\right\rangle \\
&+ \frac{1}{2}\left\langle \nabla_T W, (k^2+\lambda)T\right\rangle\}ds \\
&= \int_0^L \{T\left\langle \nabla_T W, \nabla_T T\right\rangle - \left\langle \nabla_T W, \nabla_T^2 T\right\rangle\}ds + \int_0^L \left\langle R(\nabla_T T, T)T, W\right\rangle ds \\
&+ \int_0^L \left\langle \nabla_T W, \Lambda T\right\rangle ds \\
&= \left\langle \nabla_T W, \nabla_T T\right\rangle |_0^L - \int_0^L T\left\langle W, \nabla_T^2 T\right\rangle ds + \int_0^L \left\langle W, \nabla_T^3 T\right\rangle ds \\
&+ \int_0^L \left\langle R(\nabla_T T, T)T, W\right\rangle ds + \int_0^L T\left\langle W, \Lambda T\right\rangle ds - \int_0^L \left\langle W, \nabla_T \Lambda T\right\rangle ds \\
&= [\left\langle \nabla_T W, \nabla_T T\right\rangle + \left\langle W, -\nabla_T^2 T + \Lambda T\right\rangle]_0^L + \int_0^L \left\langle E, W\right\rangle ds,
\end{aligned}$$

where $\Lambda = \frac{\lambda - 3k^2}{2}$ and

$$E = \nabla_T^3 T - \nabla_T(\Lambda T) + R(\nabla_T T, T)T$$

is the Euler-Lagrange operator.

We assume that $(M, \langle\, , \,\rangle)$ is of constant curvature C. Then

$$R(X,Y)Z = C(\langle Y, Z\rangle X - \langle X, Z\rangle Y),$$

so that $R(\nabla_T T, T)T = C\nabla_T T$ and the Euler-Lagrange operator writes down as follows

$$E = \nabla_T^3 T - \nabla_T(\Lambda_C T),$$

where$\Lambda_C = \Lambda - C$.

To compute E we use the Frenet equations. Then we find

$$\begin{aligned}
E &= \nabla_T^3 T - \nabla_T(\Lambda_C T) \\
&= \nabla_T(\nabla_T^2 T - \Lambda_C T) \\
&= \nabla_T(\nabla_T(kN) - \Lambda_C T) \\
&= \nabla_T(k_s N + k\nabla_T N - \Lambda_C T) \\
&= \nabla_T\left(k_s N - k^2 T + k\tau B - \frac{\lambda - 2C - 3k^2}{2} T\right) \\
&= \frac{2k_{ss} + k^3 - \lambda k + 2Ck - k\tau^2}{2} N + (2k_s\tau + k\tau_s)B.
\end{aligned}$$

The curve γ is an elastica provided that $E = 0$, said otherwise, γ is an elastica if and only if the following system of differential equations holds

$$\begin{aligned}
2k_{ss} + k^3 - \lambda k + 2Ck - k\tau^2 &= 0 \\
2k_s\tau + k\tau_s &= 0.
\end{aligned}$$

From the second equation we get

$$k^2\tau = a \text{ constant}.$$

Eliminating τ from the first equation and integrating we have

$$k_s^2 + \frac{k^4}{4} + (C - \frac{\lambda}{2})k^2 + \frac{a^2}{k^2} = A.$$

Letting $u = k^2$, this becomes

$$u_s^2 + u^3 + 4(C - \frac{\lambda}{2})u^2 - 4Au + 4A^2 = 0,$$

whose solutions are (see [Si])

1. $u = k^2 =$ constant and $\tau =$ constant: helices and circles.
2. $k = k_0 \operatorname{sech}(\frac{k_0}{2w}s)$ and $\tau = 0$: borderline elastica.

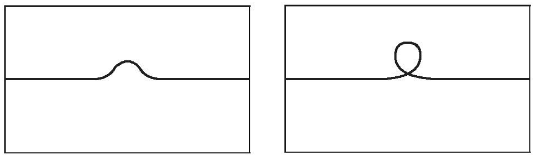

Figure 4: Borderline elastica

3. $k = k_0 \operatorname{dn}(\frac{k_0}{2w}s, p)$ and $\tau = 0$: orbitlike elastica.

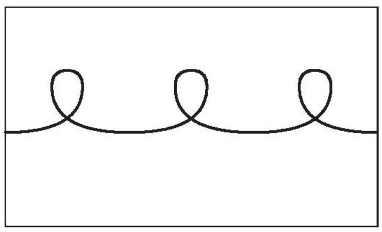

Figure 5: Orbitlike elastica

4. $k = k_0 \operatorname{cn}(\frac{k_0}{2w}s, p)$ and $\tau = 0$: wavelike elastica (see [Br]).

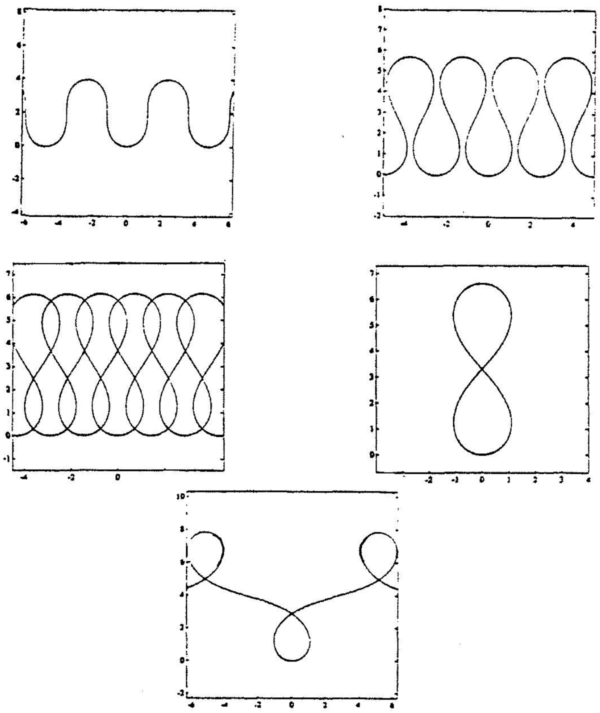

Figure 6: Wavelike elastica

4 Elastica and Willmore functional.

Let $I(M^2, \mathbb{R}^3) = \{\phi : M^2 \to \mathbb{R}^3\}$ be the set of immersions of a surface M^2 in the Euclidean 3-space. The Willmore functional is defined on $I(M^2, \mathbb{R}^3)$ by

$$\mathcal{W}(\phi) = \int_M H^2 \, dA,$$

where H^2 stands for the mean curvature of the surface in the 3-space.

The extremals of $\mathcal{W}(\phi)$ are called Willmore surfaces. In order to classify them in $\mathbb{R}^3$, the starting result is

Theorem 1 Willmore's theorem ([W1], [W2]) *Let $S \subset \mathbb{R}^3$ be a compact surface. Then we have*

(i) $\mathcal{W}(S) \geq 4\pi$; and

(ii) $\mathcal{W}(S) = 4\pi$ if and only if $S = \mathbb{S}^2(r)$.

To leave $\mathbb{R}^3$, let S be a surface in a Riemannian manifold $(M, \langle\ ,\ \rangle)$. Now the Willmore functional writes down as

$$\mathcal{W}(S) = \int_S (H^2 + R)\, dA,$$

where H^2 stands for the mean curvature of S in M and R is the sectional curvature of M along the tangent bundle TS.

A crucial fact is that the Willmore functional is conformally invariant. Then $\mathbb{R}^3$ and $\mathbb{S}^3$ have the same Willmore surfaces, being minimal surfaces the trivial ones.

A test surface. It is well known that the Clifford torus T_C is minimal, then Willmore, in $\mathbb{S}^3$. Furthermore, it is easy to see that

$$\mathcal{W}(T_C) = 2\,\pi^2.$$

Let $I(T, \mathbb{R}^3)$ be the set of immersions of a torus T in the Euclidean 3-space. The Willmore functional is now defined by

$$\mathcal{W}(\phi) = \int_T H_\phi^2\, dA_\phi.$$

We then have

The Willmore conjecture ([W1], [W2]), which states that $\mathcal{W}(\phi) \geq 2\,\pi^2$, equality holding if and only if $\phi(T)$ is conformal to the Clifford torus.

Now we get a second key point

Pinkall's theorem ([Pi]). Willmore tori in $\mathbb{S}^3$ are obtained by lifting, via the Hopf mapping, closed elastic curves in $\mathbb{S}^2$.

To see that let $\Pi : \mathbb{S}^3(1) \to \mathbb{S}^2(\frac{1}{2})$ be the usual Hopf fibration. Let $\gamma : I \subset \mathbb{R} \to \mathbb{S}^2(\frac{1}{2})$ be a unit speed curve and let $\bar{\gamma}$ be its horizontal lift. Then $M_\gamma = \Pi^{-1}(\gamma)$ is a flat surface, which we will call the Hopf tube over γ, parameterized by

$$\phi(s,t) = e^{it}\bar{\gamma}(s).$$

When γ is closed in $\mathbb{S}^2(\frac{1}{2})$ of length L, enclosing an area A, then the Hopf tube is a flat torus isometric to $\mathbb{R}^2/\Gamma$, Γ being the lattice spanned by $\{(0, 2\pi), (L, 2A)\}$.

The Euler-lagrange equation for Willmore tori in $\mathbb{S}^3$ is

$$\Delta^D H = |A|^2 H - 2\langle H, H\rangle H,$$

which becomes

$$2\bar{k}'' + \bar{k}^3 + 4\bar{k} = 0,$$

$\bar{k}$ standing for the curvature of the lifting $\bar{\gamma}$. Therefore M_γ is Willmore in $\mathbb{S}^3$ if and only if γ is an elastica in $\mathbb{S}^2(\frac{1}{2})$.

This was the key point to design a programme to find Willmore surfaces in the anti de Sitter space $\mathbb{H}^3_1$, as well as Willmore-Chen submanifolds (see next section) in pseudo-Riemannian space forms.

In [BFLM1] we exhibit a new method to find Willmore tori in spaces endowed with pseudo-Riemannian warped product metrics, whose fibres are homogeneous spaces. The chief points are the invariance of the involved variational problems with respect to the conformal changes of the metrics on the ambient spaces and the Palais principle of symmetric criticality [Pa]. They allow us to relate the variational problems with that of generalized elastic curves in the conformal structure of the base space. Among others applications we get a rational one-parameter family of Willmore tori in the standard anti De Sitter 3-space shaped on an associated family of closed free elastic curves in the once punctured standard 2-sphere. As an application of a general approach to our method, we give nice examples of pseudo-Riemannian 3-spaces which are foliated with leaves being non-trivial Willmore tori. More precisely, the leaves of this foliation are Willmore tori which are conformal to non-zero constant mean curvature flat tori.

Our main results are sketched as follows:

(i) Willmore tori in the 3-sphere $\mathbb{S}^3$.

Let $\pi : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$ be the usual Hopf fibration, which is a Riemannian submersion relative to canonical metrics on both spheres. For any unit speed curve $\gamma : I \subset \mathbb{R} \to \mathbb{S}^2(1/2)$, we can talk about horizontal lifts $\bar{\gamma}(s)$ of $\gamma(s)$ and obtain unit speed curves in $\mathbb{S}^3$. All these curves define the complete lift $M_\gamma = \pi^{-1}(\gamma)$ of γ. This is a flat surface which we will call the Hopf tube over γ. It is easy to see that M_γ can be parametrized by

$$\Psi(s,t) = e^{it}\bar{\gamma}(s), \tag{1}$$

Ψ being a mapping $I \times \mathbb{R} \to \mathbb{S}^3$ and $\bar{\gamma}$ a fixed horizontal lift of γ. If γ is a closed curve in $\mathbb{S}^2(1/2)$ of length L enclosing an oriented area A, then its Hopf tube M_γ is a flat torus (the Hopf torus over γ) which is isometric to $\mathbb{R}^2/\Gamma$, Γ being the lattice generated by $(0, 2\pi)$ and $(L, 2A)$. Then M_γ *is a Willmore surface in* $\mathbb{S}^3$*, if and only if* γ *is a* 4*-elastica in* $\mathbb{S}^2(1/2)$.

(ii) Willmore tori in non-standard anti De Sitter 3-space.

Let $\pi : (M, g) \to (B, h)$ be a pseudo-Riemannian submersion. We can define a very interesting deformation of the metric g by changing the relative scales of B and the fibres. More precisely, it is defined the canonical variation g_t, $t > 0$, of g by setting

$$\begin{aligned} g_t|_{\mathcal{V}} &= t^2\, g|_{\mathcal{V}}\,, \\ g_t|_{\mathcal{H}} &= g|_{\mathcal{H}}\,, \\ g_t(\mathcal{V}, \mathcal{H}) &= 0, \end{aligned}$$

where $\mathcal{V}$ and $\mathcal{H}$ stand for vertical and horizontal distributions, respectively, associated with the submersion. Thus we obtain a one-parameter family of pseudo-Riemannian submersions $\pi_t : (M, g_t) \to (B, h)$ with the same horizontal distribution $\mathcal{H}$, for all $t > 0$. Relative to O'Neill invariants A^t and T^t of these pseudo-Riemannian submersions, we will just recall a couple of properties. First, if g has totally geodesic fibres ($T \equiv 0$), the same happens for g_t, for all $t > 0$. Furthermore,

$$A^t_Y U = t^2 A_Y U, \tag{2}$$

for any $Y \in \mathcal{H}$ and $U \in \mathcal{V}$.

Now we consider the canonical variation of the Hopf fibration $\pi = \pi_0 : \mathbb{H}^3_1 \to \mathbb{H}^2(-1/2)$ to get a one-parameter family of pseudo-Riemannian submersions $\pi_t : (\mathbb{H}^3_1, g_t) \to (\mathbb{H}^2(-1/2), g_0)$. Let γ be a unit speed curve immersed in $\mathbb{H}^2(-1/2)$. Set $\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma)$. Then $\mathcal{T}_{\gamma,t}$ is a Lorentzian flat surface immersed in $\mathbb{H}^3_1$, that will be called the Lorentzian Hopf tube over γ. As the fibres of π_t are $\mathbb{H}^1_1$, which topologically are circles, then $\mathcal{T}_{\gamma,t}$ is a Hopf torus in $(\mathbb{H}^3_1, g_t)$, provided that γ is a closed curve. It is obvious that the group $G = \mathbb{S}1$ naturally acts through isometries on $(\mathbb{H}^3_1, g_t)$, for all $t > 0$, getting $(\mathbb{H}^2(-1/2), g_0)$ as the orbit space.

Now, let $\pi_t : (\mathbb{H}^3_1, g_t) \to (\mathbb{H}^2(-1/2), g_0)$, $t > 0$, be the canonical variation of the pseudo-Riemannian Hopf fibration. Let γ be a closed immersed curve in $(\mathbb{H}^2(-1/2), g_0)$ and $\mathcal{T}_{\gamma,t} = \pi_t^{-1}(\gamma)$ its Lorentzian Hopf torus. *Then $\mathcal{T}_{\gamma,t}$ is a Willmore surface in $(\mathbb{H}^3_1, g_t)$ if and only if γ is an elastica in $(\mathbb{H}^2(-1/2), g_0)$ with Lagrange multiplier $\lambda = -4t^2$.*

In [BF1] we obtain isoperimetric inequalities for the Willmore energy of Hopf tori in a wide class of conformal structures on the three sphere. This class includes, on the one hand, the family of conformal Berger spheres and, on the other hand, a one parameter family of Lorentzian conformal structures. This allows us to give the best possible lower bound of Willmore energies concerning isoareal Hopf tori.

The main result states as follows:

Let α be an immersed closed curve in $\mathbb{S}^2(1/2) = (\mathbb{S}^2, g)$ with length L, then the Willmore energy of $\mathbf{S}_\alpha$ in $(\mathbb{S}^3, [\bar{g}^\varepsilon_r])$ satisfies

$$\mathcal{W}^\varepsilon_r(\mathbf{S}_\alpha) \geq \max\left\{2\pi r^2[\pi + L(\varepsilon r^2 - 1)]; 2\pi r^2\left[\frac{\pi^2}{L} + L(\varepsilon r^2 - 1)\right]\right\},$$

with equality holding if and only if α is a circle of $\mathbb{S}^2(1/2)$ and so $\mathbf{S}_\alpha$ is a rotational torus with area $2\pi r L$ in $(\mathbb{S}^3, \bar{g}^\varepsilon_r)$.

As a consequence, we give some applications. For instance, choosing a point $q_o \in \mathbb{S}^3$, we use the stereographic projection $\mathbf{E}_o : \mathbb{S}^3 - \{q_o\} \to \mathbb{E}^3$. Then take $L_o > 0$ and consider the subclass of tori $\mathbf{E}_o(\mathcal{T}_o) = \{\mathbf{E}_o(\mathbf{S}_\alpha) : \text{Length}(\alpha) = L_o\}$. Then, we get the best possible lower bound

$$\mathcal{W}_o(\mathbf{E}_o(\mathbf{S}_\alpha)) \geq \max\left\{2\pi^2, \frac{2\pi^3}{L}\right\}$$

for the Willmore energy in the class $\mathbf{E}_o(\mathcal{T}_o)$, with equality holding if and only if $\mathbf{E}_o(\mathbf{S}_\alpha)$ is an anchor ring with known radii.

The complete classification of homogeneous three spaces is well known for some time. Of special interest are those with rigidity four which appear as Riemannian submersions with geodesic fibers over surfaces with constant curvature. Consequently their geometries are completely encoded in two values, the constant curvature, c, of the base space and the so called bundle curvature, r. In [BFG] we obtain the complete classification of equivariant Willmore surfaces in homogeneous three spaces with rigidity four. All these surfaces appear by lifting elastic curves of the base space. Once more, the qualitative behavior of these surfaces is encoded in the above mentioned parameters (c, r). The case where the fibres are compact is obtained as a special case of a more general result that works, via the principle of symmetric criticality, for bundle-like conformal structures in circle bundles. However, if the fibres are not compact, a different approach is necessary. We compute the differential equation satisfied by the equivariant Willmore surfaces in conformal homogeneous spaces with rigidity of order four and then we reduce directly the symmetry to obtain the Euler Lagrange equation of $4r^2$-elasticae in surfaces with constant curvature, c. We also work out the solving natural equations and the closed curve problem for elasticae in surfaces with constant curvature. It allows us to give explicit parametrizations of Willmore surfaces and Willmore tori in those conformal homogeneous 3-spaces.

In [BFG] we give the complete classification of equivariant Willmore surfaces in three dimensional conformal homogeneous spaces having 4-dimensional isometry group, no matter if the fibres are compact or not. In both cases, the original problem becomes one about elastic curves in $B(c)$, for which we use the machinery developed in [LS] and [Si]. The field equation for these curves, and so their qualitative behavior, is completely encoded in the parameters (c, r) that determine the homogeneous structure as we have described in the section 6. Our main results can be summarized as follows:

(1) *The family of equivariant Willmore surfaces in the conformal* $\mathbf{E}(c, r)$ *with* $c \geq 2r^2$ *is made up of the following surfaces:*

(1.1) *Minimal surfaces obtained by lifting geodesics.*

(1.2) *A one-parameter class of surfaces obtained by lifting wavelike elastic curves.*

(2) *The family of equivariant Willmore surfaces in the conformal* $\mathbf{E}(c, r)$ *with* $c < 2r^2$ *is made up of the following surfaces:*

(2.1) *Minimal surfaces obtained by lifting geodesics.*

(2.2) *Surfaces with constant mean curvature* $\sqrt{2(2r^2 - c)}/2$ *shaped on circles with curvature* $\sqrt{2(2r^2 - c)}$.

(2.3) *A one-parameter class of surfaces built on orbitlike elastic curves.*

(2.4) *A one-parameter class of surfaces built on wavelike elastic curves.*

(2.5) *A surface shaped on a borderlike elastic curve.*

Up to now we have worked in codimension one, but this restriction can be dropped as follows.

5 Elastica and Willmore-Chen functional.

Let $I\left((M^m, g), (\bar{M}, \bar{g})\right)$ be the set of immersions between two Riemannian manifolds. The Willmore-Chen functional is defined on $I\left((M^m, g), (\bar{M}, \bar{g})\right)$ by

$$\mathcal{WC}(\phi) = \int_M (H^2 - \tau_e)^{\frac{m}{2}} \, dA,$$

where H^2 stands for the mean curvature of M in $\bar{M}$ and τ_e is the extrinsic scalar curvature (see [Ch]).

Then the extremals of $\mathcal{WC}(\phi)$ are called Willmore-Chen submanifolds.

Problem. Look for Willmore surfaces and Willmore-Chen submanifolds in pseudo-Riemannian space forms.

The solution of this problem will show us a natural and nice connection between Geometry and Physics.

In [BFLM1] we find Willmore-Chen submanifolds in spaces endowed with pseudo-Riemannian warped product metrics, whose fibres are homogeneous spaces. We obtain rational one-parameter families of Willmore-Chen submanifolds in standard pseudo-hyperbolic spaces. We introduce a new method to construct critical points of the Willmore-Chen functional in the pseudo-hyperbolic space $\mathbb{H}^n_r = \mathbb{H}^n_r(-1)$. First we will write $\mathbb{H}^n_r$ as a warped product with base space the standard hyperbolic space $\mathbb{H}^{n-r}$. Then we will use the conformal invariance of the Willmore-Chen variational problem to make a conformal change of the canonical metric of $\mathbb{H}^n_r$. Next we use the Palais principle of symmetric criticality to reduce the problem to a variational one for closed curves in the once punctured standard $(n-r)$-sphere. Then we show

(1) *Let γ be a fully immersed closed curve in the hyperbolic space $\mathbb{H}^{n-r}$. The tube $\Upsilon_\gamma = \Phi(\gamma \times \mathbb{S}^r)$ in $(\mathbb{H}^r, h_0)$ is a Willmore-Chen submanifold if and only if γ is a generalized free elastica in the once punctured unit sphere $(\Sigma^{n-r}, d\sigma^2)$. In particular, $n - r \leq 3$.*

(2) *Let r be any natural number. For any non zero rational number q, there exists an $(r+1)$-dimensional Willmore-Chen submanifold $\Upsilon_\gamma = \Phi(\gamma\times\mathbb{S}^r)$ in the pseudo-hyperbolic space $(\mathbb{H}^{r+3}_r, h_0)$, γ being an r-generalized free elastic closed helix in the once punctured unit 3-sphere $(\Sigma^3, d\sigma^2)$.*

In [BFLM2] we exhibit a criterion for a reduction of variables for Willmore-Chen submanifolds in conformal classes associated with generalized Kaluza-Klein metrics on flat principal fibre bundles. Our method relates the variational problem of Willmore-Chen with an elasticity functional defined for closed curves in the base space. The main ideas involve the extrinsic conformal invariance of theWillmore-Chen functional, the large symmetry group of generalized Kaluza- Klein metrics and the Palais principle of symmetric criticality. We also obtain interesting families of elasticae in both lens spaces and surfaces of revolution (Riemannian and Lorentzian). We give applications to the contruction of explicit examples of isolated Willmore-Chen submanifolds, discrete families of Willmore-Chen submanifolds and foliations whose leaves are Willmore- Chen submanifolds. The main result states as follows:

Let G be an m-dimensional compact Lie group endowed with a bi-invariant metric. Let (Γ,ω) be a flat principal connection on a principal fibre bundle $P(M,G)$. Let $\bar{h} = \Phi_\varepsilon(h,u)$ be a generalized Kaluza-Klein metric on $P(M,G,\omega)$ and $\mathcal{C}(\bar{h})$ its conformal class. Given an immersed closed curve γ in M, then N_γ is a Willmore-Chen submanifold in $(P,\mathcal{C}(\bar{h}))$ if and only if γ is an m-generalized elastica in $(M,u^{-2}h)$.

In [BFL] we deal with string theories and M-theories on backgrounds of the form $AdS\times M$, M being a compact principal $U(1)$-bundle. These configurations are the natural settings to study Hopf T-dualities [MLP], and so to define duality chains connecting different string theories and M-theories. There is an increasing great interest in studying those properties (physical or geometrical) which are preserved along the duality chains. For example, it is known that Hopf T-dualities preserve the black hole entropies. In this paper we consider a two-parameter family of actions which constitutes a natural variation of the conformal total tension action (also known as Willmore-Chen functional in Differential Geometry). Then, we show that the existence of wide families of solutions (in particular compact solutions) for the corresponding motion equations is preserved along those duality chains. In particular, we exhibit ample classes of Willmore-Chen submanifolds with a reasonable degree of symmetry in a wide variety of conformal string theories and conformal M-theories, that in addition are solutions of a second variational problem known as the area-volume isoperimetric problem. These are good reasons to refer those submanifolds as the best worlds one can find in a conformal universe. The method we use to obtain this invariant under Hopf T-dualities is based on the Palais principle of symmetric criticality. However, it is used in a two-fold sense. First to break symmetry and so to reduce variables. Second to gain rigidity in direct approaches to integrate the Euler-Lagrange equations. The existence of generalized elastic curves is also important in the explicit exhibition of those configurations. The relationship between solutions and elasticae can be regarded as a holographic property.

6 Interplay between Geometry and Physics (see [B2])

We first note that the Hopf map is more than a simple tool. It provides excellent applications in different contexts in Physics, as we can see in the nice paper [U] by H. K. Urbantke.

Secondly, the Willmore functional has a certain universality regarding its physical applications: from strings and branes to membranes and vesicles, because all of them are extremals of a certain action. We are interested in knowing what kind of action.

In that line, Poisson and Sophie Germaine proposed that action should be an even and symmetric function of the principal curvatures of the surface.

In the seventies, thinking about membranes, Canham and Helfrich proposed a new model, now based on a quadratic function of the principal curvatures, given by

$$\mathcal{CH}(S) = \int_S (a + bH^2 + cG)\, dA,$$

H and G standing for the mean and Gauss curvatures, respectively, of the surface, and $a, b, c \in \mathbb{R}$.

As the topology of membranes does not change by fluctuations, then the Gauss-Bonnet theorem reduces the Canham-Helfrich functional to

$$\mathcal{CH}(S) = \int_S (a + bH^2)\; dA.$$

Furthermore, as minimality and compacity are not good mates, we can assume that $b \neq 0$. Then, by taking $b = 1$, the Canham-Helfrich functional is nothing but a modified Willmore funcional.

Thirdly, it seems that strings theories will play a key role to understand physical world. A string theory is carry out in a non-flat spacetime, where strings (curves) evolve generating surfaces (worldsheets). The problem now is looking for the action describing the dynamics.

The most widely accepted nowadays is that of Kleiner and Polyakov given by

$$\mathcal{KP}(S) = a \int_S dA + b \int_S H^2\; dA,$$

which strongly sounds Willmore again.

7 Elastica and two-dimensional $O(2,1)$ nonlinear sigma model

In [BF2] the two-dimensional $O(2,1)$ nonlinear sigma model with boundary is considered. We calibrate the size of its space of field configurations by exhibiting new and wide classes of solutions. We first construct solutions by evolving, under a certain group of transformations, free elastic curves in any surface, either Riemannian or Lorentzian, of constant curvature. Furthermore, we show that any null scroll can provide a solution of this sigma model. This surprising phenomenon, which obviously has no Euclidean counterpart, guarantees the existence of an ample class of solutions which are generated by null (or lightlike) curves evolving through null ruling flows.

Our main results are

(1) *The solutions of the two dimensional* $\mathbf{O}(2,1)$ *nonlinear sigma model are just the Willmore surfaces.*

(2) *A surface* $\mathbf{S}_\gamma = \gamma \times \mathbb{S}^1$ *is a solution of the two dimensional* $\mathbf{O}(2,1)$ *nonlinear sigma model if and only if its profile curve,* γ*, is a clamped free elastic curve in the unit sphere.*

(3) *There exist wide classes of solutions of the two dimensional* $\mathbf{O}(2,1)$ *nonlinear sigma model obtained from elastic curves in any surface, Riemannian or Lorentzian, with constant curvature.*

(4) *Every null scroll is a Willmore surface in* $\mathbb{L}^3$.

(5) *Null scrolls provide solutions of the two dimensional* $\mathbf{O}(2,1)$ *nonlinear sigma model.*

8 The Plyushchay model

The integrand of the elastica functional

$$\mathcal{F}(\gamma) = \int_\gamma (k^2 + \lambda)\, ds$$

can be modified to get a new functional

$$\mathcal{P}(\gamma) = \int_\gamma f(k_i)\, ds$$

depending on the curvatures of γ. Then could you find the critical points of $\mathcal{P}$? Furthermore, could you find the physical meaning of them?

As for (ii) remember that we wish to consider the general functional

$$\mathcal{P}(\gamma) = \int_\gamma f(k_i)\, ds$$

and look for its critical points.

Let $(M, \langle\ ,\ \rangle)$ be an n-dimensional Riemannian manifold. As above write

$$\mathcal{C} = \{\gamma : [a,b] \to M\}$$

and the simplest functional

$$\mathcal{F}_m : \mathcal{C} \to \mathbb{R},$$

for any $m \in \mathbb{R}$ defined by

$$\mathcal{F}_m(\gamma) = \int_\gamma (k(s) + m)\ ds.$$

This is known as the Plyushchay functional (or Plyushchay model, see [Pl]), because he was the first to use it to study trajectories of relativistic particles.

Remark 3 (1) When $m = 0$ then $\mathcal{F}_0(\gamma)$ is nothing but the total curvature of curves in $\mathcal{C}$. If $M = \mathbb{R}^2$ then $k = \theta'$. Now, if γ is closed then

$$\mathcal{F}_0(\gamma) = \int_\gamma k(s)\, ds = 2\pi\, i(\gamma),$$

$i(\gamma) \in \mathbb{Z}$ being the rotation index of γ and $\mathcal{F}_0$ is constant on any homotopy class of curves.

If $\mathcal{C}$ is the space of clamped curves curves, i. e., $\gamma_z(a) = \gamma_z(b)$, ${\gamma_z}'(a) = \vec{u}$ and ${\gamma_z}'(b) = \vec{v}$, then

$$\mathcal{F}_0(\gamma) = \varphi_0 + 2\pi\, \#(\text{interior loops}),$$

where $\varphi_0 = \text{angle}(\vec{u}, \vec{v})$. Therefore, $\mathcal{F}_0$ is also constant on any homotopy class of clamped curves.

Summarizing, the variational problem associated with $\mathcal{F}_0$ on $\mathbb{R}^2$ has no physical interest.

(2) What about $\mathcal{F}_0$ when M^2 is a surface in $\mathbb{R}^3$? Take now $\mathcal{C}$ the set of one-to-one closed curves in M^2 and let D be a disc in $\mathbb{R}^2$. Consider the space of embeddings $\{\Phi : D \to M^2\}$

Then we have

$$\int_\gamma k(s)\, ds + \int_{\Phi(D)} K\, dA = 2\pi.$$

(3) As for $\mathcal{F}_0$ on $M = \mathbb{R}^3$, some classical results are known.

(3.1) If $\gamma \subset \mathbb{R}^3$ is one-to-one and closed, then

$$\int_\gamma k(s)\, ds \geq 2\pi,$$

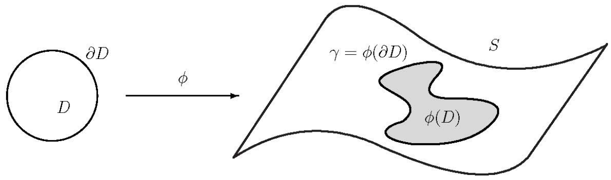

Figure 7: An embedding

equality holding if and only if γ is planar and convex.

(3.2) If $\gamma \subset \mathbb{R}^3$ is one-to-one, closed and knotted, then

$$\int_\gamma k(s)\,ds \geq 4\pi.$$

To look for the critical points of $\mathcal{F}_m(\gamma) = \int_\gamma (k(s) + m)\ ds$ in the general background, let

$$\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$$

be a variation of γ defined by

$$\Gamma(z,t) = \exp_{\gamma(t)} zW(t),$$

$W(t)$ being a vector field along γ. Then

$$\begin{aligned} \Gamma(0,t) &= \gamma(t) \\ \frac{\partial}{\partial z}\Big|_{z=0} \Gamma(z,t) &= W(t). \end{aligned}$$

and

$$
\begin{aligned}
\delta\mathcal{F}_m(W) &= \frac{d}{dz}\Big|_{z=0}\mathcal{F}_m(\Gamma(z)) \\
&= \frac{\partial}{\partial z}\Big|_{z=0}\mathcal{F}_m(\Gamma(z,t)) \\
&= \frac{\partial}{\partial z}\int_a^b k(z,t)\, v(z,t)\, dt \\
&= \int_a^b W(k)\, v\, dt + \int_a^b k\, W(v)\, dt \\
&= \int_\gamma \langle \Omega(\gamma), W\rangle\ ds + [\mathcal{B}(\gamma, W)]_0^L,
\end{aligned}
$$

where

$$\Omega(\gamma) = \nabla_T^2 N + \nabla_T\left((k-m)T\right) + R(N,T)T$$

stands for the Euler-Lagrange operator and

$$\mathcal{B}(\gamma, W) = \langle \nabla_T W, N\rangle + \langle W, mT + \tau B\rangle$$

is the boundary operator.

Let $\mathcal{C}$ be the set of clamped curves defined by $\mathcal{C} = \{\gamma : [a,b] \to M \,/\, \gamma(a) = p, \gamma(b) = q, \gamma'(a) = \vec{u}, \gamma'(b) = \vec{v}\}$. Then $T_\gamma\mathcal{C} = \{W \,\text{along}\, \gamma \,:\, W(a) = W(b) = 0\}$, so that

$$[\mathcal{B}(\gamma, W)]_0^L = 0.$$

Summarizing, γ **is a critical point of** $\mathcal{F}_m$ **if and only if** $\Omega(\gamma) = 0$**.**

The condition $\Omega(\gamma) = 0$ is called the Euler-Lagrange equation of the variational problem.

By using the Frenet equations, the condition $\Omega(\gamma) = 0$, in a space form $M^n(C)$, reads as follows

$$
\begin{aligned}
\tau^2 + mk &= C, \\
\tau_s' &= 0, \\
\tau\eta &= 0,
\end{aligned}
$$

where $\eta \perp \{T, N, B\}$.

As a first consequence we have that $\tau =$ constant, as well as $\eta = 0$. Then the critical points of this model live in a 3-dimensional totally geodesic submanifold. Furthermore, when

(i) When $m \neq 0$, then the critical points form a 1-parameter family of helices $\{(k, \tau) \in \mathbb{R}^2 \, : \, mk + \tau^2 = C\}$.

(ii) When $m = 0$ we only know that $\tau^2 = C$, i. e., the critical points are living in $\mathbb{S}^3(C)$.

Without loss of generality, we can take $C = 1$ and state the following problem.

Problem. Look for $\tau^2 = 1$ curves in $\mathbb{S}^3(1)$.

To get an answer we recall the Hopf map to find that the lifting of any curve in $\mathbb{S}^2(\frac{1}{2})$ provides a curve in $\mathbb{S}^3(1)$ with $\tau^2 = 1$.

Going for a walk in the realm of Lorentzian world, it is easy to see that the extremals of this variational problem is given by the one-parameter family $\{(k, \tau) \in \mathbb{R}^2 \, : \, \varepsilon_2 mk - \tau^2 = C\}$, where $\varepsilon_2 = \langle N, N \rangle$ is the causal character of N. So they are living in the anti de Sitter world $\mathbb{H}^3_1(-1)$. Then in the anti de Sitter world the dynamics of a system of particles governed by the action $\int_\gamma (k(s) + m)\, ds$ is also described by helices.

The beauty of the model governed by actions of the form

$$\int_\gamma f\left(k_1(s), \cdots, k_n(s)\right)\, ds$$

lies in the fact that the degree of freedom that were added in the classical method is actually encoded in the geometry of the particle paths.

Then in [BFJL1] we consider the motion of relativistic particles described by an action that is linear in the torsion (second curvature) of the particle path. The Euler-Lagrange equations and the dynamical constants of the motion associated with the Poincaré group, the mass and the spin of the particle, are expressed in terms of the curvatures of the embedded worldline. The moduli spaces of solutions are completely exhibited in 4-dimensional background spaces and in the 5-dimensional case we explicitly obtain the curvatures of the worldline.

In [FGJL] we deal with the motion of relativistic particles described by an action which is a function of the curvature and torsion of the particle path. The Euler-Lagrange equations and the dynamical constants of the motion are given in a simple way in terms of a suitable coordinate system. The moduli spaces of solutions in a three-dimensional pseudo-Riemannian space form are completely exhibited.

In [BFJL2] models describing relativistic particles, where Lagrangian densities depend linearly on both the curvature and the torsion of the trajectories, are revisited in $D = 3$ Lorentzian spacetimes with constant curvature. The moduli spaces of trajectories are completely and explicitly determined. Trajectories are Lancret curves including ordinary helices. To get the geometric integration of the solutions, we design algorithms that essentially involve the Lancret program as well as the notions of scrolls and Hopf tubes. The most interesting and consistent models appear in anti de Sitter spaces, where the Hopf mappings, both the standard and the Lorentzian ones, play an important role. The moduli subspaces of closed solitons in anti de Sitter settings are also obtained. Our main tool is the isoperimetric inequality in the hyperbolic plane.

The mass spectra of these models are also obtained. In anti de Sitter backgrounds, the characteristic feature is that the presence of real gravity makes that, under reasonable conditions, these physical spectra always present massive states. This fact has no equivalent in flat spaces where spectra necessarily present tachyonic sector. Furthermore, the existence of systems with only massive states, in anti de Sitter geometry, solves an early stated problem in spaces with a non trivial gravity.

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