BIRKHOFF INTEGRAL FOR MULTI-VALUED FUNCTIONS

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Dedicated to Professor John Horvath on the occasion of his 80th birthday

ABSTRACT. The aim of this paper is to study Birkhoff integrability for multi-valued maps $F : \Omega \longrightarrow cwk(X)$, where (Ω, Σ, μ) is a complete finite measure space, X is a Banach space and $cwk(X)$ is the family of all non-empty convex weakly compact subsets of X. It is shown that the Birkhoff integral of F can be computed as the limit for the Hausdorff distance in $cwk(X)$ of a net of Riemann sums $\sum_n \mu(A_n)F(t_n)$. We link Birkhoff integrability with Debreu integrability, a notion introduced to replace sums associated to correspondences when studying certain models in Mathematical Economics. We show that each Debreu integrable multi-valued function is Birkhoff integrable and that each Birkhoff integrable multi-valued function is Pettis integrable. The three previous notions coincide for finite dimensional Banach spaces and they are different even for bounded multi-valued functions when X is infinite dimensional and X^* is assumed to be separable. We show that when F takes values in the family of all non-empty convex norm compact sets of a separable Banach space X, then F is Pettis integrable if, and only if, F is Birkhoff integrable; in particular, these Pettis integrable F's can be seen as single-valued Pettis integrable functions with values in some other adequate Banach space. Incidentally, to handle some of the constructions needed we prove that if X is an Asplund Banach space, then $cwk(X)$ is separable for the Hausdorff distance if, and only if, X is finite dimensional.

1. INTRODUCTION AND PRELIMINARY RESULTS

A great deal of work about measurable and integrable multifunctions was made in the last decades. Some pioneering and highly influential ideas and notions around the matter were inspired by problems arising in Control Theory and Mathematical Economics. We can cite the papers by Aumann [2] and Debreu [10], the monographs by Castaing and Valadier [8], Klein and Thompson [23], and the survey by Hess [18].

Henceforth $F : \Omega \longrightarrow cwk(X)$ will be a multi-valued function from a complete finite measure space (Ω, Σ, μ) into the family of all non-empty convex weakly compact subsets $cwk(X)$ of the Banach space X.

The notion of Debreu integrability introduced in 1967 is a multi-valued counterpart to Bochner integrability. Despite the theory of integration developed by Debreu in [10] dealt with functions taking values in the family $ck(X)$ of all non-empty convex norm compact subsets of X, it is readily seen, as pointed out by Byrne in [5, p. 246], that this theory extends to the case of $cwk(X)$-valued functions. Debreu integral is defined by means of a certain embedding of $cwk(X)$ into a Banach space. The brief explanation below includes some preliminary results, e.g. Lemma 1.1, that will be needed in the subsequent sections.

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The family $\mathcal{C}$ of all non-empty bounded closed subsets of X is a metric space with the Hausdorff distance [17], given by

$$h(A,B) := \inf\{\eta > 0 : A \subset B + \eta B_X,\ B \subset A + \eta B_X\},$$

where B_X denotes the closed unit ball of X. Since the underlying metric in X is complete, the space $(\mathcal{C}, h)$ is complete too, see [8, Theorem II.3] or [23, Corollary 4.3.12 (i)]. It is easily proved that $ck(X)$ (resp. $cwk(X)$) is a closed subspace of $(\mathcal{C}, h)$ [23, Corollary 4.3.12 (v)] (resp. bear in mind that the set of all convex elements of $\mathcal{C}$ is closed [23, Corollary 4.3.12 (iii)], and use Grothendieck's characterization of weak compactness [11, Lemma 2, p. 227]). For a bounded set $B \subset X$ and each x^* in the dual space X^*, we write

$$\delta^*(x^*, B) := \sup\{x^*(x) : x \in B\}.$$

We have the following embedding result.

Lemma 1.1 (Theorems II.18 and II.19 in [8])**.** *Let $\ell_\infty(B_{X^*})$ be the Banach space of bounded real valued functions defined on B_{X^*} endowed with the supremum norm $\|\cdot\|_\infty$. Then, the map $j : cwk(X) \longrightarrow \ell_\infty(B_{X^*})$ given by $j(A) := \delta^*(\cdot, A)$ satisfies the properties below:*

(i) $j(A + B) = j(A) + j(B)$ *for every* $A, B \in cwk(X)$*;*
(ii) $j(\lambda A) = \lambda j(A)$ *for every* $\lambda \geq 0$ *and every* $A \in cwk(X)$*;*
(iii) $h(A,B) = \|j(A) - j(B)\|_\infty$ *for every* $A, B \in cwk(X)$*;*
(iv) $j(cwk(X))$ *is closed in* $\ell_\infty(B_{X^*})$*.*

The multi-valued function $F : \Omega \longrightarrow cwk(X)$ is *Debreu integrable* [23, Definition 17.2.3 and Proposition 17.2.4] if, and only if, the composition $j \circ F$ is Bochner integrable. In this case, the Debreu integral of F in Ω is the unique element $(D)\int_\Omega F\,d\mu \in cwk(X)$ such that $j((D)\int_\Omega F\,d\mu)$ is the Bochner integral of $j \circ F$. In fact, Debreu integrability does not depend on the particular embedding j considered, see [23, Proposition 17.2.4], and in order to define Debreu integral we can use any map $i : cwk(X) \longrightarrow Y$, into a Banach space Y, as long as properties (i)-(iv) in Lemma 1.1 are fulfilled. The existence of such kind of embedding was first proved by Rådström, [29]. For information about the Debreu integral we refer the reader to [5, 10, 20], [23, Chapter 17], [18, Section 3] and the references therein.

Given a multi-valued function $F : \Omega \longrightarrow cwk(X)$, we write $\delta^*(x^*, F)$ to denote the real valued function given by $\delta^*(x^*, F)(t) := \delta^*(x^*, F(t))$, $t \in \Omega$. When X is a separable Banach space the function F is said to be *Pettis integrable* if $\delta^*(x^*, F) \in \mathcal{L}^1(\mu)$ for every $x^* \in X^*$ and for every $A \in \Sigma$ there is $(P)\int_A F\,d\mu \in cwk(X)$ such that

$$\delta^*\Big(x^*, (P)\int_A F\,d\mu\Big) = \int_A \delta^*(x^*, F)\,d\mu, \quad x^* \in X^*.$$

The notion of Pettis integrable multifunction was first considered in [8, Chapter V, §4] and has been pretty recently studied in [1, 14, 19, 32, 33].

It is known that, for separable X, a multi-valued function $F : \Omega \longrightarrow cwk(X)$ is scalarly measurable (i.e. $\delta^*(x^*, F)$ is measurable for every $x^* \in X^*$) if, and only if, F is Effros measurable (i.e. $\{t \in \Omega : F(t) \cap U \neq \emptyset\} \in \Sigma$ for every open set $U \subset X$), see e.g. [3, Corollary 4.10 (a)]. In this case F admits at least one strongly measurable selector, by the selection theorem due to Kuratowski and Ryll-Nardzewski, see e.g. [8, Theorem III.6]. The books [8, 23] and the papers [3, 18, 20, 21] are convenient references on measurability properties of multi-valued functions.

In particular, if X is separable and $F : \Omega \longrightarrow cwk(X)$ is a Pettis integrable multi-valued function, then F is scalarly measurable and therefore F admits strongly measurable selectors. Moreover, each strongly measurable selector of F is Pettis integrable and we have

$$(1) \qquad (P)\int_A F\, d\mu = \Big\{ \text{(Pettis)} \int_A f\, d\mu : \ f \text{ is a Pettis integrable selector of } F \Big\}$$

for every $A \in \Sigma$, [32, Theorem 3.2] and [33]. When F is Debreu integrable, $(D)\int_A F\, d\mu$ is described as in (1) but using Bochner integrable selectors instead of Pettis integrable ones, i.e. Debreu integral coincides with Aumann integral, see [23, Theorem 17.3.2] and [5].

This article is organized in the following manner. In Section 2 we offer the definition of Birkhoff integral for multi-valued functions $F : \Omega \longrightarrow cwk(X)$ using the embedding j given in Lemma 1.1. We give two different characterizations of Birkhoff integrability. The first one is given in exclusive terms of $(cwk(X), h)$, Proposition 2.6, and it is used to show that the notion of Birkhoff integrability does not depend on the embedding j, Corollary 2.7. The second one is for bounded multi-functions, Proposition 2.9, and uses Bourgain property that we studied in relationship to Birkhoff integrability for single-valued functions in [7]. This characterization is used in the examples at the end of the paper.

Section 3 has two different subsections. The first one is devoted to study positive results about the relationship between Debreu, Birkhoff and Pettis integrability. Proposition 3.1 establishes that each Debreu integrable multi-valued function is Birkhoff integrable and that each Birkhoff integrable one is Pettis integrable. The three previous notions coincide for finite dimensional Banach spaces, Theorems 3.2 and 3.4. In fact Theorem 3.4 is a bit better: Birkhoff integrability coincides with Pettis integrability when the multi-valued function F takes values in $ck(X)$; in particular, such an F is Pettis integrable if, and only if, the single-valued function $j \circ F$ is Pettis integrable, Proposition 3.5, which is, to the best of our knowledge, a new characterization for Pettis integrable multi-valued maps. To end up this part we prove that if X is an Asplund Banach space, then $cwk(X)$ is separable for the Hausdorff distance if, and only if, X is finite dimensional, Corollary 3.7. The last subsection of the paper is devoted to provide examples showing that Birkhoff integrability for multi-valued maps lies strictly between Debreu and Pettis integrability when X is infinite dimensional and X^* is separable, even when we deal with bounded multi-valued functions, Examples 3.10 and 3.12.

Terminology. Our Banach spaces are assumed to be real and referred to by letters X, Y and Z; if the norm is explicitly needed we shall write $\|\cdot\|$. The weak topology of Y is denoted by w, and w^* denotes the weak* topology of the dual Y^*. For a given set $S \subset Y$ we use $\|\cdot\|$-diam$(S) := \sup_{y,y' \in S} \|y - y'\|$. A set $B \subset B_{Y^*}$ is norming if $\|y\| = \sup\{|\langle y^*, y\rangle| : y^* \in B\}$ for every $y \in Y$. The topology of pointwise convergence in $\mathbb{R}^\Omega$ is denoted by $\tau_p(\Omega)$. $\mathcal{L}^1(\mu)$ stands for the space of real μ-integrable functions defined on Ω and $L^1(\mu)$ for the associated Banach space of equivalence classes with its usual norm $\|\cdot\|_1$. We denote by λ the Lebesgue measure on the σ-algebra $\mathcal{L}$ of all Lebesgue measurable subsets of $[0,1]$. For the theory of Bochner and Pettis integrals we refer the reader to [12] and [26].

2. SERIES OF SETS IN BANACH SPACES AND BIRKHOFF INTEGRABLE MULTI-VALUED FUNCTIONS

Recall that a single-valued function f defined on Ω with values in a Banach space Y is *summable* with respect to a given countable partition $\Gamma = (A_n)$ of Ω in Σ [4] if $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set

$$J(f,\Gamma) := \Big\{\sum_n \mu(A_n) f(t_n) : \ t_n \in A_n\Big\}$$

is made up of unconditionally convergent series. The function f is said to be *Birkhoff integrable* [4] if for every $\varepsilon > 0$ there is a countable partition Γ of Ω in Σ for which f is summable and $\|\cdot\|$-diam$(J(f,\Gamma)) \leq \varepsilon$. In this case, the Birkhoff integral $(B)\int_\Omega f\, d\mu$ of f is the only point in the intersection

$$\bigcap\{\overline{\mathrm{co}(J(f,\Gamma))} : f \text{ is summable with respect to } \Gamma\}.$$

We stress that Birkhoff integrability lies strictly between Bochner and Pettis integrability, [4], [27] and [28]. If f is Birkhoff integrable then we have the equality $(B)\int_\Omega f\, d\mu =$ (Pettis) $\int_\Omega f\, d\mu$ and both integrals are, from now onwards, simply written as $\int_\Omega f\, d\mu$.

In a similar way as Debreu integral was defined, we introduce below the natural extension of Birkhoff integral to the case of multi-valued functions.

Definition 2.1. *Let $F : \Omega \longrightarrow cwk(X)$ be a multi-valued function. We say that F is Birkhoff integrable if the single-valued function $j \circ F : \Omega \longrightarrow \ell_\infty(B_{X^*})$ is Birkhoff integrable.*

If F is Birkhoff integrable, then for every $A \in \Sigma$ the restriction $F|_A$ is Birkhoff integrable with respect to the restriction of μ to the σ-algebra $\{E \cap A : E \in \Sigma\}$, because the same holds true for the single-valued Birkhoff integrable function $j \circ F$ [4, Theorem 14], and

$$\int_A j \circ F\, d\mu \in \mu(A) \cdot \overline{\mathrm{co}(j \circ F(A))}.$$

Since $j(cwk(X))$ is a closed convex cone in $\ell_\infty(B_{X^*})$ by Lemma 1.1, we conclude that $\int_A j \circ F\, d\mu \in j(cwk(X))$. Therefore there is a unique $(B)\int_A F\, d\mu \in cwk(X)$, that will be called *Birkhoff integral* of F on A, that satisfies

$$j\Big((B)\int_A F\, d\mu\Big) = \int_A j \circ F\, d\mu.$$

The remaining of the section is devoted to prove two different characterizations of Birkhoff integrability, Propositions 2.6 and 2.9. To get started we need some previous machinery about convergent series of sets in Banach spaces.

Given a sequence $B_1, B_2, \dots$ of subsets of X, the symbol $\sum_{n=1}^\infty B_n$ denotes a formal series. The series $\sum_{n=1}^\infty B_n$ is said to be *unconditionally convergent* provided that for every choice $b_n \in B_n$, $n \in \mathbb{N}$, the series $\sum_{n=1}^\infty b_n$ is unconditionally convergent in X. In this case we define

$$\sum_{n=1}^\infty B_n := \Big\{\sum_{n=1}^\infty b_n : \ b_n \in B_n, \ n \in \mathbb{N}\Big\}.$$

If we agree to write $\|B\| := \sup\{\|x\| : x \in B\}$ when $B \subset X$, then $\sum_{n=1}^\infty B_n$ is unconditionally convergent if, and only if, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|\sum_{i\in S} B_i\| \leq \varepsilon$ for every finite set $S \subset \mathbb{N} \setminus \{1,\dots,N\}$, see [4, p. 362]. Indeed, the *if* part is clear and we prove the *only if* part by contradiction. Suppose that there is $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there is a finite set $S \subset \mathbb{N} \setminus \{1, \dots, N\}$ such that $\| \sum_{i \in S} B_i \| > \varepsilon$. Then there exist an infinite sequence (S_k) of pairwise disjoint nonempty finite subsets of $\mathbb{N}$ and choices $b_n \in B_n$, $n \in S_k$, $k \in \mathbb{N}$, such that $\| \sum_{n \in S_k} b_n \| > \varepsilon$ for every $k \in \mathbb{N}$. Fix $b_n \in B_n$ for every $n \in \mathbb{N} \setminus \bigcup_{k=1}^{\infty} S_k$. Then the family $(b_n)_{n \in \mathbb{N}}$ is not summable and therefore $\sum_{n=1}^{\infty} b_n$ can not be unconditionally convergent (see e.g. [9, Theorem 10.7]).

Lemma 2.2. *Let (B_n) be a sequence in $cwk(X)$ such that $\sum_n B_n$ is unconditionally convergent. Then $\sum_n B_n \in cwk(X)$.*

Proof. Clearly $\sum_n B_n$ is convex. To see that $\sum_n B_n$ is weakly compact let us consider the mapping

$$S : \prod_n B_n \longrightarrow X, \quad S((b_n)_n) := \sum_n b_n.$$

Let $\mathfrak{T}$ be the product topology in $\prod_n B_n$ obtained when each B_n is endowed with the restriction of the weak topology of X. We claim that S is $\mathfrak{T}$-to-weak continuous. Indeed, fix $(b_n)_n \in \prod_n B_n$ and $U \in \mathcal{U}$, where $\mathcal{U}$ is the family of all neighborhoods of 0 in the weak topology of X. There exist $\varepsilon > 0$ and $V \in \mathcal{U}$ such that $2\varepsilon B_X + V \subset U$. Since $\sum_n B_n$ is unconditionally convergent, there is $N \in \mathbb{N}$ such that $\| \sum_{i \in S} B_i \| \leq \varepsilon$ for every finite set $S \subset \mathbb{N} \setminus \{1, \dots, N\}$. Fix $W_1, \dots, W_N \in \mathcal{U}$ such that $\sum_{n=1}^{N} W_n \subset V$. Define $H_n := B_n \cap (b_n + W_n)$ for every $1 \leq n \leq N$, $H_n := B_n$ for every $n > N$ and $H := \prod_n H_n$. Then H is a $\mathfrak{T}$-neighborhood of $(b_n)_n$ such that for each $(b'_n)_n \in H$

$$\begin{aligned} S((b'_n)_n) = \sum_n b'_n = \sum_{n=1}^{N} b'_n + \sum_{n>N} b'_n &\in \sum_{n=1}^{N} b_n + \sum_{n>N} b'_n + \sum_{n=1}^{N} W_n \\ &\subset \sum_n b_n + \sum_{n>N} (b'_n - b_n) + V \subset \sum_n b_n + 2\varepsilon B_X + V \subset \sum_n b_n + U. \end{aligned}$$

Since $(b_n)_n \in \prod_n B_n$ and $U \in \mathcal{U}$ are arbitrary, S is $\mathfrak{T}$-to-weak continuous. Finally, since $(\prod_n B_n, \mathfrak{T})$ is compact (by Tychonoff's theorem [22, Theorem 13, p. 143]), $S(\prod_n B_n) = \sum_n B_n$ is weakly compact. The proof is over. $\square$

The following result can be obtained as a consequence of the previous lemma and Propositions 2.3 and 2.5 in [13]; from the proof we give below it becomes clear that the role played by j in our definition of Birkhoff integrability can be also played by any embedding i from $cwk(X)$ into a Banach space Y fulfilling (i)-(iv) in Lemma 1.1.

Lemma 2.3. *Let (B_n) be a sequence in $cwk(X)$. The following conditions are equivalent:*

(i) *$\sum_n B_n$ is unconditionally convergent;*
(ii) *there is $B \in cwk(X)$ with the following property: for every $\varepsilon > 0$ there is a finite set $P \subset \mathbb{N}$ such that $h(\sum_{n \in Q} B_n, B) \leq \varepsilon$ for every finite set $Q \subset \mathbb{N}$ such that $P \subset Q$;*
(iii) *$\sum_n j(B_n)$ is unconditionally convergent in $\ell_\infty(B_{X^*})$.*

In this case, $\sum_n B_n = B$ and $j(\sum_n B_n) = \sum_n j(B_n)$.

Proof. (i)⇒(ii) Note that $B := \sum_n B_n$ belongs to $cwk(X)$ by Lemma 2.2. Fix $\varepsilon > 0$. Since $\sum_n B_n$ is unconditionally convergent, there is $N \in \mathbb{N}$ such that $\| \sum_{n \in S} B_n \| \leq \varepsilon$ for every finite set $S \subset \mathbb{N} \setminus P$, where $P := \{1, \dots, N\}$. Take any finite set $Q \subset \mathbb{N}$ with $P \subset Q$. Then

$$\sum_{n \in Q} B_n \subset B + \varepsilon B_X \quad \text{and} \quad B \subset \sum_{n \in Q} B_n + \varepsilon B_X,$$

hence $h(\sum_{n\in Q} B_n, B) \leq \varepsilon$. This proves (i)⇒(ii).

Assume now that (ii) holds, meaning that the net

$$\Big\{\sum_{n\in Q} B_n : \ Q \subset \mathbb{N},\ Q \text{ finite}\Big\}$$

converges in $(cwk(X), h)$. In particular, for $\varepsilon > 0$, there is a finite set $P \subset \mathbb{N}$ such that $h(\sum_{n\in Q} B_n, \sum_{n\in P} B_n) \leq \varepsilon$ for every finite set $Q \subset \mathbb{N}$ with $P \subset Q$. Take any finite set $S \subset \mathbb{N} \setminus P$. By Lemma 1.1 we have

$$\Big\|\sum_{n\in S} B_n\Big\| = h\Big(\sum_{n\in S} B_n, \{0\}\Big) = \Big\|j\Big(\sum_{n\in S} B_n\Big)\Big\|_\infty$$
$$= \Big\|j\Big(\sum_{n\in S\cup P} B_n\Big) - j\Big(\sum_{n\in P} B_n\Big)\Big\|_\infty = h\Big(\sum_{n\in S\cup P} B_n, \sum_{n\in P} B_n\Big) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary the series $\sum_n B_n$ is unconditionally convergent and we have proved that (ii)⇒(i).

To realize that (ii) and (iii) are actually equivalent we simply note that the computations above yield

$$\Big\|\sum_{n\in S} B_n\Big\| = h\Big(\sum_{n\in S} B_n, \{0\}\Big) = \Big\|j\Big(\sum_{n\in S} B_n\Big)\Big\|_\infty = \Big\|\sum_{n\in S} j(B_n)\Big\|_\infty,$$

for each finite set $S \subset \mathbb{N}$.

The equality $\sum_n B_n = B$ follows from the proof (i)⇒(ii) and once this is known the equality $j(\sum_n B_n) = \sum_n j(B_n)$ follows from Lemma 1.1. □

Remark 2.4. Observe that each one of the three equivalent statements in Lemma 2.3 is equivalent to the following:

(iv) *For any embedding i from $cwk(X)$ into a Banach space Y satisfying properties* (i)-(iv) *in Lemma 1.1, the series $\sum_n i(B_n)$ is unconditionally convergent in Y.*

In this case, $i(\sum_n B_n) = \sum_n i(B_n)$.

In [7, Proposition 2.6] we exhibited the following characterization of Birkhoff integrability for single-valued functions. As usual, we say that a partition Γ of Ω in Σ is finer than another one Γ_0, if each element of Γ is contained in some element of Γ_0.

Proposition 2.5. *A single-valued function f defined on Ω with values in a Banach space Y is Birkhoff integrable if, and only if, there is $y \in Y$ with the following property: for every $\varepsilon > 0$ there is a countable partition Γ_0 of Ω in Σ such that for every countable partition $\Gamma = (A_n)$ of Ω in Σ finer than Γ_0 and any choice $T = (t_n)$ in Γ (i.e. $t_n \in A_n$ for every n), the series $\sum_n \mu(A_n)f(t_n)$ converges unconditionally and*

$$\Big\|\sum_n \mu(A_n)f(t_n) - y\Big\| \leq \varepsilon.$$

In this case, $y = \int_\Omega f\, d\mu$.

As an easy consequence of Lemmas 1.1 and 2.3 and Proposition 2.5 we have the following characterization.

Proposition 2.6. *Let $F : \Omega \longrightarrow cwk(X)$ be a multi-valued function. The following conditions are equivalent:*

(i) *F is Birkhoff integrable;*

(ii) *there is $B \in cwk(X)$ with the following property: for every $\varepsilon > 0$ there is a countable partition Γ_0 of Ω in Σ such that for every countable partition $\Gamma = (A_n)$ of Ω in Σ finer than Γ_0 and any choice $T = (t_n)$ in Γ, the series $\sum_n \mu(A_n)F(t_n)$ is unconditionally convergent and*

$$h\Big(\sum_n \mu(A_n)F(t_n), B\Big) \leq \varepsilon.$$

In this case, $B = (B)\int_\Omega F\, d\mu$.

If we bear in mind Remark 2.4 we will convince ourselves that the following holds.

Corollary 2.7. *The notions of Birkhoff integrability and Birkhoff integral for multi-valued functions do not depend on the particular embedding i from $cwk(X)$ into a Banach space Y chosen, as long as i satisfies properties* (i)-(iv) *in Lemma 1.1.*

The characterization of Birkhoff integrability that closes the section, Proposition 2.9, is for bounded multi-valued functions. We recall some definitions first.

Definition 2.8. *Let $\mathcal{F}$ be a family in $\mathbb{R}^\Omega$.*

(i) *We say that $\mathcal{F}$ has Bourgain property (with respect to μ)* [30] *if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $B_1, \dots, B_n \subset A$, $B_i \in \Sigma$, with $\mu(B_i) > 0$, such that for every $f \in \mathcal{F}$*

$$\inf_{1 \leq i \leq n} |\cdot|\text{-diam}(f(B_i)) \leq \varepsilon.$$

(ii) *We say that $\mathcal{F}$ has Birkhoff property (with respect to μ)* [7] *if for every $\varepsilon > 0$ there is a countable partition $\Gamma = (A_n)$ of Ω in Σ such that for each $t_k, t_k' \in A_k$, $k \in \mathbb{N}$, we have*

$$\Big|\sum_{k=1}^m \mu(A_k)f(t_k) - \sum_{k=1}^m \mu(A_k)f(t_k')\Big| \leq \varepsilon$$

for every $m \in \mathbb{N}$ and every $f \in \mathcal{F}$.

We notice that if $\mathcal{F}$ has Birkhoff property, then its pointwise closure $\overline{\mathcal{F}}^{\tau_p(\Omega)}$ and its absolutely convex hull $\operatorname{aco}(\mathcal{F})$ also have Birkhoff property. If $\mathcal{F}$ has Birkhoff property, then $\mathcal{F}$ has Bourgain property. The converse holds if $\mathcal{F}$ is uniformly bounded, see [7, Lemma 2.3].

Observe that for every $x^* \in B_{X^*}$ and $t \in \Omega$ we have

$$\delta^*(x^*, F)(t) = \langle e_{x^*}, j \circ F(t)\rangle,$$

where $e_{x^*} \in B_{\ell_\infty(B_{X^*})^*}$ is defined by $\langle e_{x^*}, g\rangle := g(x^*)$ for every $g \in \ell_\infty(B_{X^*})$. Given a multi-valued function $F : \Omega \longrightarrow cwk(X)$, we fix the following terminology

$$W_F := \{\delta^*(x^*, F) :\ x^* \in B_{X^*}\} \subset \mathbb{R}^\Omega.$$

Proposition 2.9. *Let $F : \Omega \longrightarrow cwk(X)$ be a bounded (for the Hausdorff distance) multi-valued function. The following conditions are equivalent:*

(i) *F is Birkhoff integrable;*
(ii) *W_F has Birkhoff property;*
(iii) *W_F has Bourgain property.*

Proof. Since F is h-bounded, the function $j \circ F : \Omega \to \ell_\infty(B_{X^*})$ is bounded. On the other hand, since the set $B := \{e_{x^*} : x^* \in B_{X^*}\} \subset B_{\ell_\infty(B_{X^*})^*}$ is norming, the proposition straightforwardly follows from Theorem 2.4 in [7] applied to $j \circ F$ and B. $\square$

3. Birkhoff integrability versus Debreu and Pettis integrability

This section is devoted to study the relationship between the notions of Birkhoff, Debreu and Pettis integrability, see Subsection 3.1. We also provide some examples that show that Birkhoff integrability lies strictly between Debreu and Pettis integrability even when bounded multi-valued maps are considered, see Subsection 3.2.

3.1. **Positive results.** In our first result below the connection between Birkhoff and Aumann integrals is made clear too.

Proposition 3.1. *Let $F : \Omega \longrightarrow cwk(X)$ be a multi-valued function.*

(i) *If F is Debreu integrable, then F is Birkhoff integrable and $(B)\int_\Omega F\,d\mu = (D)\int_\Omega F\,d\mu$.*

Assuming that X is separable we have:

(ii) *If F is Birkhoff integrable, then F is Pettis integrable and for every $A \in \Sigma$ we have $(B)\int_A F\,d\mu = (P)\int_A F\,d\mu$. Moreover, F admits strongly measurable selectors, each of them being Birkhoff integrable, and for every $A \in \Sigma$ we have*

$$(B)\int_A F\,d\mu = \Big\{\int_A f\,d\mu : \ f \text{ is a Birkhoff integrable selector of } F\Big\}. \tag{2}$$

Proof. Statement (i) follows from the very definitions of the integrals involved.

Let us assume that X is separable and let us prove (ii). Since $j \circ F$ is Pettis integrable, we have $\delta^*(x^*, F) = \langle e_{x^*}, j\circ F\rangle \in \mathcal{L}^1(\mu)$ for every $x^* \in B_{X^*}$ and

$$\begin{aligned}\delta^*\Big(x^*, (B)\int_A F\,d\mu\Big) &= \langle e_{x^*}, j\Big((B)\int_A F\,d\mu\Big)\rangle \\ &= \langle e_{x^*}, \int_A j\circ F\,d\mu\rangle = \int_A \langle e_{x^*}, j\circ F\rangle\,d\mu = \int_A \delta^*(x^*, F)\,d\mu\end{aligned}$$

for every $A \in \Sigma$ and every $x^* \in B_{X^*}$. Consequently F is Pettis integrable and for every $A \in \Sigma$ the equality $(P)\int_A F\,d\mu = (B)\int_A F\,d\mu$ holds.

As pointed out in the introduction, since $F : \Omega \longrightarrow cwk(X)$ is scalarly measurable, F admits strongly measurable selectors. In addition, since F is even Pettis integrable, each strongly measurable selector of F is Pettis integrable and equality (1) in page 3 holds. Finally, for separable Banach spaces Birkhoff and Pettis integrability coincide for single-valued functions [27], hence equality (2) follows from equality (1) in page 3 and the proof is over. □

To prove the next result we will use that the space $(ck(X), h)$ is separable if X is separable, see [8, Theorem II.8].

Theorem 3.2. *If X is finite dimensional, then a multi-valued function $F : \Omega \longrightarrow ck(X)$ is Debreu integrable if, and only if, F is Birkhoff integrable.*

Proof. The *only if* part is statement (i) in Proposition 3.1. The proof of the *if* part is as follows. Assume that F is Birkhoff integrable. In order to establish that F is Debreu integrable we have to show that $j \circ F$ is Bochner integrable. Since $j(ck(X))$ is separable and $j \circ F$ is scalarly measurable, $j \circ F$ is strongly measurable by Pettis's measurability theorem [12, Theorem 2, p. 42]. Therefore, the proof will be finished when proving

$$\int_\Omega \|j \circ F\|_\infty\,d\mu < \infty. \tag{3}$$

Given any set $A \subset \Omega$, write $\mathbb{F}(A) = \bigcup_{t \in A} F(t)$. Observe that

$$\|j \circ F(A)\| := \sup\{\|j(F(t))\|_\infty : t \in A\} = \sup\{\|F(t)\| : t \in A\} = \|\mathbb{F}(A)\|.$$

Using that $j \circ F$ is Birkhoff integrable, we find a countable partition (A_m) of Ω in Σ such that $j \circ F(A_m)$ is bounded in $\ell_\infty(B_{X^*})$, whenever $\mu(A_m) > 0$, with the series $\sum_m \mu(A_m) j \circ F(t_m)$ unconditionally convergent for every choice (t_m) in (A_m). Hence $\|\mathbb{F}(A_m)\| < \infty$ whenever $\mu(A_m) > 0$, and the series $\sum_m \mu(A_m)\mathbb{F}(A_m)$ is unconditionally convergent in X by Lemma 2.3. In finite dimensional Banach spaces every unconditionally convergent series is absolutely convergent and we conclude that

$$\sum_{\mu(A_m)>0} \mu(A_m)\|j \circ F(A_m)\| = \sum_{\mu(A_m)>0} \mu(A_m)\|\mathbb{F}(A_m)\| < \infty,$$

which proves (3) and finishes the proof. □

The lemma below is well known, at least for the case $B = B_{Y^*}$, see [26, Theorem 5.2]. However, we include a proof for the sake of completeness and because our approach is easily extended to a far more general context that shall be explained at the end of the subsection. Recall that a set $\mathcal{F} \subset \mathcal{L}^1(\mu)$ is uniformly integrable if $\mathcal{F}$ is $\|\cdot\|_1$-bounded and for every $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_{f \in \mathcal{F}} \int_E |f|\, d\mu \leq \varepsilon$ whenever $\mu(E) \leq \delta$. Equivalently, the canonical image of $\mathcal{F}$ in $L^1(\mu)$ is relatively weakly compact, see [16, 247C].

Lemma 3.3. *Let Y be a separable Banach space and $B \subset B_{Y^*}$ a norming set. Let $f : \Omega \longrightarrow Y$ be a function such that $Z_{f,B} = \{\langle y^*, f\rangle : y^* \in B\}$ is a uniformly integrable subset of $\mathcal{L}^1(\mu)$. Then f is Pettis integrable.*

Proof. Since B is norming, the Hahn-Banach separation theorem applies to obtain that the absolutely convex hull of B, $\mathrm{aco}(B)$, is w^*-dense in B_{Y^*}. The separability of Y implies that (B_{Y^*}, w^*) is metrizable, hence $\mathrm{aco}(B)$ is w^*-sequentially dense in B_{Y^*}. Therefore the uniformly integrable set $\mathrm{aco}(Z_{f,B})$ is $\tau_p(\Omega)$-sequentially dense in $Z_f = \{\langle y^*, f\rangle : y^* \in B_{Y^*}\}$. An appeal to Vitali's theorem [16, 246J (a)] establishes that Z_f is a uniformly integrable subset of $\mathcal{L}^1(\mu)$.

In order to see that f is Pettis integrable we only have to check that the canonical map

$$T : B_{Y^*} \longrightarrow L^1(\mu), \quad T(y^*) = \langle y^*, f\rangle,$$

is w^*-to-w continuous [26, Theorem and Remark 4.3]. To this end, fix $C \subset B_{Y^*}$ and take any $y^* \in \overline{C}^{w^*}$. Since Y is separable, there is a sequence (y_n^*) in C that w^*-converges to y^*. Therefore $(T(y_n^*))$ is a sequence in Z_f converging pointwise to $T(y^*)$ and, since Z_f is uniformly integrable, another appeal to Vitali's theorem ensures us that

$$\lim_n \|T(y_n^*) - T(y^*)\|_1 = 0.$$

In particular, $T(y^*) \in \overline{T(C)}^w$. Hence $T(\overline{C}^{w^*}) \subset \overline{T(C)}^w$ for every $C \subset B_{Y^*}$. It follows that T is w^*-to-w continuous and f is Pettis integrable. □

To prove Theorem 3.4 and Proposition 3.5 below we will use the following characterization of Pettis integrable multi-valued functions: *a multi-valued function $F : \Omega \longrightarrow cwk(X)$ is Pettis integrable if, and only if, $W_F = \{\delta^*(x^*, F) : x^* \in B_{X^*}\}$ is a uniformly integrable subset of $\mathcal{L}^1(\mu)$*, see [32, Theorem 3.2] and [33].

Theorem 3.4. *Assume that X is separable. Let $F : \Omega \longrightarrow cwk(X)$ be a multi-valued function such that $F(\Omega)$ is h-separable (e.g. $F(\Omega) \subset ck(X)$). Then F is Birkhoff integrable if, and only if, F is Pettis integrable.*

Proof. In view of Proposition 3.1 it only remains to show the *if* part. Assume that F is Pettis integrable. We begin with the proof of the claim below.

Claim.- The single-valued function $j \circ F : \Omega \longrightarrow \ell_\infty(B_{X^*})$ is Pettis integrable.

From the h-separability of $F(\Omega)$ we deduce that $Y := \overline{\mathrm{span}}(j \circ F(\Omega))$ is a closed separable subspace of $\ell_\infty(B_{X^*})$ in which $j \circ F$ takes its values. Let us notice that the set $B := \{e_{x^*}|_Y : x^* \in B_{X^*}\} \subset B_{Y^*}$ is norming. By the Pettis integrability of F, the family $W_F = Z_{j\circ F,B}$ is a uniformly integrable subset of $\mathcal{L}^1(\mu)$ and Lemma 3.3 tells us that $j \circ F$ is Pettis integrable, as claimed. Since Birkhoff and Pettis integrability coincide for single-valued functions with values in a separable Banach space [27], it follows that $j \circ F$ is Birkhoff integrable. The proof is complete. □

Given a separable Banach space X and a multi-valued map $F : \Omega \longrightarrow cwk(X)$ we could not find in the literature any reference to prior study about the relationship between F being Pettis integrable and $j \circ F$ being Pettis integrable too. In Proposition 3.5 we analyze this matter.

Proposition 3.5. *Assume that X is separable and let $F : \Omega \longrightarrow cwk(X)$ be a multi-valued function. Let us consider the following statements:*

(i) *$j \circ F$ is Pettis integrable;*
(ii) *F is Pettis integrable.*

Then (i) *always implies* (ii) *and in this case $j((P)\int_A F\, d\mu) = \int_A j \circ F\, d\mu$ for every $A \in \Sigma$. If moreover $F(\Omega)$ is h-separable (e.g. $F(\Omega) \subset ck(X)$) then* (ii) *implies* (i).

Proof. Assume that (i) holds. We know that $W_F = \{\langle e_{x^*}, j \circ F\rangle : x^* \in B_{X^*}\}$ is a uniformly integrable subset of $\mathcal{L}^1(\mu)$, see [26, Corollary 4.1], and therefore the multi-valued function F is Pettis integrable.

Moreover, for every $A \in \Sigma$ and every $x^* \in B_{X^*}$ we have

$$\begin{aligned}\langle e_{x^*}, \int_A j \circ F\, d\mu\rangle &= \int_A \langle e_{x^*}, j \circ F\rangle\, d\mu \\ &= \int_A \delta^*(x^*, F)\, d\mu = \delta^*\Big(x^*, (P)\int_A F\, d\mu\Big) = \langle e_{x^*}, j\Big((P)\int_A F\, d\mu\Big)\rangle.\end{aligned}$$

Hence $j((P)\int_A F\, d\mu) = \int_A j \circ F\, d\mu$ for every $A \in \Sigma$.

The implication (ii) ⇒(i) when $F(\Omega)$ is h-separable is a straightforward consequence of Theorem 3.4. The proof is finished. □

The reader is well aware at this point of the role played by the hypothesis "*$F(\Omega)$ is h-separable*" in the implication (ii)⇒(i) above: we have to fulfill the requirements in Lemma 3.3 that is used in the proof of Theorem 3.4. In other words, we have to ensure that $Y := \overline{\mathrm{span}}(j \circ F(\Omega))$ is a closed separable subspace of $\ell_\infty(B_{X^*})$. So in order to get possible extensions of Proposition 3.5 two natural questions arise:

(A) ¿when is $cwk(X)$ h-separable?
(B) ¿can we extend Lemma 3.3 for a class of Banach spaces wider than the class of separable ones?

Question (B) has a pretty reasonable answer. Lemma 3.3 can be extended to those Banach spaces Y with dual unit ball satisfying the following property:

> *For every subset C of B_{Y^*}, if $y^* \in \overline{C}^{w^*}$, then there is a sequence in C that w^*-converges to y^* – shortly, (B_{Y^*}, w^*) is angelic (Fremlin).*

Whereas the class of Banach spaces with w^*-angelic dual unit ball is difficult to handle (there is no intrinsic characterization of spaces in this class) there are, however, notorious wide subclasses of it with pretty good properties, as for instance, the class of weakly countably K-determined Banach spaces – this class properly extends the classes of separable and weakly compactly generated Banach spaces, see [31]. From the above, it follows that Lemma 3.3 extends, in particular, to the class of weakly countably K-determined Banach spaces. Since every Banach space is weakly countably K-determined provided that it contains a total weakly countably K-determined subset [31, Théorème 3.6], our Theorem 3.4 is true under the, *a priori*, more general assumption of *weakly countably K-determination* for $j \circ F(\Omega)$ instead of separability. Unfortunately, Lemma 3.6 shows that this extension is futile. This lemma also gives an answer to previous question (A) for Banach spaces with separable dual. The lemma will be used once again later in the paper.

Recall that a topological space (T, τ) is said to be *countably K-determined* if there is a separable metric space M and an upper semi-continuous multi-valued map $F : M \to 2^T$ such that $F(m)$ is compact for each $m \in M$ and $T = \bigcup\{F(m) : m \in M\}$. Here the multi-valued map F is called *upper semi-continuous* if for each $m \in M$ and for each open subset U of T such that $F(m) \subset U$ there exists a neighborhood V of m with $F(v) \subset U$ for each $v \in V$. Every countably K-determined topological space is Lindelöf, see [15].

Lemma 3.6. *Assume that X is separable. Let T be any subset of $cwk(X)$. The following statements are equivalent:*

(i) *T is h-separable;*
(ii) *$j(T)$ is countably K-determined with the weak topology w induced by $\ell_\infty(B_{X^*})$.*

If, moreover, X^* is separable and $T = cwk(X)$, then each of the above is equivalent to:

(iii) *X is finite dimensional.*

Proof. The implication (i)⇒(ii) is obvious because (i) implies that $j(T)$ is separable for the topology $\mathfrak{T}$ induced by the norm $\ell_\infty(B_{X^*})$, after Lemma 1.1, and $(j(T), w)$ is a continuous image of $(j(T), \mathfrak{T})$.

The other way around. Assume (ii) holds. Observe first that for each $A \in cwk(X)$ the function $\delta^*(., A) : B_{X^*} \to \mathbb{R}$ given by $\delta^*(., A)(x^*) = \delta^*(x^*, A)$ is bounded and continuous when B_{X^*} is endowed with the topology induced by the Mackey topology τ (with respect to the dual pair $\langle X, X^*\rangle$) in X^* [24, Mackey-Arens Theorem §21.4.(2)]. If we denote by $C_b(B_{X^*})$ the space of real bounded and continuous functions on (B_{X^*}, τ), our previous comment is rephrased as $j(cwk(X)) \subset C_b(B_{X^*})$. Then the topology induced by the weak topology of $\ell_\infty(B_{X^*})$ in $j(T)$ coincides with the topology induced by the weak topology of the Banach space $(C_b(B_{X^*}), \|\cdot\|_\infty)$. Since X is separable there is a countable set $C \subset B_{X^*}$ such that $\overline{C}^{w^*} = B_{X^*}$. If D is the set of convex combinations of C with rational coefficients, then we deduce that $B_{X^*} = \overline{D}^\tau$ –bear in mind that the dual of the locally convex space (X^*, τ) is again X and thus w^* and τ have the same closed convex sets [24, §20.8.(6)]. The topology $\tau_p(D)$ on $C_b(B_{X^*})$ of pointwise convergence on D is metrizable and coarser than the weak topology of the Banach space $C_b(B_{X^*})$. Hence $j(T)$ is w-countably K-determined and has a metrizable coarser topology ν. Since $(j(T), w)$ is Lindelöf, its continuous metrizable image $(j(T), \nu)$ is Lindelöf too, thus second countable. Now [6, Theorem 8] applies to ensure us that $(j(T), w)$ is separable. Thus $j(T)$ is separable for the topology induced by the norm of $C_b(B_{X^*})$ (or by the norm of $\ell_\infty(B_{X^*})$), and consequently a new appeal to Lemma 1.1 tells us that T is h-separable and the implication (ii)⇒(i) has been established.

For X finite dimensional we know that $cwk(X) = ck(X)$ is h-separable and so to finish the proof of the lemma we only have to prove that if $(cwk(X), h)$ and X^* are separable, then X is finite dimensional. This is proved by contradiction. If we assume that X is infinite dimensional and X^* is separable, the Ovsepian-Pelczynski theorem [25, Theorem 1.f.4] ensures us of the existence of an infinite countable shrinking Markushevich basis of X which is bounded, i.e. a sequence $\{(x_n, x_n^*)\}_{n\in\mathbb{N}} \subset X \times X^*$ such that

(i) $x_n^*(x_m) = \delta_{n,m}$ (the Kronecker symbol) for every $n, m \in \mathbb{N}$;
(ii) $\overline{\text{span}}\{x_n\}_{n\in\mathbb{N}} = X$;
(iii) $\overline{\text{span}}^{\|\cdot\|}\{x_n^*\}_{n\in\mathbb{N}} = X^*$;
(iv) $\sup_{n\in\mathbb{N}} \|x_n\|\|x_n^*\| < \infty$.

We can assume (normalize!) that (x_n) is bounded and that $x_n^* \in B_{X^*}$ for every $n \in \mathbb{N}$. Observe that (x_n) is weakly convergent to 0, hence the set $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact and, by the Krein-Smulian theorem [12, Theorem 11, p. 51], for every $\emptyset \neq P \subset \mathbb{N}$ the set

$$C_P := \overline{\text{co}}\{x_n : n \in P\}$$

belongs to $cwk(X)$. We claim that

$$h(C_P, C_Q) \geq 1 \quad \text{whenever } P \neq Q. \tag{4}$$

Indeed, assume that $Q \not\subset P$ and fix $n \in Q \setminus P$. Let $\eta > 0$ be such that $C_Q \subset C_P + \eta B_X$ and fix $\varepsilon > 0$. Since $x_n \in C_Q$, there is $y \in \text{co}\{x_m : m \in P\}$ such that $\|x_n - y\| \leq \eta + \varepsilon$. But $n \notin P$, hence $1 = x_n^*(x_n - y) \leq \|x_n - y\| \leq \eta + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\eta \geq 1$ and therefore $h(C_P, C_Q) \geq 1$, as claimed. Finally, being the collection of all non-empty subsets of $\mathbb{N}$ uncountable, it follows that the space $(cwk(X), h)$ is not separable. The proof is over. □

Given a closed subspace $Z \subset X$, we have $cwk(Z) = \{B \in cwk(X) : B \subset Z\}$ and the Hausdorff distance (relative to the metric space Z) between two arbitrary elements $B, B' \in cwk(Z)$ is exactly $h(B, B')$. As a consequence we get the following result.

Corollary 3.7. *Assume that X^* has the Radon-Nikodým property (i.e. X is Asplund). The following conditions are equivalent:*

(i) $cwk(X)$ *is h-separable;*
(ii) X *is finite dimensional.*

Proof. It follows straightforwardly from Lemma 3.6, since the dual Z^* of each closed separable subspace $Z \subset X$ is separable, see [12, Theorem 6, p. 195]. □

We stress that the hypothesis X^* separable in the implication (i)⇒(iii) in Lemma 3.6 cannot be weakened to X separable: indeed, $X = \ell_1$ with its natural norm is an infinite dimensional separable Banach space with Schur's property [11, p. 85], thus $cwk(X) = ck(X)$ is h-separable.

3.2. **Examples.** It is well known that the notions of Bochner and Birkhoff integrability coincide for *bounded* single-valued functions defined on Ω with values in a separable Banach space. However, when bounded multi-valued maps are considered the previous equivalence does not hold in general, see Example 3.10 below. For the proof we need Lemmas 3.8 and 3.9.

Lemma 3.8. *The family $\mathcal{Q} := \{\chi_{[0,s)} : s \in [0,1]\} \cup \{\chi_{[s,1]} : s \in [0,1]\} \subset \mathbb{R}^{[0,1]}$ has Birkhoff property with respecto to λ.*

Proof. From the equality $\chi_{[s,1]} = \mathbf{1} - \chi_{[0,s)}$, $s \in [0,1]$, we deduce that $\mathcal{Q}$ has Birkhoff property if $\{\chi_{[0,s)} : s \in [0,1]\}$ does. We prove the latter. Fix $\varepsilon > 0$ and choose $n \in \mathbb{N}$ large enough such that $2/n \leq \varepsilon$. Set $A_i := [(i-1)/n, i/n)$ for every $1 \leq i \leq n-1$ and $A_n := [(n-1)/n, 1]$. Given arbitrary $t_i, t_i' \in A_i$, $1 \leq i \leq n$, for each $s \in [0,1]$ we have

$$\begin{aligned}\Big|\sum_{i=1}^{n} \lambda(A_i)\chi_{[0,s)}(t_i) - \sum_{i=1}^{n} \lambda(A_i)\chi_{[0,s)}(t_i')\Big| &= \frac{1}{n} \cdot \Big|\sum_{i=1}^{n} \big(\chi_{[0,s)}(t_i) - \chi_{[0,s)}(t_i')\big)\Big| \\ &= \frac{1}{n} \cdot \Big|\sum_{i=1}^{n} \big(\chi_{(t_i,1]}(s) - \chi_{(t_i',1]}(s)\big)\Big| \\ &\leq \frac{1}{n} \cdot \Big|\sum_{t_i < t_i'} \chi_{(t_i,t_i']}(s)\Big| + \frac{1}{n} \cdot \Big|\sum_{t_i > t_i'} \chi_{(t_i',t_i]}(s)\Big| \leq \frac{2}{n} \leq \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have proved that $\{\chi_{[0,s)} : s \in [0,1]\}$ has Birkhoff property. $\square$

From now on $\{q_1, q_2, \dots\}$ is a fixed enumeration of $\mathbb{Q} \cap [0,1]$. Given $b_1, \dots, b_N \in \mathbb{R}$, we define $h_{b_1,\dots,b_N} : [0,1] \longrightarrow \mathbb{R}$ by the formula

$$h_{b_1,\dots,b_N}(t) := \max\big(\{b_n :\ 1 \leq n \leq N,\ q_n \leq t\} \cup \{0\}\big).$$

Lemma 3.9. *For any $r > 0$ the family*

$$\mathcal{H}_r := \{h_{b_1,\dots,b_N} :\ b_1, \dots, b_N \in [-r,r],\ N \in \mathbb{N}\} \subset \mathbb{R}^{[0,1]}$$

has Birkhoff property with respecto to λ.

Proof. We first prove that $\mathcal{H}_r \subset \mathrm{aco}(3r\mathcal{Q})$, where $\mathcal{Q}$ is the family defined in Lemma 3.8. Fix $b_1, \dots, b_N \in [-r,r]$. Choose a permutation σ of $\{1, \dots, N\}$ such that

$$q_{\sigma(1)} < q_{\sigma(2)} < \cdots < q_{\sigma(N)}$$

and define

$$c_i := \max\big(\{b_{\sigma(j)} :\ 1 \leq j \leq i\} \cup \{0\}\big), \quad 1 \leq i \leq N.$$

Notice that $0 \leq c_1 \leq c_2 \leq \cdots \leq c_N \leq r$ and that we have

$$\begin{aligned}h_{b_1,\dots,b_N} &= \sum_{i=1}^{N-1} c_i \chi_{[q_{\sigma(i)}, q_{\sigma(i+1)})} + c_N \chi_{[q_{\sigma(N)},1]} \\ &= \sum_{i=1}^{N-1} c_i \big(\chi_{[0,q_{\sigma(i+1)})} - \chi_{[0,q_{\sigma(i)})}\big) + c_N \chi_{[q_{\sigma(N)},1]} \\ &= -c_1 \chi_{[0,q_{\sigma(1)})} + \sum_{i=2}^{N-1} (c_{i-1} - c_i)\chi_{[0,q_{\sigma(i)})} + c_{N-1}\chi_{[0,q_{\sigma(N)})} + c_N \chi_{[q_{\sigma(N)},1]}.\end{aligned}$$

On the other hand

$$\begin{aligned}&|-c_1| + \sum_{i=2}^{N-1} |c_{i-1} - c_i| + |c_{N-1}| + |c_N| \\ &\qquad = c_1 + \sum_{i=2}^{N-1} (c_i - c_{i-1}) + c_{N-1} + c_N = 2c_{N-1} + c_N \leq 3r.\end{aligned}$$

Therefore $h_{b_1,\dots,b_N} \in \mathrm{aco}(3r\mathcal{Q})$. It follows that $\mathcal{H}_r \subset \mathrm{aco}(3r\mathcal{Q})$.

Since by Lemma 3.8 the family $\mathcal{Q}$ has Birkhoff property, the family $3r\mathcal{Q}$ also does. Therefore, $\mathcal{H}_r \subset \mathrm{aco}(3r\mathcal{Q})$ has Birkhoff property, and the proof finishes. □

Example 3.10. *If X is infinite dimensional and X^* is separable, then there exists a bounded Birkhoff integrable multi-valued function $F : [0,1] \longrightarrow cwk(X)$ which is not Debreu integrable.*

Proof. As in the proof of Lemma 3.6, take $\{(x_n, x_n^*)\}_{n\in\mathbb{N}}$ an infinite countable shrinking Markushevich basis of X such that $r := \sup_{n\in\mathbb{N}} \|x_n\| < \infty$ and $x_n^* \in B_{X^*}$ for every $n \in \mathbb{N}$. The same line of arguments we did for the proof of (i)⇒(iii) in Lemma 3.6 ensures that the bounded multi-valued function F defined by

$$F(t) := \overline{\mathrm{aco}}\{x_n : \ q_n \le t\}, \quad t \in [0,1].$$

takes values in $cwk(X)$.

We first prove that F is not Debreu integrable by showing that F is not λ-essentially h-separably valued. This follows from the fact that for any $t \neq s$ in $[0,1]$, we have

$$\{n \in \mathbb{N} : q_n \le t\} \neq \{n \in \mathbb{N} : q_n \le s\}$$

and therefore $h(F(t), F(s)) \ge 1$ (see the proof of Lemma 3.6; inequality (4) also holds for absolutely convex hulls).

We now prove that F is Birkhoff integrable. According to Proposition 2.9 we only have to check that $W_F = \{\delta^*(x^*, F) : \ x^* \in B_{X^*}\}$ has Birkhoff property. Define

$$\mathcal{G} := \{\delta^*(x^*, F) : \ x^* \in B_{X^*} \cap \mathrm{span}\{x_m^* : m \in \mathbb{N}\}\}.$$

We claim that $\mathcal{G} \subset \mathcal{H}_r$, where $\mathcal{H}_r$ is the family defined in Lemma 3.9. Indeed, given $x^* = \sum_{n=1}^N a_n x_n^* \in B_{X^*}$, we have $x^*(x_n) = a_n$ and $|a_n| \le \|x_n\| \le r$ for every $1 \le n \le N$. Moreover, for each $t \in [0,1]$ we have

$$\begin{aligned}
\delta^*(x^*, F(t)) &= \sup\Big\{x^*(x) : x \in \mathrm{aco}\{x_m : \ q_m \le t\}\Big\} \\
&= \sup\Big\{\sum_{n=1}^N \sum_{q_m \le t} a_n \lambda_m \delta_{n,m} : \ \sum_{q_m \le t} |\lambda_m| \le 1, \ \lambda_m = 0 \text{ for all but finitely many } m\Big\} \\
&= \sup\Big\{\sum_{\substack{n=1 \\ q_n \le t}}^N a_n \lambda_n : \ \sum_{q_m \le t} |\lambda_m| \le 1, \ \lambda_m = 0 \ \text{ for all but finitely many } m\Big\}.
\end{aligned}$$

It is now clear that

$$\delta^*(x^*, F(t)) = \max\big(\{|a_n| : \ 1 \le n \le N, \ q_n \le t\} \cup \{0\}\big) = h_{|a_1|,\dots,|a_N|}(t)$$

for every $t \in [0,1]$. Since $x^* \in B_{X^*} \cap \mathrm{span}\{x_m : m \in \mathbb{N}\}$ is arbitrary, we conclude that $\mathcal{G} \subset \mathcal{H}_r$, as we claimed.

From the above and Lemma 3.9 we deduce that $\mathcal{G}$ has Birkhoff property, hence its pointwise closure $\overline{\mathcal{G}}^{\tau_p([0,1])}$ has Birkhoff property too. In order to finish the proof we will see that $W_F \subset \overline{\mathcal{G}}^{\tau_p([0,1])}$. To this end fix $x^* \in B_{X^*}$. Since $\mathrm{span}\{x_n^* : n \in \mathbb{N}\}$ is norm dense in X^*, there is a sequence (y_n^*) in $B_{X^*} \cap \mathrm{span}\{x_n^* : n \in \mathbb{N}\}$ converging to x^* for the dual norm. For each $t \in [0,1]$ we have $\lim_n y_n^*(x) = x^*(x)$ uniformly for $x \in F(t)$, hence $\lim_n \delta^*(y_n^*, F(t)) = \delta^*(x^*, F(t))$. Thus $\delta^*(x^*, F)$ belongs to $\overline{\mathcal{G}}^{\tau_p([0,1])}$. Therefore, $W_F \subset \overline{\mathcal{G}}^{\tau_p([0,1])}$ has Birkhoff property and the proof ends. □

Given a closed subspace $Z \subset X$ and a multi-valued function $F : \Omega \longrightarrow cwk(Z)$, it is easy to see that F is Birkhoff (resp. Debreu) integrable if, and only if, F is Birkhoff (resp. Debreu) integrable when viewed as a $cwk(X)$-valued function. In this case the respective integrals coincide. Bearing this in mind, Example 3.10 yields the following result.

Corollary 3.11. *Assume that X^* has the Radon-Nikodým property (i.e. X is Asplund). The following conditions are equivalent:*

(i) *every bounded Birkhoff integrable multi-valued function $F : [0,1] \longrightarrow cwk(X)$ is Debreu integrable;*
(ii) *X is finite dimensional.*

We have already mentioned that Birkhoff and Pettis integrability coincide for single-valued functions with values in a separable Banach space. In general, for multi-valued functions Pettis integrability is strictly weaker than Birkhoff integrability, as we show next.

Example 3.12. *If X is infinite dimensional and X^* is separable, then there exists a bounded Pettis integrable multi-valued function $F : [0,1] \longrightarrow cwk(X)$ which is not Birkhoff integrable.*

Proof. Let us consider the complete probability space $(\{0,1\}^{\mathbb{N}}, \Sigma, \mu)$ obtained after completing the usual product probability measure on $\mathrm{Borel}(\{0,1\}^{\mathbb{N}})$, i.e. the denumerable product of the measure ν on $\{0,1\}$ given by $\nu(\{0\}) = \nu(\{1\}) = \frac{1}{2}$. It is well known that $(\{0,1\}^{\mathbb{N}}, \Sigma, \mu)$ and $([0,1], \mathcal{L}, \lambda)$ are isomorphic as measure spaces, see [16, 254K], and therefore, in order to have the claimed example, it is sufficient to find a bounded Pettis integrable multi-valued function $F : \{0,1\}^{\mathbb{N}} \longrightarrow cwk(X)$ which is not Birkhoff integrable (with respect to μ).

We already know that X admits an infinite countable shrinking Markushevich basis $\{(x_n, x_n^*)\}_{n\in\mathbb{N}}$ such that $\{x_n : n \in \mathbb{N}\}$ is bounded and $x_n^* \in B_{X^*}$ for every $n \in \mathbb{N}$, and that we can define a bounded multi-valued function $F : \{0,1\}^{\mathbb{N}} \longrightarrow cwk(X)$ by

$$F(z) := \begin{cases} \overline{\mathrm{aco}}\{x_n : \ z_n = 1\} & \text{if } z = (z_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}} \setminus \{\mathbf{0}\} \\ \{0\} & \text{if } z = \mathbf{0} := (0,0,\dots) \end{cases}$$

(see the proof of Example 3.10).

On the one hand, *F is not Birkhoff integrable*. In order to prove this it suffices to check that the family $\{\delta^*(x_n^*, F) : n \in \mathbb{N}\} \subset W_F$ does not have Bourgain property and then use Proposition 2.9. Let us notice that for each $n \in \mathbb{N}$ the function $f_n := \delta^*(x_n^*, F)$ satisfies

$$f_n(z) = \delta^*(x_n^*, F(z)) = \sup\Big\{x_n^*(x) : \ x \in \mathrm{aco}\{x_m : \ z_m = 1\}\Big\} = z_n$$

for every $z \in \{0,1\}^{\mathbb{N}} \setminus \{\mathbf{0}\}$, with $f_n(\mathbf{0}) = 0$.

We will prove that $\{f_n : n \in \mathbb{N}\}$ does not have Bourgain property by contradiction. Suppose that $\{f_n : n \in \mathbb{N}\}$ has Bourgain property. Then there are $A_1, \dots, A_m \in \Sigma$ of positive μ-measure such that

$$\mathbb{N} = \bigcup_{i=1}^{m} \{n \in \mathbb{N} : \ |\cdot|\text{-diam}(f_n(A_i)) < 1\}.$$

Hence there is $1 \le i \le m$ such that

$$\begin{aligned} P &:= \{n \in \mathbb{N} : \ |\cdot|\text{-diam}(f_n(A_i)) < 1\} \\ &= \{n \in \mathbb{N} : \ f_n^{-1}(\{0\}) \cap A_i = \emptyset \ \text{ or } \ f_n^{-1}(\{1\}) \cap A_i = \emptyset\} \end{aligned}$$

is infinite. Since $z_n = z'_n$ for every $z, z' \in A_i$ and every $n \in P$, we have $A_i \subset \prod_{n=1}^{\infty} T_n$, where T_n is a singleton for every $n \in P$ and $T_n = \{0,1\}$ whenever $n \in \mathbb{N} \setminus P$. Since P is infinite and $\nu(T_n) = \frac{1}{2}$ for every $n \in P$, it follows that $\mu(A_i) \leq \mu(\prod_{n=1}^{\infty} T_n) = 0$, a contradiction which proves that F is not Birkhoff integrable.

On the other hand, in order to establish that F *is Pettis integrable* we only need to show that W_F is a uniformly integrable subset of $\mathcal{L}^1(\mu)$, as we pointed out before Theorem 3.4. Since F is bounded, W_F is uniformly bounded and the proof will be finished when proving that $\delta^*(x^*, F)$ is measurable for every $x^* \in B_{X^*}$. We begin with a particular case.

Claim.- $\delta^*(y^*, F)$ is measurable for every $y^* \in \operatorname{span}\{x_n^* : n \in \mathbb{N}\}$.

Indeed, fix $y^* \in \operatorname{span}\{x_n^* : n \in \mathbb{N}\}$ and write $y^* = \sum_{n=1}^{N} \alpha_n x_n^*$, $\alpha_i \in \mathbb{R}$. Notice that for every $z \in \{0,1\}^{\mathbb{N}} \setminus \{\mathbf{0}\}$ we have

$$\delta^*(y^*, F(z)) = \sup\Big\{y^*(x) : \ x \in \operatorname{aco}\{x_m : \ z_m = 1\}\Big\}$$

$$= \sup \Big\{\sum_{n=1}^{N} \sum_{z_m=1} \alpha_n \lambda_m \delta_{n,m} : \ \sum_{z_m=1} |\lambda_m| \leq 1, \ \lambda_m = 0 \text{ for all but finitely many } m\Big\}$$

$$= \sup \Big\{ \sum_{\substack{n=1 \\ z_n=1}}^{N} \alpha_n \lambda_n : \ \sum_{z_m=1} |\lambda_m| \leq 1, \ \lambda_m = 0 \text{ for all but finitely many } m\Big\}.$$

It is now easy to see that

$$\delta^*(y^*, F(z)) = \begin{cases} \max\{|\alpha_n| : \ 1 \leq n \leq N, \ z_n = 1\} & \text{if } z \in A \\ 0 & \text{if } z \in \Omega \setminus A, \end{cases}$$

where $A := \bigcup_{n=1}^{N} \{z \in \{0,1\}^{\mathbb{N}} : z_n = 1\}$. Since the coordinate projections $z \mapsto z_n$ are continuous, $\delta^*(y^*, F)$ is measurable, as we claimed.

Finally, fix $x^* \in B_{X^*}$. Since $\overline{\operatorname{span}}^{\|\cdot\|}\{x_n^*\}_{n \in \mathbb{N}} = X^*$, there is a sequence (y_n^*) in $\operatorname{span}\{x_n^* : n \in \mathbb{N}\}$ converging to x^* for the dual norm. Therefore, for each $z \in \{0,1\}^{\mathbb{N}}$ we have $\lim_n y_n^*(x) = x^*(x)$ uniformly for $x \in F(z)$, and thus $\lim_n \delta^*(y_n^*, F(z)) = \delta^*(x^*, F(z))$. By Claim above each $\delta^*(y_n^*, F)$ is measurable, hence $\delta^*(x^*, F)$ is measurable and the proof is over. $\square$

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