Relativity and the Equivalence Principle

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Outline of this lecture

- We introduce and discuss the *Equivalence Principle* which is a foundational principle of the General Relativity theory.

- The EP is stated in various versions and is related to the notion of *free fall* in a gravitational field, the geodesics of curved spacetime and how the laws of physics are perceived by different observers.

- Towards formulating the EP we discuss the notion of inertial frames, Galilean relativity, Lorentz invariance and fictitious inertial forces.

- We will see how the notion of an inertial frame becomes local and discuss the *general Relativity Principle* and the *Principle of general covariance*.

- Finally, we discuss two important physical consequences of the EP: *bending of light* and *gravitational frequency shift*.

Reference frames & Observers

- The statement “a body is moving” is meaningful only when a *coordinate system* has been specified. Then, a change in the body’s position means that its coordinates change.

- The notion of a *reference frame* is synonymous with a coordinate system. We can also simply identify a reference frame with an *observer*.

- The key question is this:

Are the laws of physics the same for different observers?

✓ For reference frames related by simple spatial translations and rotations the answer is trivially “yes”.

✓ However, for observers in *relative motion* the answer is far from clear.

Inertial frames

- *Inertial frames* (or inertial observers) are those reference frames that are in state of constant rectilinear motion with respect to one another. These frames are not accelerating.

- In an inertial frame, Newton's first law (the law of inertia) is satisfied: *a freely moving body has a constant velocity*.

- We could also say that in an inertial frame the laws of mechanics take their simplest form.

- The *Relativity Principle*:

  the laws of physics take the same form in all inertial frames.

- Based on the above, one can eventually arrive at the Lorentz transformation and the theory of Special Relativity!

Galilean invariance

- We know that the laws of Newtonian mechanics remain *invariant* between inertial frames (assuming velocity-independent forces). Newton's laws do not change under the *Galilean spacetime transformation*:

$$t' = t + \text{constant}, \qquad \mathbf{x}' = \mathbf{x} - \mathbf{v}t$$

(the unprimed/primed coordinates refer to the inertial frames F/F´)

- This is the statement of the *Galilean principle of relativity*. This principle entails a *relativity of velocity*.

- In non-inertial (=accelerating) frames Newtonian dynamics is modified by the introduction of "ficticious" inertial forces (e.g. the centrifugal force).

Lorentz invariance

- On the other hand, the laws of *electromagnetism* are *not* Galilean-invariant (the speed of light *c* is the same for all inertial observers).

- They are the same in all inertial frames provided that the transformation between them is the *Lorentz spacetime transformation*:

$$\mathbf{x}' = \gamma(\mathbf{x} - \mathbf{v}t) - (\gamma - 1)[\mathbf{x} - (\mathbf{x} \cdot \mathbf{v})\mathbf{v}/v^2], \qquad t' = \gamma(t - \mathbf{x} \cdot \mathbf{v}/c^2)$$

where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$

- Lorentz invariance again implies a relativity of velocity, but leaves acceleration as an absolute quantity. Newton's laws are still valid but we must now use 4-vectors, a body's proper time τ and the notion of Minkowski spacetime:

$$m\alpha^\mu = F^\mu, \qquad \alpha^\mu = d^2x^\mu/d\tau^2$$

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x} = \text{invariant}$$

The special nature of gravity

- In defining an inertial frame one has to look for a *force-free* situation (so that Newton's first law is valid).

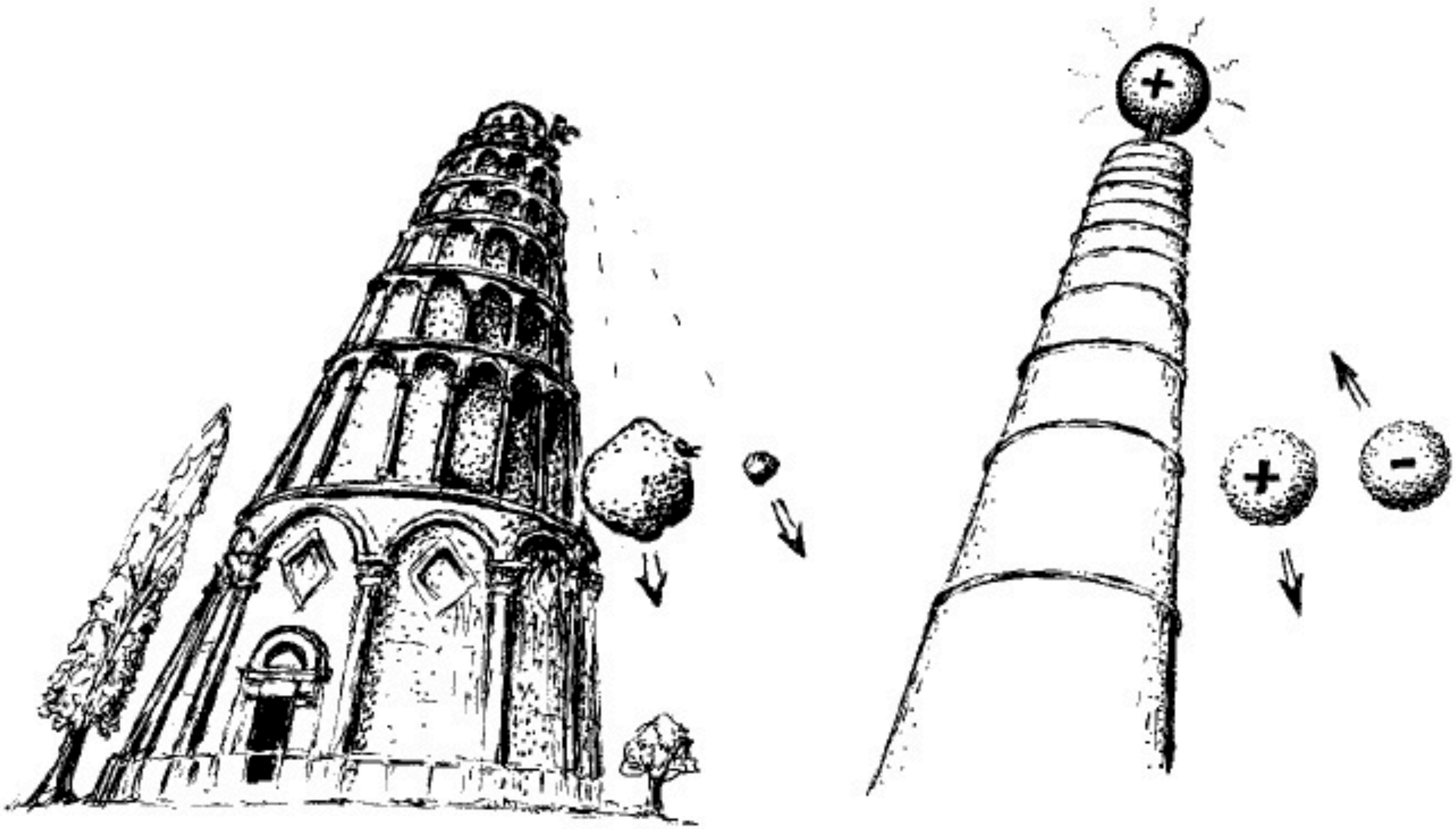
- This is easy to achieve in the case of *electromagnetic* forces: one simply has to choose bodies with no charge (or any other higher electromagnetic moment). This can be seen from the Coulomb law formula:

$$F = \frac{Qq}{r^2} \longrightarrow a = \frac{q}{m}\frac{Q}{r^2}$$

- In the case of the *gravitational* field this is not possible since *all* bodies are affected by gravity. Moreover, the corresponding gravitational acceleration is independent of mass. Therefore it cannot be eliminated by letting the mass go to zero.

$$F = \frac{GMm}{r^2} \longrightarrow a = \frac{GM}{r^2}$$

Galileo's free fall experiment for gravity and electromagnetism



Newton's equivalence principle

- At this point we need to clarify the difference between different kinds of mass. The equation of motion for a body with *inertial* mass m_i in a gravitational field described by a potential $\Phi(\mathbf{x},t)$ is:

$$m_i \ddot{\mathbf{x}} = -m_g \nabla \Phi$$

where $\mathbf{x}(t)$ is the body's position at time t and m_g is the body's *gravitational* mass (which can be thought as a gravitational "charge").

- In principle, the two masses m_i, m_g express completely different properties and we would expect them to be distinct.

- Newton's *equivalence principle*: $m_i = m_g$

- *The equality of the two kinds of mass has been verified experimentally to a very high precision!*

Universality of free fall

$$m_i = m_g \Leftrightarrow \text{ free fall motion is universal}$$

- The universality of free fall is usually called *Galileo's principle*:

  In a given gravitational field the path in space and time of a test body is fully determined by its initial position and velocity. In particular, the path is independent of the body's mass and composition (a "test body" is a body with a negligible effect on the background gravitational field)

- In other words, the motion is fully described by:

$$\ddot{\mathbf{x}} = \mathbf{g} = \nabla\Phi \qquad \mathbf{x}_0 = \mathbf{x}(t=0),\ \mathbf{v}_0 = \dot{\mathbf{x}}(t=0)$$

- In Newtonian theory the universality of free fall (or the equality of inertial and gravitational mass) is taken as an axiom, without additional insight.

Gravity as Geometry: geodesics

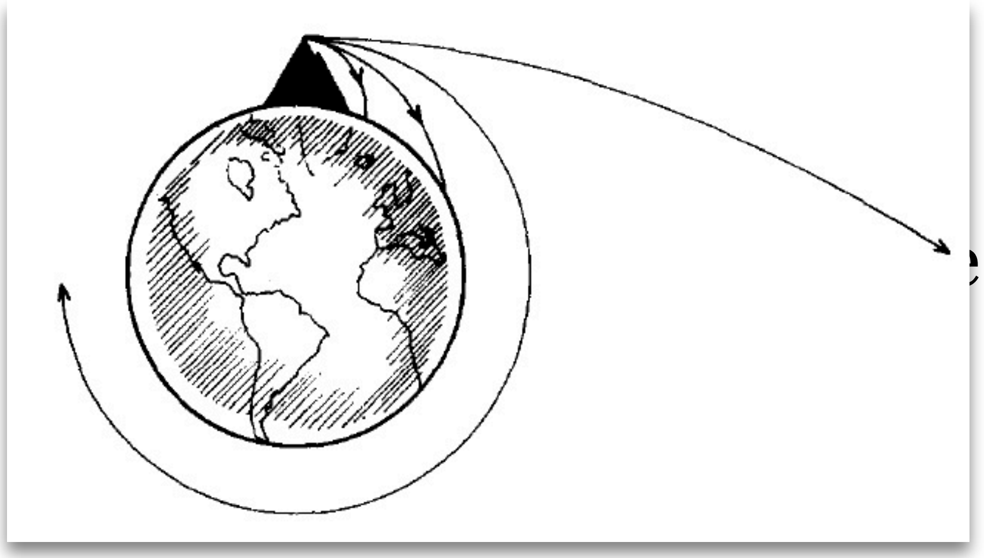
- Contemplating the mystery of Galileo's principle, one could suggest:

universality of free fall ⟷ A property of curved spacetime, rather than a property of mass

- At the same time we know from *differential geometry* that a surface, or a more general *n*-dimensional space, has one natural set of "rails": the *geodesics*.

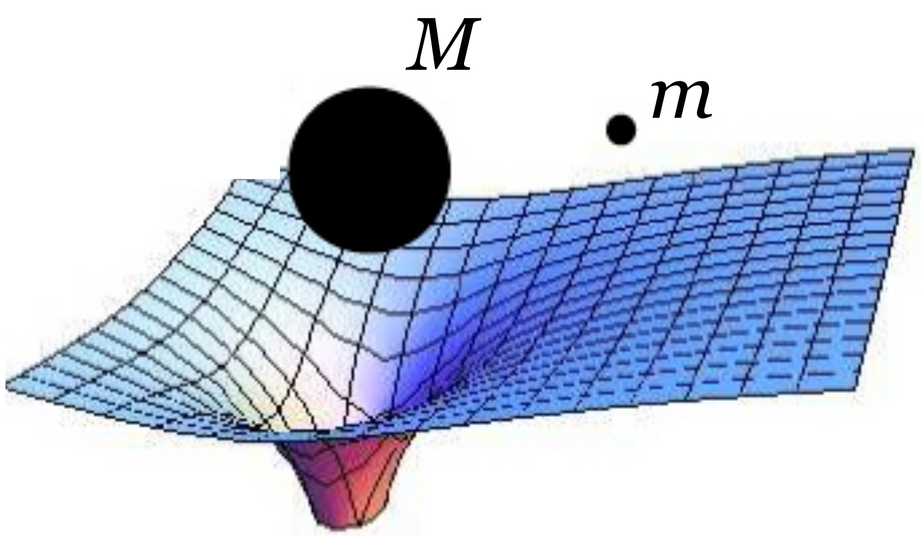
- These are the *straightest* paths; moreover there is a *unique geodesic at a point and at a given direction*.

✓ The uniqueness of geodesics is a property of the *four*-dimensional spacetime, not of *three*-dimensional space (see figure).



The *Weak* Equivalence Principle

- In the theory of General Relativity one abandons the notion of a gravitational force. Instead the notion of a *curved spacetime* is used.



The interaction between a big mass *M* and a small body of mass *m* is described as the motion of *m* in the spacetime curved by *M*.

- The motion of *m* is independent of its mass and composition because it follows the geodesic “rails” of the curved spacetime.

- Galileo’s principle of the universality of free fall is now called the *weak equivalence principle* and makes *no* reference to m_i, m_g.

WEP: The motion of (non-rotating) test bodies with negligible self-gravity is independent of their properties

Inertial forces

- We have already mentioned that Newton's laws are modified in a accelerating (non-inertial) reference frame.

- For example, in a frame rotating with angular velocity $\mathbf{\Omega}$ Newton's 2nd law is modified by the inclusion of the centrifugal and Coriolis inertial forces:

In F (inertial frame): $m\mathbf{a} = \mathbf{F}$

In F´(non-inertial frame): $m\mathbf{a}' = \mathbf{F} + \mathbf{F}_{\text{inertial}}$

$$\mathbf{F}_{\text{inertial}} = m\left[-2\mathbf{\Omega} \times \mathbf{v}' - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}')\right]$$

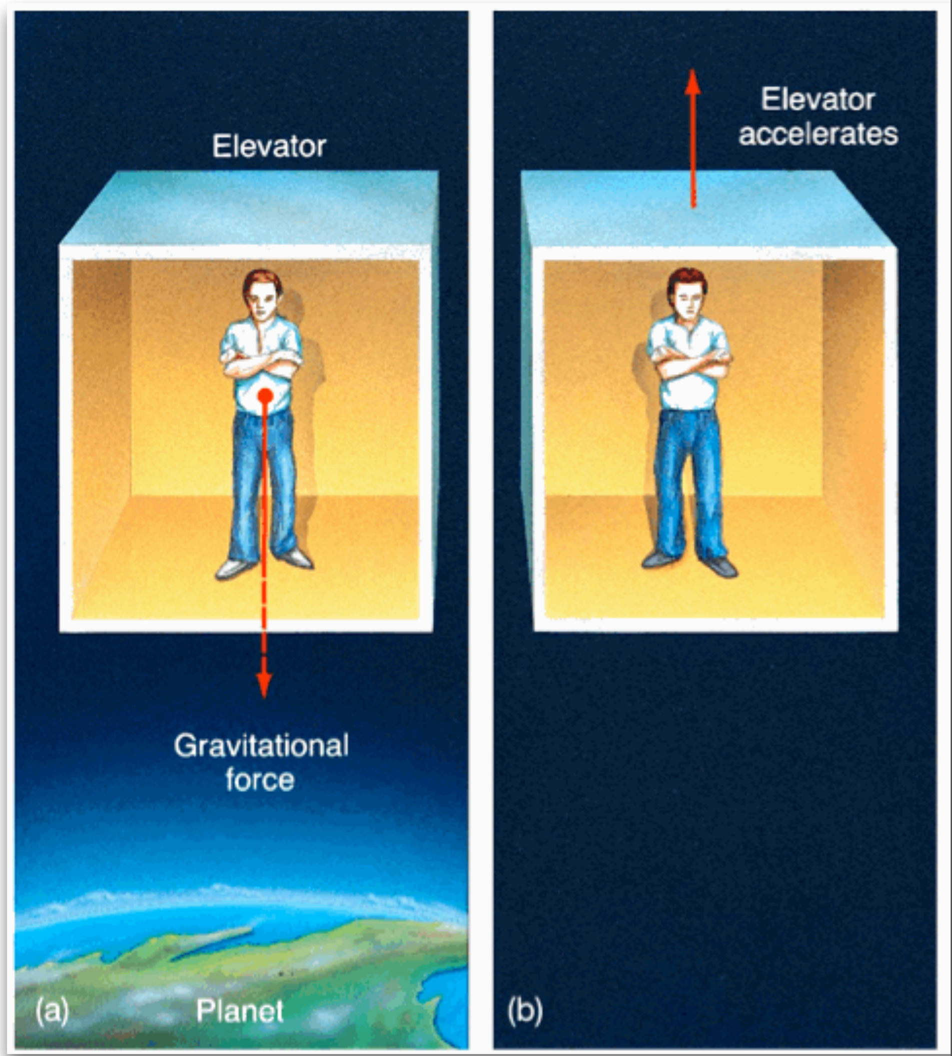
- These forces are "fictitious" in the sense that they appear due to the fact that we are in a non-inertial frame. The key property of all inertial forces is that they are always proportional to the body's mass. This is also a property of the gravitational force.

- *Is there a connection between these two kinds of force*?

Apparent gravitational fields

- Consider a static *homogeneous* gravitational field (as would be Earth's field when observed over *sufficiently small regions*). In this field a body of mass m would feel a gravitational force:

$$\mathbf{F}_{\text{grav}} = -\frac{GMm}{R^2}\hat{\mathbf{z}} = -mg\hat{\mathbf{z}}$$



- Next, consider a region without gravity and in it an elevator (our laboratory!) moving with acceleration $\mathbf{a} = a\hat{\mathbf{z}}$. In the reference frame of the elevator a body would feel an inertial force:

$$\mathbf{F}_{\text{inertial}} = -ma\hat{\mathbf{z}}$$

- The accelerating observer would see an apparent gravitational field with $\mathbf{g} = -\mathbf{a} = -a\hat{\mathbf{z}}$.

- If $a = g$ the accelerating frame is *equivalent* to the static homogeneous gravitational field.

The Equivalence Principle

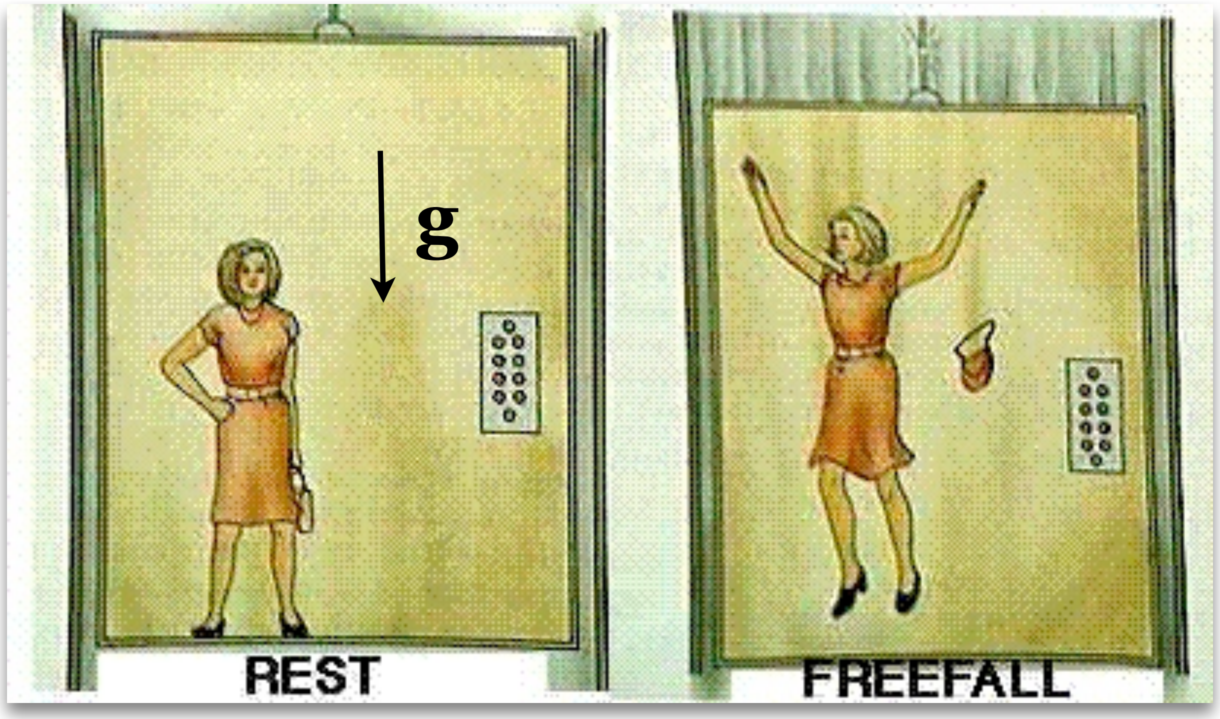
- Having established the equivalence between inertial forces and homogeneous gravitational fields, we consider a *freely falling* elevator in a static homogeneous gravitational field. This is *Einstein's famous elevator thought experiment*.

- The elevator's frame has acceleration $\mathbf{a} = \mathbf{g}$ and Newton's 2nd law becomes:

$$m\ddot{\mathbf{x}}' = \mathbf{F}_{\text{grav}} + \mathbf{F}_{\text{inertial}} = m(-\mathbf{g} + \mathbf{a}) = 0$$

- Gravity has been effectively "cancelled" and the *freely falling frame has become an inertial frame*!

- This equivalence between inertial and gravitational forces is encapsulated in *Einstein's Equivalence Principle*.



EP: A state of rest in homogeneous gravitational field is physcially equivalent to a state of uniform acceleration in gravity-free space.

Einstein’s elevator in action!

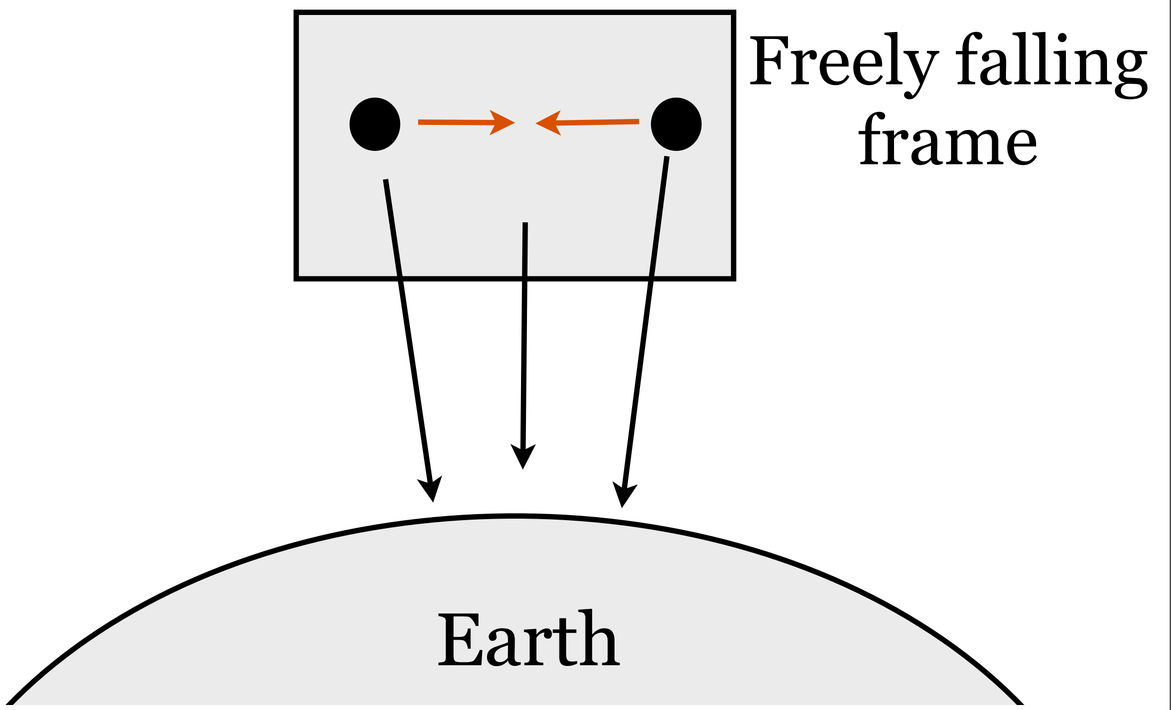
NASA’s “zero-gravity” astronaut training in a “freely falling” airplane

Local inertial frames

- The equivalence of a freely falling frame with an inertial frame is a *local* notion.

- This is because gravitational fields are *never* completely homogeneous. Imagining a large freely falling Einstein elevator, two free bodies in it (as in the figure) will be observed to approach each other. The inertial forces cannot completely eliminate an inhomogeneous gravitational field.

- In this sense, we can now distinguish between *apparent* gravitational fields (these are the homogeneous ones) and *true* gravitational fields (these cannot be completely cancelled by inertial forces).



Curvature & local flatness

- In General Relativity the gravitational field is represented by a curved spacetime with *metric* $g_{\alpha\beta}$. The criterion for the presence of gravity at a given point is a non-zero curvature at that point. The spacetime curvature is encoded in the *Riemann tensor* $R_{\alpha\beta\gamma\delta}$ (this is the generalization of the more familiar Gaussian curvature *K* of two-dimensional surfaces).

$$R_{\alpha\beta\gamma\delta} \neq 0 \Longleftrightarrow \text{ gravitational field}$$

- Homogeneous gravitational fields have $R_{\alpha\beta\gamma\delta} = 0$, this also implies that such a field is not "true".

- As known from differential geometry the local neighborhood of point on a curved surface can be approximated by a plane. In a similar way a curved spacetime can be *locally* approximated by a flat spacetime (i.e. the Minkowski metric of Special Relativity), that is, locally we can set $g_{\alpha\beta} = \eta_{\alpha\beta}$, $g_{\alpha\beta,\gamma} = 0$.

- This *local inertial frame* is the frame of a freely falling observer at the given spacetime point (i.e. an Einstein elevator).

The *Strong* Equivalence Principle

- The existence of local inertial frames in a gravitational field forms the basis of Einstein's *Strong Equivalence Principle*:

> **SEP**: Locally and at any point of spacetime physics is that of special relativity (Lorentz invariant) and is not affected by the presence of a gravitational field.

- We can quantify the notion of "locally": the spatial size *D* of the local inertial frame is much smaller that the typical lengthscale *L* of the gravitational field, $R_{\alpha\beta\gamma\delta} \sim 1/L^2$. Hence, in a region *D* << *L* the field is almost homogeneous and can be "cancelled" by inertial forces.

- *The SEP implies the WEP*: the behavior of freely falling test-bodies in a gravitational field is locally (i.e. in a local inertial frame) indistinguishable from that of a free body. Since the latter is universal, also freely falling bodies should behave in a universal way.

The general Relativity Principle

- The SEP has been confirmed experimentally to a very high precision.

- The identification of gravity with curved spacetime has led us to a *new definition of an inertial frame*: it is the local freely falling frame. This definition replaces the *global* inertial frames of non-gravitational physics (but includes them as a limiting case for a flat spacetime $R_{\alpha\beta\gamma\delta} = 0$).

- An observer in this local frame does not see a gravitational field and therefore can find himself in a force-free situation (up to a certain precision).

- These newly defined inertial frames are, in general, *accelerated* with respect to each other (whereas in Special Relativity inertial frames move with constant velocity with respect to each other).

- According to the SEP, in all these local inertial frames the laws of physics are the same. This is the *general Relativity Principle*.

Physics is the same for *all* observers

- General relativity goes beyond the general Relativity Principle and requires that the laws of physics are the same relative to *all* frames (=all observers), not just relative to the inertial frames.

- This is accomplished by *formulating all physical laws in a tensorial form*.

- The equivalence of all observers with respect to physical laws is called the *principle of general covariance*.

- An example of the covariant (=tensorial) way of writing equations is the expression for geodesic motion (free fall):

$$u^\nu \nabla_\nu u^\mu = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0, \qquad u^\mu = \frac{dx^\mu}{d\tau}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} \left[g_{\beta\lambda,\alpha} + g_{\alpha\lambda,\beta} - g_{\alpha\beta,\lambda} \right]$$

Applications of the Equivalence Principle

- There are two very important physical effects that follow directly from the EP without making any reference to the equations of General Relativity.

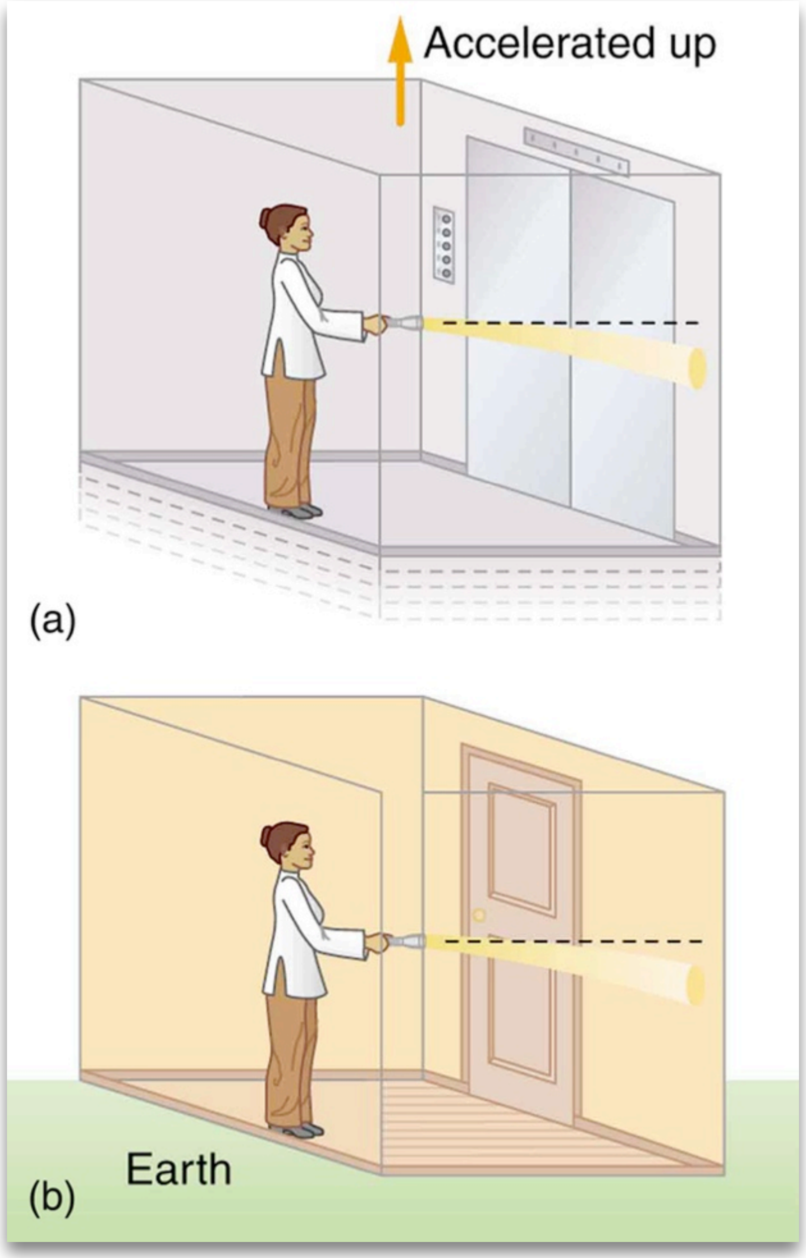
$$\boxed{\text{Equivalence Principle}} \quad \Longrightarrow \quad \boxed{\begin{array}{c}\text{Bending of light} \\ \& \\ \text{Gravitational frequency shift}\end{array}}$$

- Both effects can be revealed by using Einstein's elevator thought experiment and are discussed in the following slides.

Bending of light by gravity

- Consider an Einstein elevator initially at rest in a gravity-free region. A light ray is seen to propagate along a straight line (the elevator is an inertial frame).

- Subsequently the elevator is accelerated upwards as in figure (a). Due to the relative motion between the light beam and the elevator, the trajectory would bend downwards. Relative to the initial inertial frame, light still propagates along a straight line.

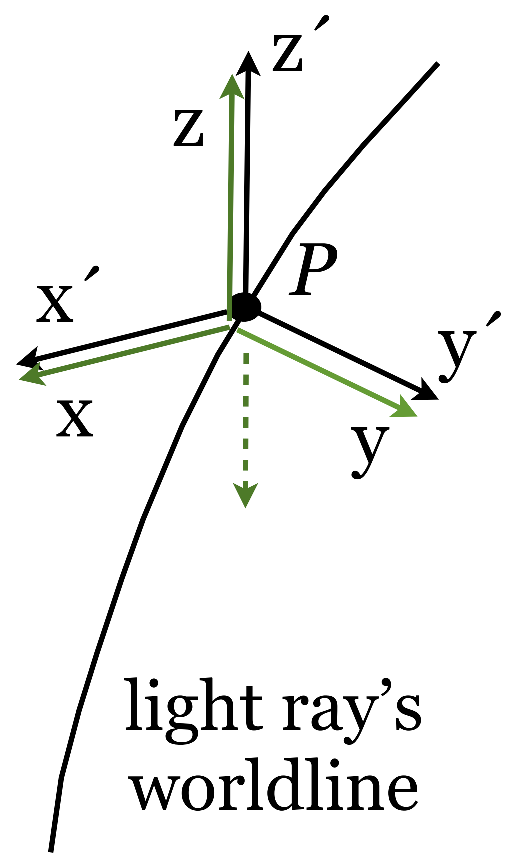


- According to the EP, the accelerated frame is indistinguishable from a homogeneous gravity field.

- Therefore, relative to an observer at rest in a homogeneous gravitational field, the light beam is deflected downwards. Gravity attracts light!

Bending of light: rigorous analysis (I)

- Consider a light ray propagating in a curved spacetime. At a point *P* along its path we set up a local frame F´ with coordinates (t', x', y', z'). At the same point P we also have a freely falling frame F with coordinates (t, x, y, z), momentarily at rest with respect to F´.



- The two sets of coordinates are related as:

$$\left.\begin{aligned} x' &= x \\ y' &= y \\ z' &= z - \frac{1}{2}gt^2 \end{aligned}\right\} + \text{ terms of third and higher order in } x, y, z, t.$$

(g = gravitational acceleration at P as seen in F´)

- F is a local inertial frame: relative to it the light path is a straight line at angle θ with respect to the horizontal x-axis:

$$x = ct\cos\theta \qquad z = ct\sin\theta$$

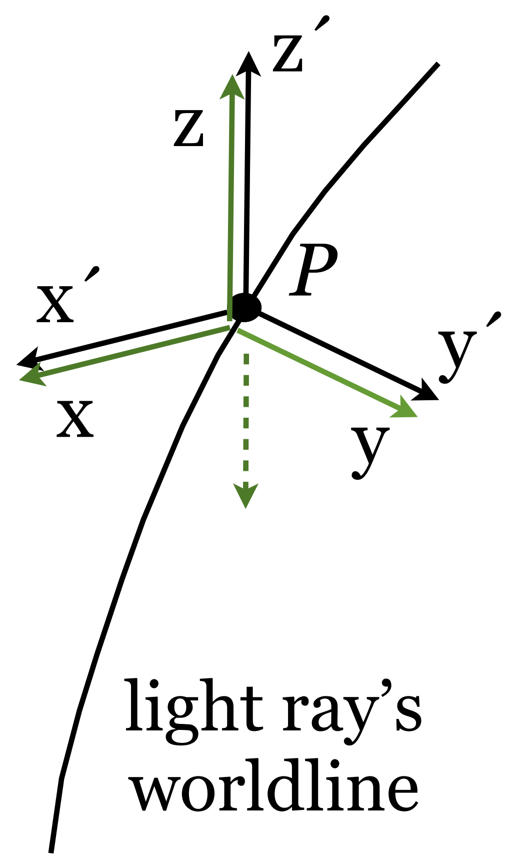
Bending of light: rigorous analysis (II)

- The trajectory of light as seen in F´ is not a straight line:

$$z'(x') = x' \tan\theta - \frac{g}{2c^2}\frac{x'^2}{\cos^2\theta} + \mathcal{O}(x'^3)$$

- The *curvature* of this trajectory at P, (t', x', y', z') = (0,0,0,0), can be calculated with the help of the familiar formula:

$$\kappa = \frac{d^2z'/dx'^2}{[1+(dz'/dx')^2]^{3/2}} = -\frac{g}{c^2}\cos\theta$$



- This result is *exact*.

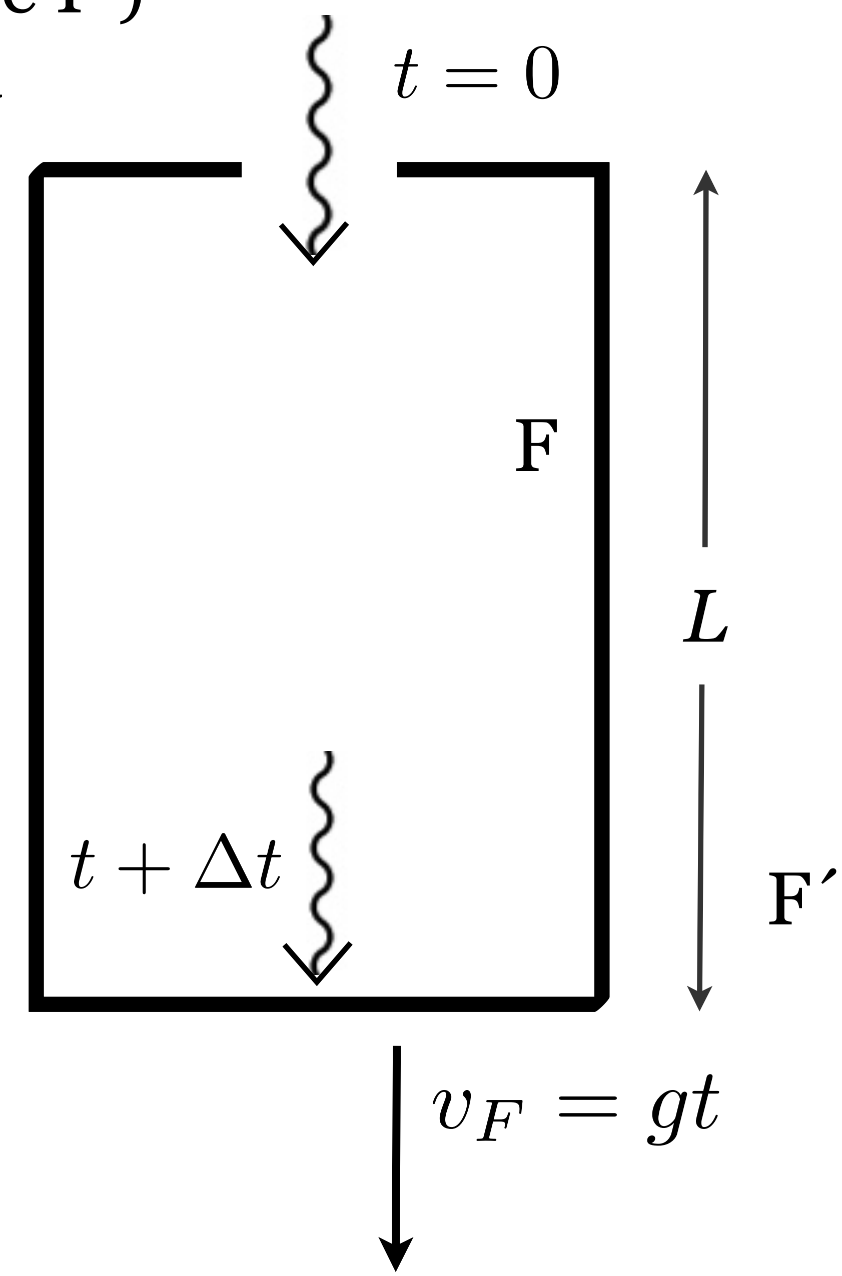
Gravitational frequency shift

- We again consider an Einstein elevator (inertial frame F) and at the moment it is released to a free fall we let a light beam enter through the ceiling.

- The light will reach the floor after a time

$$\Delta t = \frac{L}{c} + \mathcal{O}(v/c)$$

at which time F is moving with velocity (g is the local gravitational acceleration) $v_F = g\Delta t$.



- In F the frequency of the light ν remains unaltered. However, relative to F the observer F´(who is at rest in the gravitational field) moves with velocity v_F towards the light beam. From the usual Doppler formula:

$$\frac{\Delta\nu}{\nu} = \frac{v_F}{c} = \frac{gL}{c^2}$$

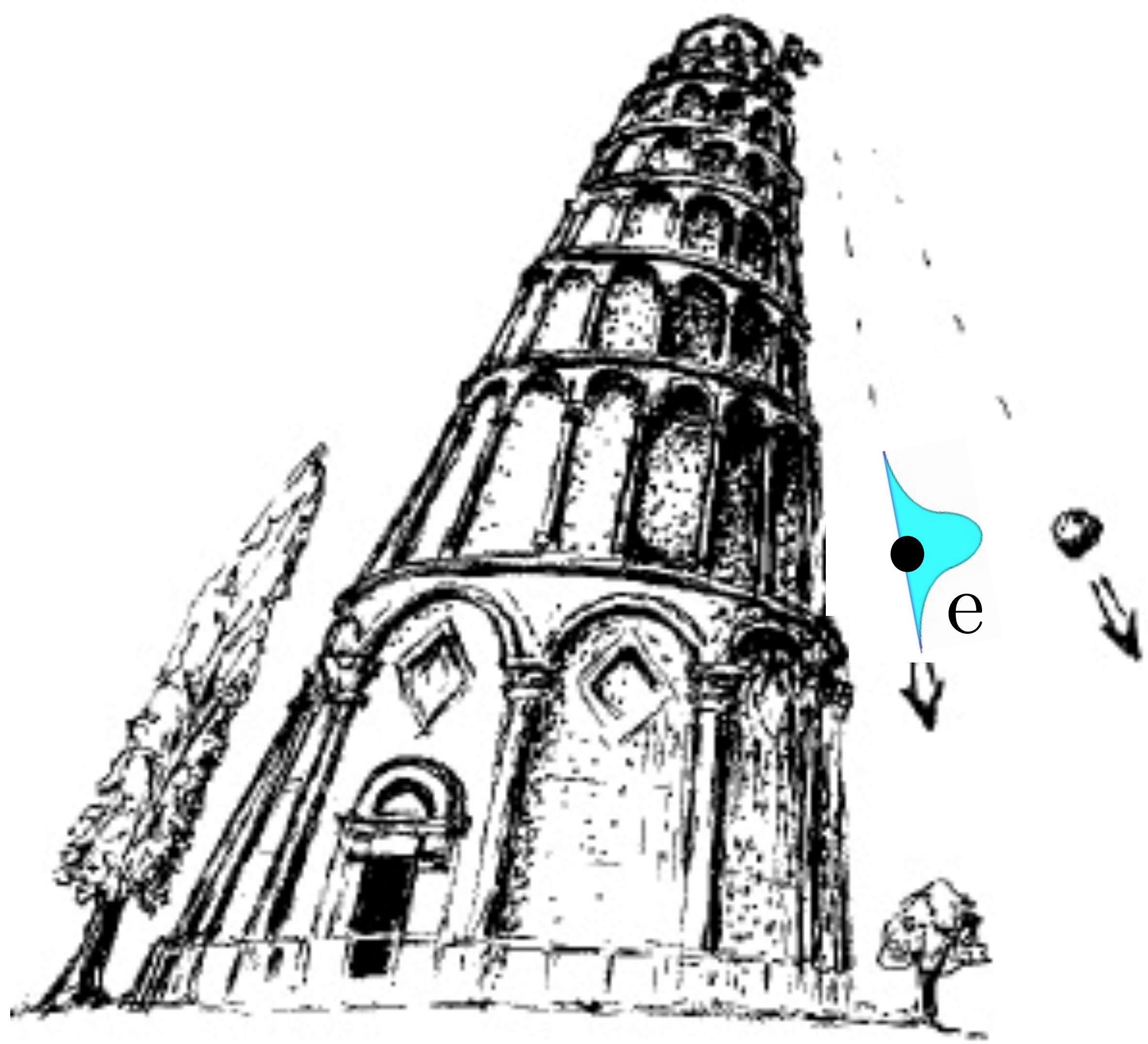
blueshift (redshift) for light propagating downwards (upwards)

Epilogue: EP & Quantum physics

- This can be considered as the “homework” part of the lecture.
- So far we have discussed the Equivalence Principle as an essential part of classical (i.e. non-quantum) physics.
- After having been exposed to the various versions and meanings of the EP, you are asked *to speculate on its compatibility with quantum mechanics*.

Quantum Galileo free fall experiment

- An quantum particle (e.g. an electron) is dropped together with a "classical" particle. Do they fall together? Is the path of the quantum particle independent of its mass? (hint: use the uncertainty principle)



The non-inertial Schrödinger equation

- The Schrödinger equation for a free particle of mass m in an inertial frame is:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$

- Next, transform this to a uniformly accelerated frame (assume an acceleration $\mathbf{a} = g\hat{\mathbf{z}}$ along the vertical z-axis).

- Compare the result against the Schrödinger equation in a homogeneous gravitational field $\mathbf{g} = -g\hat{\mathbf{z}}$:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + mgz\psi$$

- Is the Equivalence Principle satisfied?