FÍSICA DEL COSMOS 2014/15

PROBLEM SHEET FC5 – SOLUTIONS

10. Write down the dynamical equation for the flat FLRW models in terms of $(\Omega_r, \Omega_m, \Omega_\Lambda, \Omega_k)$ in the case when there is no radiation $\Omega_r = 0$ but both Ω_Λ and Ω_m are present. How large would Ω_Λ have to be for the universe to be accelerating at the present time and how small would then Ω_m have to be?

    SOLUTION
    Flat model: $k = 0 = \Omega_k$ and no radiation $\Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k = 1 = \Omega_m + \Omega_\Lambda$.
    With $\Omega_m = \frac{8\pi G}{3H_0^2}\rho$ and $\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}$, the Friedmann equation: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$ becomes:
    $\left(\frac{\dot{a}}{H_0}\right)^2 = \frac{\Omega_m}{a} + \Omega_\Lambda a^2$ and, after derivation with respect to time: $\frac{\ddot{a}}{H_0^2} = \frac{-\Omega_m}{2a^2} + \Omega_\Lambda a$.
    An accelerating universe, $\ddot{a} > 0$ implies $\Omega_\Lambda > \frac{\Omega_m}{2a^3}$.
    Therefore, considering that $\Omega_m = 1 - \Omega_\Lambda$ and that at present time $a \equiv 1$, the sought conditions are:
    $$\Omega_\Lambda > \frac{1}{3} \quad \text{and} \quad \Omega_m < \frac{2}{3} \quad .$$

11. Assume that you are monitoring the shape of a cube of cosmological dimensions (consisting of test particles) under the influence of cosmic expansion. You observe that with time the cube does not retain its initial shape, becoming elongated along the z-axis, while each $z = \text{const}$ slice retains its square x − y shape. Write down a spatially homogeneous spacetime line element consistent with this observation.

    SOLUTION
    Recall the RW metric for flat 3-space: $ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right]$.
    As you know, this describes isotropic/homogeneous expansion. In this spacetime the cube remains a cube, albeit growing in size. It is then not too hard to come up with an anisotropic (but still homogeneous) metric with respect to the direction of the z-axis:
    $$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2\right] + \tilde{a}^2(t)\,dz^2 \quad .$$
    The cube will be distorted in the way described in the exercise if $\tilde{a}(t) > a(t)$.
    Note that the special case $\tilde{a} = Ca$ where C = constant takes us back to the initial equation after a trivial rescaling of the z coordinate.

12. You are asked to estimate the cosmic time at which matter decoupled from radiation (recombination era, formation of first hydrogen atoms). You can assume a matter-dominated universe 13.8 Gyr old and a photon gas obeying a black body thermal distribution.
    You are reminded of Wien's law for the radiation wavelength at the peak energy of the black

body spectrum: $\lambda_{\max}=\frac{0.3}{T(\mathrm{K})}\mathrm{cm}$, and you are also given that the photon energy at $\lambda_{\max}$ is roughly 10 times lower than the ionization energy (13.6 eV) of the hydrogen atom (this accounts for the high energy "tail" $\lambda<\lambda_{\max}$ of the Planckian black body spectrum.

SOLUTION

In a matter-dominated universe, the scale factor scales with time as follows: $a(t)\propto t^{2/3}$ and therefore the relationship between time and wavelength can be written as: $\frac{\lambda_0}{\lambda_{rec}}=\frac{a_0}{a_{rec}}=\left(\frac{t_0}{t_{rec}}\right)^{2/3}$.

Through Wien's law with $T_0=3\,\mathrm{K}$ we obtain $\lambda_0=\frac{0.3}{3}=0.1\,\mathrm{cm}$.

For the radiation wavelength at the recombination era, we use the given hint about photon energy E_{rec} , so that: $\lambda_{rec}=\frac{hc}{E_{rec}}=10^{-4}\mathrm{cm}=10^{-3}\lambda_0$.

Knowing that the current age of the universe is estimated to be $t_0=13.8\,\mathrm{Gyr}$, we find the recombination era:

$$t_{rec}=t_0\left(\frac{\lambda_{rec}}{\lambda_0}\right)^{3/2}\simeq 400\,000\,\mathrm{yr} \quad .$$