

Newton's laws, because we know that Newton's laws work quite well for apples and planets.) Theories that describe the mechanics of waves are called *wave mechanics* or *quantum mechanics*. One such theory was presented in 1925 by the German physicist *Erwin Schrödinger*. In his theory the information about the motion of a particle is contained in a function, called a *wave function*. Schrödinger's interpretation of the wave function was that it is related to the *probability* of finding a particle in a particular place with a particular momentum. This replaced the absolute determinism of classical physics, with the statement that we can only predict where a particle is *likely* to be, but not *exactly* where it will be. However, we can predict the average positions and momenta of a large group of particles, and it is these average properties that we see (and measure) in our everyday world. Many physicists (including Einstein) were not comfortable with this probabilistic interpretation, but quantum theory has been very successful in predicting the outcome of a wide variety of experiments. We will pick up on some of the threads of the quantum revolution later in this book.

### 3.4 | Formation of spectral lines

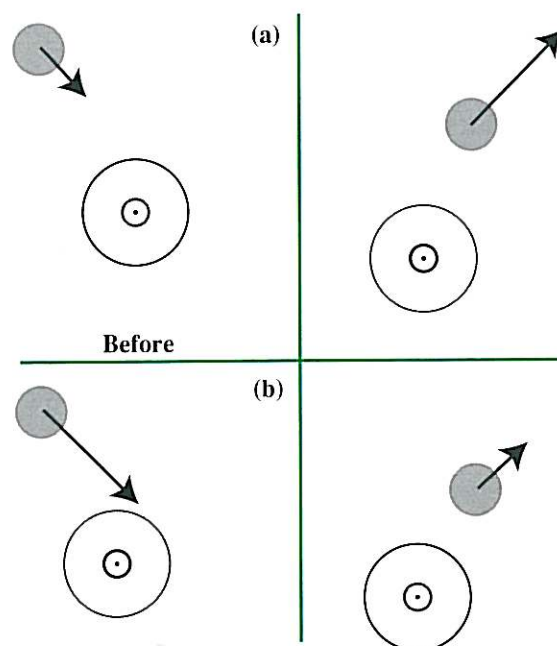
Now that we have some idea of how atoms can emit or absorb radiation, we can return to stellar spectra. The first point to realize is that in a star we are not talking about the radiation from a single hydrogen atom, but from a large number of them. We see a strong  $H\alpha$  absorption line in stars because many photons are removed from the continuum by this process. It is clear, however, that having a lot of hydrogen does not assure us of a strong  $H\alpha$  absorption. In order for such absorption to take place, a significant number of atoms must be in level 2, ready to absorb a photon. If all the hydrogen is in level 1, you will not see the Balmer series, no matter how much hydrogen is present.

#### 3.4.1 Excitation

In general, the strength of a particular transition (emission or absorption) will depend on the number of atoms in the initial state for that transi-

tion. The number of atoms per unit volume in a given state is called the *population* of that state. In this section we look at the factors that determine the populations of the various states. We refer to processes that can alter the populations as *excitation* processes. We have already seen one type of excitation process – the emission and absorption of photons. Electrons can jump to a higher level when a photon is absorbed or they can jump to a lower level when a photon is emitted.

Populations can also be changed by collisions with other atoms, as illustrated in Fig. 3.6. For example, atom 1 can be in state  $i$ . It then undergoes a collision with atom 2, and makes a transition to a higher state,  $j$ . In the process the kinetic energy of atom 2 is decreased by the difference between the energies of the two states in atom 1,  $E_j - E_i$ . The reverse process is also possible, with atom 2 gaining kinetic energy and atom 1 dropping from state  $j$  to state  $i$ .



**Fig 3.6.** Collisional excitation. In each case, the left frame shows the atoms before the collision and the right frame shows them after. In each frame, the occupied level is indicated by a heavier line. (a) To a lower state. After the collision, atom 1 is in a lower state and atom 2 is moving faster. (b) To a higher state. After the collision, atom 1 is in a higher state and atom 2 is moving slower.

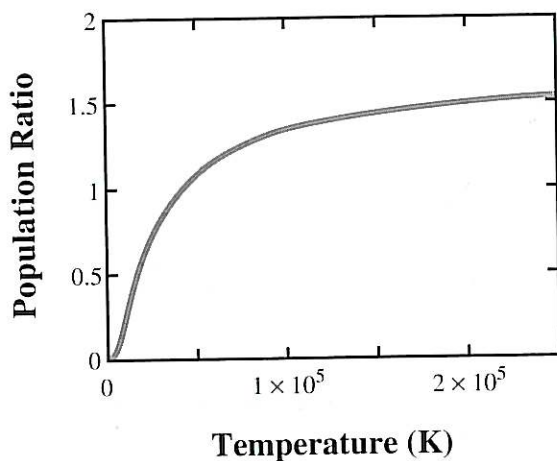
The collisional excitation rates will depend on the kinetic temperature of the gas. The higher the temperature the faster the atoms are moving. For atoms of kinetic temperature  $T_k$  the average kinetic energy per atom is  $(3/2)kT_k$ . As the temperature increases more energy is available for collisions. This makes higher energy states easier to reach. Also, since the particles are moving faster, they spend less time between collisions. There are more collisions per second.

When a gas is in thermodynamic equilibrium (which we discussed in the previous chapter), with a kinetic temperature  $T_k$ , the ratios of the level populations are given by a *Boltzmann distribution*. If we let  $n_i$  and  $n_j$  be the populations of levels  $i$  and  $j$ , respectively, their ratio is given by

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} e^{-[(E_j - E_i)/kT_k]} \quad (3.9)$$

In this equation  $g_i$  and  $g_j$  are called *statistical weights*. They are needed because certain energy levels are actually groupings of sublevels that have the same energy. The statistical weight of a level is just a count of the number of sublevels in that level. Typically,  $g$  are small integers.

To help us understand the Boltzmann distribution, Fig. 3.7 shows how the ratio of populations for an atom with just two levels depends on temperature. When the temperature is zero, all



**Fig 3.7.** Level populations as a function of temperature for a two-level system. In this case we have put in energies and statistical weights (3, 5) for the  $n = 2$  and  $n = 3$  states of hydrogen (first Balmer transition).

the atoms are in the ground state, so the ratio is zero. As the temperature increases, the quantity in square brackets gets smaller, so the exponent becomes less negative, and the ratio increases. If we let  $T_k$  go to infinity the ratio of populations approaches the ratio of statistical weights. For a given temperature, increasing the energy separation between the two levels makes the exponent more negative, lowering the ratio. This makes sense, since the greater the energy separation, the harder it is to excite the atom to the higher level.

The Boltzmann distribution provides us with a convenient reference point, even for a system that is not in thermodynamic equilibrium. For any given population ratio  $n_j/n_i$ , we can always find some value of  $T$  to plug into equation (3.9) to make the equation correct. We call such a temperature the *excitation temperature*. When they are not in equilibrium, each pair of levels can have a different excitation temperature. In thermodynamic equilibrium all excitation temperatures are equal to each other and to the kinetic temperature.

### 3.4.2 Ionization

If we know the temperature in the atmosphere of a star, we can use the Boltzmann equation to predict how many atoms will be in each state,  $i$ , and predict the strengths of various spectral lines. However, there is still an additional effect that we have not taken into account – *ionization*. If the temperature is very high, some of the colliding particles will have kinetic energies greater than the ionization energy of the atom, so the electron will be torn away in the collision. Once a hydrogen atom is ionized, it can no longer participate in line emission or absorption.

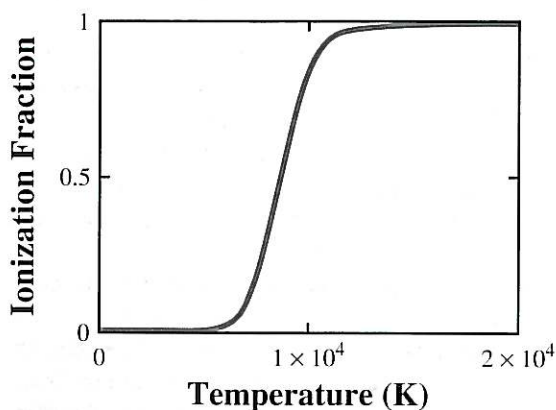
When the gas is ionized, electrons and positive ions will sometimes collide and recombine. When the total rate of ionizations is equal to the total rate of recombinations, we say that the gas is in *ionization equilibrium*. If the gas is in thermal equilibrium and ionization equilibrium, then the *Saha equation* tells us the relative abundances of various ions. We let  $n(X_r)$  and  $n(X_{r+1})$  be the densities of the  $r$  and  $r + 1$  ionization states, respectively, of element  $X$ . (For example, if  $r = 0$ , then we are comparing the neutral species and the first ionized state.) The ionization energy to go from  $r$  to  $r + 1$  is  $E_{ion}$ . The electron density is  $n_e$ ,

and the kinetic temperature is  $T_k$ . Finally,  $g_r$  and  $g_{r+1}$  are the statistical weights of the ground electronic states of  $X_r$  and  $X_{r+1}$  (assuming that most of each species is in the ground electronic state). The Saha equation tells us that

$$\frac{n_e n(X_{r+1})}{n(X_r)} = \frac{2 g_{r+1}}{g_r} \left( \frac{2\pi m_e k T_k}{h^2} \right)^{3/2} e^{-[E_r/kT_k]} \quad (3.10)$$

The Saha equation has the same exponential energy dependence as the Boltzmann distribution. However, there is an additional factor of  $T_k^{3/2}$ . This comes from the fact that a free electron has more states available to it at higher  $T_k$  than at lower  $T_k$ . In addition there is a factor of  $n_e$  on the left. This is because a higher abundance of electrons leads to a higher rate of recombinations, driving down the fraction of atoms that are ionized. Just as we did with the excitation temperature in the Boltzmann equation, we can define an *ionization temperature*  $T_i$ , which makes the Saha equation correct, even if the gas is not in thermodynamic equilibrium.

In this equation  $n_e$  is the number of electrons from all sources, since any electron can combine with a hydrogen ion (for example) no matter where that electron came from (hydrogen, helium, etc.) In many situations, virtually all of the ions are hydrogen. That is because hydrogen is by far the most abundant element, and because the next most abundant element, helium, is very hard to ionize. In that case, the number of electrons is equal to the number of positive ions,  $n_+$ ,



**Fig 3.8.** The ratio of electrons to the total number of hydrogen atoms (neutral plus ion), for an electron density appropriate to stars like the Sun.

Table 3.1. Ionization energies (eV).		
Atom	Singly ionized	Doubly ionized
H	13.6	—
He	24.6	54.4
C	11.3	24.4
N	14.5	29.6
O	13.6	35.1
Na	5.1	47.3
K	4.3	31.8
Ca	6.1	11.9
Fe	7.9	16.2

so the left side of equation (3.10) simplifies to  $n_e^2/n_0$ , where  $n_0$  is the number of neutrals. This extra factor of  $n_e$  makes even this simpler form of the Saha equation harder to solve for  $(n_e/n_0)$  than the Boltzmann equation is to solve for the ratio of level populations. In Fig. 3.8, we show the ratio  $n_e/n_0$  as a function of temperature, for a value of  $n_e$  reasonable for stars like the Sun.

The ionization energies of some common atoms are given in Table 3.1. This table is useful in deciding which ions you are likely to encounter at various temperatures. In designating ionized atoms, there is a shorthand that has been adopted. The roman numeral I is used to designate the neutral species, II the singly ionized species, III the doubly ionized species, and so on. For example, neutral hydrogen is H(I), ionized hydrogen ( $H^+$ ) is H(II), doubly ionized carbon is C(III).

### 3.4.3 Intensities of spectral lines

We are now in a position to discuss the intensities of various absorption lines in stars. We will take  $H\alpha$  as an example to see the combined effects of excitation and ionization. At low temperatures, essentially all of hydrogen is neutral, and most of it is in the ground state. Since little H will be in the second state, there will be few chances for  $H\alpha$  absorption. The  $H\alpha$  line will be weak.

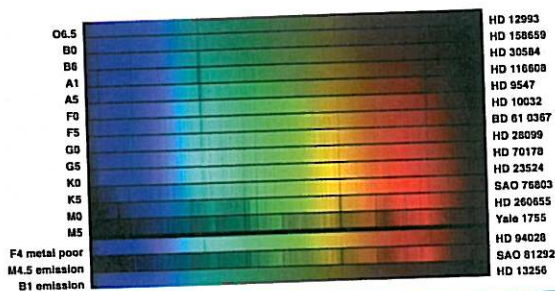
As we go to moderate temperatures, most of the hydrogen is still neutral. However, more of the hydrogen is in excited states, meaning that a reasonable amount will be in level 2.  $H\alpha$  absorption is possible. As the temperature increases, the  $H\alpha$  absorption becomes stronger.

At very high temperatures, the hydrogen becomes ionized. Since there is less neutral hydrogen, the H $\alpha$  line becomes weaker. This explains why the H $\alpha$  line is strongest in middle-temperature stars, and why the original scheme of classifying by hydrogen line strengths did not produce a sequence ordered in temperature.

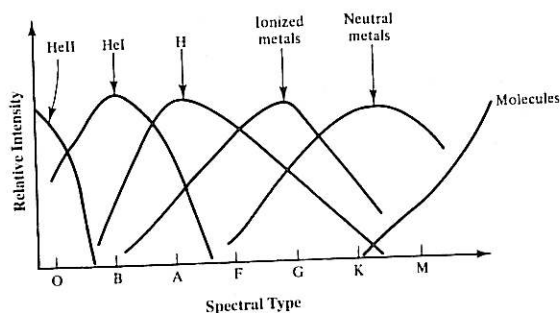
We can apply a similar analysis to other elements. The details will differ because of different energy level structures and different ionization energies. It should be noted that, after hydrogen and helium, the abundances of the elements fall off drastically (see Appendix F for the abundances of the elements). In fact, astronomers often refer to hydrogen, helium and 'everything else'. The 'everything else' are collectively called *metals*, even though many of the elements don't fit our common definition of a metal.

We now look at the properties of different spectral types, in order of increasing temperature. Sample spectra are shown in Fig. 3.9, and the behaviors of a few spectral lines are shown in Fig. 3.10.

M Temperatures in M stars are below 3500 K, explaining their red color. The temperature is not high enough to produce strong H $\alpha$  absorption, but some lines from neutral metals are seen. The stars are cool enough for simple molecules to form, and many lines are seen from molecules such as CN (cyanogen) and TiO (titanium oxide). If cool stars show strong CH lines, we designate them as C-type or 'carbon stars'. If any M



**Fig 3.9.** Samples of spectra from stars of different spectral types. The name of the star appears on the right of each spectrum, and the spectral type appears on the left. In each spectrum, the wavelength increases from left to right. Hotter stars are at the top. [NOAO/AURA/NSF]



**Fig 3.10.** The relative strengths of spectral lines from important species as a function of spectral type. Each species shows the effects of excitation and ionization. For example, the increase in H line strengths from K to A stars occurs because the increasing temperature results in more hydrogen in the  $n = 2$  (and higher) levels. However, the higher temperatures of the B and O stars ionize much of the hydrogen and the lines get much weaker.

star has strong ZrO (zirconium oxide) lines as opposed to TiO lines, we call it an S-type.

- K Temperatures range from 3500 to 5000 K. There are many lines from neutral metals. The H lines are stronger than in M stars but most of the H is still in the ground state.
- G Temperatures in the range 5000–6000 K. The Sun is a G2 star. The H lines are stronger than in K stars, as more atoms are in excited states. The temperature is high enough for metals with low ionization energies to be partially ionized. Two prominent lines are from Ca(II). When Fraunhofer studied the solar spectrum, he gave the strongest lines letter designations. These Ca(II) lines are the H and K lines in his sequence.
- F Temperatures range from 6000 to 7500 K. The H lines are a little stronger than in G stars. The ionized metal lines are also stronger.
- A Temperatures range from 7500 to 10 000 K. These stars are white–blue in color. They have the strongest H lines. Lines of ionized metals are still present.
- B Temperatures are in the range 10 000–30 000 K, and the stars appear blue. The H lines are beginning to weaken because the temperatures are high enough to ionize a significant fraction of the hydrogen. The lines of neutral and singly ionized helium begin to appear. Otherwise there are relatively few lines in the spectrum.

O Temperatures range from 30 000 to over 60 000 K, and the stars appear blue. The earliest spectral types that have been seen are O3 stars and there are very few O3 and O4 stars. The hydrogen lines fall off very sharply because of the high rate of ionization. The lines of singly ionized helium are still present, but there are very few lines overall in the visible part of the spectrum. There are several lines in the ultraviolet.

Some stars have emission as well as absorption lines in their spectra. These stars are designated with an 'e' after the spectral class, for example, Oe, Be, Ae, etc. O stars with very broad emission lines are called *Wolf-Rayet stars*. These stars probably have circumstellar material that has been ejected from the star. (Wolf-Rayet stars are not the only stars with such outflowing material.)

### 3.5 The Hertzsprung–Russell diagram

Even though we cannot study any one star (except for the Sun) in great detail, we can compensate somewhat by having a large number of stars to study. From statistical studies we learn about general trends. For example, if we find that brighter stars tend to be both hotter and larger, then any theory of stellar structure would have to explain that trend. Also, we think that any property that is common to many stars must be telling us about the laws of physics that are important in understanding the structure of stars.

One of the earliest statistical studies was carried out in 1910 independently by the Danish astronomer *Enjar Hertzsprung*, and the American astronomer *Henry Norris Russell*. They plotted the properties of stars on a diagram in which the horizontal axis is some measure of temperature (e.g. color or spectral type) and the vertical axis is some measure of luminosity. We call such a diagram a *Hertzsprung–Russell diagram*, or simply an *HR diagram*.

If a random group of stars is chosen, all at different distances, a comparison of apparent magnitudes is not very meaningful. The apparent magnitude must be corrected to give the absolute

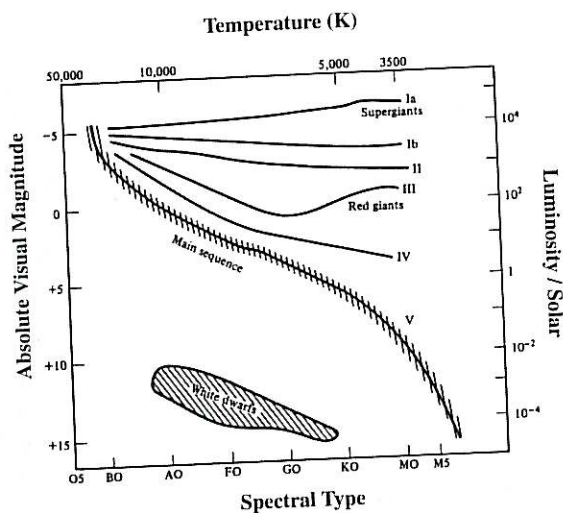
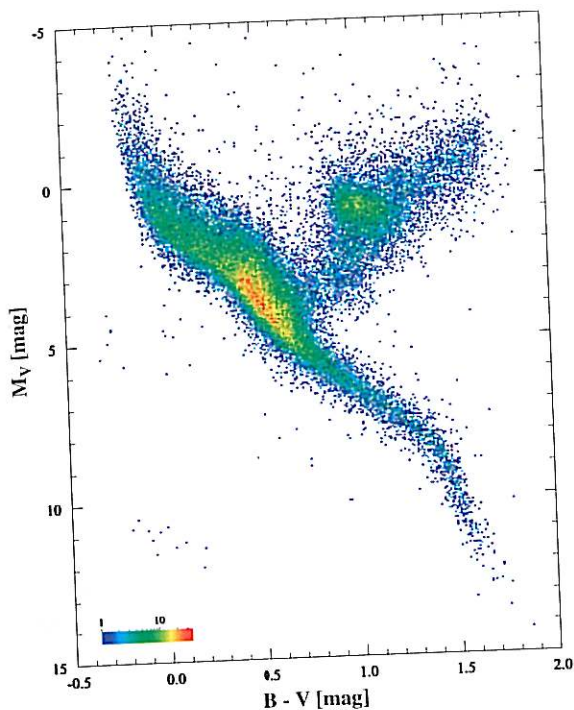
magnitude. However, if we find a group of stars all at the same distance, we can plot their apparent magnitudes, since the distance modulus would be the same for all the stars. For this purpose, we use clusters of stars.

An HR diagram for over 40 000 nearby stars is shown in Fig. 3.11(a). These stars were studied by the Hipparcos satellite, which was designed to measure trigonometric parallaxes, so distances to these stars are well known. So, apparent magnitudes can be converted into absolute magnitudes. This allows us to compare, on the same basis, the properties of stars that are not all in a cluster. The first thing we notice is that stars appear only in certain parts of the diagram. Arbitrary combinations of temperature and luminosity are not allowed. Remember, for a given temperature, the luminosity depends on the radius of the star, so the HR diagram is telling us that *arbitrary combinations of radius and temperature are not allowed*.

Most of the stars are found in a narrow band, called the *main sequence*. The significance of the main sequence is that most stars of the same temperature have essentially the same luminosity, and hence essentially the same size. This close relationship between size and temperature must be a result of the laws of physics as applied to stars. It gives us hope that we can understand stellar structure by applying the known laws. It also gives us a crucial test: any theory of stellar structure must predict the existence of the main sequence.

Not all stars appear on the main sequence. Some appear above the main sequence. This means that they are more luminous than main sequence stars of the same temperature. If two stars have the same temperature but one is more luminous, it must be larger than the other. Stars appearing above the main sequence are therefore larger than main sequence stars. We call these stars *giants*. By contrast, we call the main sequence stars *dwarfs*. We subdivide the giants into three groups: *subgiants*, *giants*, *supergiants*.

To keep track of the size of a star of a given spectral type, we append a *luminosity class* to the spectral type. The luminosity class is denoted by a roman numeral. Main sequence stars are luminosity class V. The Sun, for example, is a G2 V star. Subgiants are luminosity class IV, giants are luminosity class III. Luminosity class II stars are



**Fig 3.11.** (a) HR diagram for over 40 000 nearby stars studied by the Hipparcos satellite, designed to measure trigonometric parallaxes, so distances are known for all of these stars. In this figure, the color represents the number of stars in each category, with red being the most and blue being the least. (b) A schematic HR diagram, showing the main features of the actual diagrams. Luminosity classes are indicated by roman numerals. [(a) Michael Perryman, ESA, Hipparcos]

somewhere between giants and supergiants. Supergiants are luminosity class I. We further divide supergiants into Ia and Ib, with Ia being larger. When we look at the spectral lines from a star we can actually tell something about the size. Stars of different sizes will have different accelerations of gravity near their surface. The surface gravity affects the detailed appearance of certain spectral lines.

There are also stars that appear below the main sequence. These stars are typically 10 mag fainter than main sequence stars of the same temperature. They are clearly much smaller than main sequence stars. Since most of these are in the middle spectral types, and therefore appear white, we refer to them as *white dwarfs*. (Do not confuse dwarfs, which are main sequence stars, with white dwarfs, which are much smaller than ordinary dwarfs.)

**Example 3.3** Size of white dwarfs

(a) Suppose that some white dwarf has the same spectral type as the Sun, but has an absolute magnitude that is 10 mag fainter than the Sun. What is the ratio of the radius of the white dwarf,  $R_{wd}$ , to that of the Sun,  $R_{\odot}$ ?

**SOLUTION**

The luminosity is proportional to the square of the radius, so

$$L_{wd}/L_{\odot} = (R_{wd}/R_{\odot})^2$$

We use equation (2.2) to find the luminosity ratio for a 10 mag difference:

$$L_{wd}/L_{\odot} = 10^{(M_{\odot} - M_{wd})/2.5} = 10^{-4}$$

Combining these two results to find the ratio of the radii yields

(b)

$$R_{wd}/R_{\odot} = (L_{wd}/L_{\odot})^{1/2} = (10^{-4})^{1/2} = 10^{-2}$$

The radius of a white dwarf is 1% of the radius of the Sun!

For any cluster for which we plot an HR diagram, we only know the apparent magnitudes.