Simetría esférica. Métrica de Schwarzschild.

9

Geometría de Schwarzschild

- Solución de la ecuación de Einstein para el caso más simple y más útil: espacio vacío en el exterior de una fuente de curvatura (materia/energía) esféricamente simétrica.

$$ds^2 = -\left(1-\frac{2GM}{c^2 r}\right)(cdt)^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + sen^2\theta d\phi^2\right)$$

- Permitió realizar las pruebas clásicas de la RG: enrojecimiento gravitatorio, precesión del perihelio, desviación y retraso de luz.
- Características de la geometría:
  - Independiente del tiempo $\xi_E = (1,0,0,0)$
  - Esféricamente simétrica $\xi_L = (0,0,0,1)$

    2-superficie con t,r ctes $d\Sigma^2 = r^2\left(d\theta^2 + sen^2\theta d\phi^2\right)$ métrica indepte de ϕ
  - Masa/energía total M de la fuente de curvatura (efecto no lineal)

    $GM/c^2 r \ll 1 \;\Rightarrow\; g_{rr} = 1 + \dfrac{2GM}{c^2 r} =$ aprox. Newton campo débil estático con $g_{rr} = 1 - \dfrac{2\Phi}{c^2}$; $\Phi = \dfrac{-GM}{r}$
  - Radios especiales: r = 0 y radio de Schwarzschild $R_s = \dfrac{2GM}{c^2}$

QQ Calcúlalo para Sol, WD, NS, BH y comprueba si R>Rs

Órbitas de partículas. Conservación.

- Coordenada *r* no es distancia desde ningún centro sino algo que tiene que ver con el área en 2-esferas de t, r=R ctes:

$$dA=\sqrt{g_{\theta\theta}g_{\phi\phi}}\,d\theta d\phi=r^2\,sen\theta d\theta d\phi \qquad A=\int dA=R^2\int sen\theta d\theta\int d\phi=R^2\cdot 2\cdot 2\pi=4\pi R^2$$

- Leyes de conservación (unidades geometrizadas G=c=1 o sistema [L]):

$$-\xi_E\cdot\vec{u}=cte\equiv\tilde{E}=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}$$

$$\xi_L\cdot\vec{u}=cte\equiv\tilde{L}=r^2\,sen^2\theta\frac{d\phi}{d\tau}$$

la partícula orbita en un plano, pues, suponiendo que en un instante dado la velocidad está contenida en meridiano (libre elección de ejes), L=0 y por tanto se queda siempre en ese meridiano: $d\phi/d\tau=0\ \Rightarrow\tilde{L}=0=cte$

QQ Geodésicas 2-esfera

Reorientamos plano orbital=ecuatorial: $\theta=\frac{\pi}{2}=cte\Rightarrow u^\theta=0\Rightarrow\tilde{L}=r^2\frac{d\phi}{d\tau}$

$$\vec{u}\cdot\vec{u}=g_{\alpha\beta}u^\alpha u^\beta=-1\Rightarrow-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2+r^2\left(\frac{d\phi}{d\tau}\right)^2=-1$$ (partículas)

$$-\left(1-\frac{2M}{r}\right)^{-1}\tilde{E}^2+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2+\frac{\tilde{L}^2}{r^2}=-1\Rightarrow\quad\epsilon\equiv\frac{\tilde{E}^2-1}{2}=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)-1\right]$$

$$W_{ef}(r)\equiv\frac{-M}{r}+\frac{\tilde{L}^2}{2r^2}-\frac{M\tilde{L}^2}{r^3}\quad\Rightarrow\quad\epsilon=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+W_{ef}(r)$$

QQ ver anexo “Movimiento newtoniano bajo fuerza central gravitatoria” (GYAapuntesRG5.pdf)

Comparación caso newtoniano

QQ Pasando el potencial efectivo a unidades MLT: $W_{ef} = \frac{1}{c^2}\left(\frac{-GM}{r} + \frac{\tilde{L}^2}{2r^2} - \frac{GM\,\tilde{L}^2}{c^2 r^3}\right)$

obtenemos la siguiente correspondencia con la energía newtoniana:

$$E_{Newt} \equiv mc^2 \epsilon = \frac{1}{2} mc^2 \left(\frac{dr}{d\tau}\right)^2 + mc^2 W_{ef} = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + \left[\frac{-GMm}{r} + \frac{L^2}{2mr^2} \boxed{-\frac{GML^2}{c^2 mr^3}}\right]$$

$L = m\{\tilde{L}$¿

corrección relativista

$$E_{Newt} \equiv mc^2 \epsilon = mc^2 \left(\frac{\tilde{E}^2 - 1}{2}\right) \Rightarrow \tilde{E}^2 = 1 + \frac{2E_{Newt}}{mc^2} \simeq \left(1 + \frac{E_{Newt}}{mc^2}\right)^2 \Rightarrow \tilde{E} = 1 + \frac{E_{Newt}}{mc^2} = \frac{mc^2 + E_{Newt}}{mc^2}$$

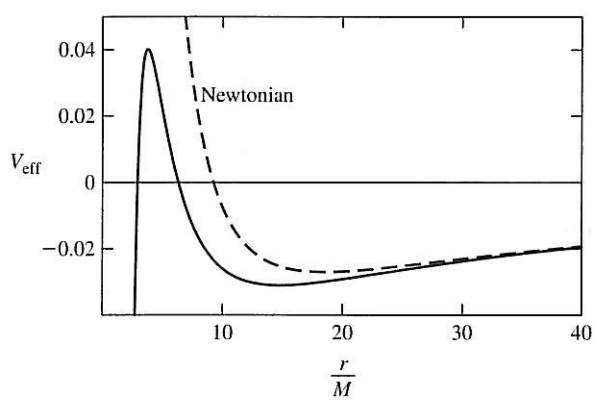

FIGURE 9.2 The relativistic and Newtonian effective potentials for radial motion compared for $\ell/M = 4.3$. The relativistic effective potential $V_{\rm eff}(r)$ is defined by (9.28) and we take the Newtonian effective potential to be the first two terms of that. The two are close for large r, as shown, but differ significantly for small r, where the $1/r^3$ term in (9.28) becomes important. In particular the infinite centrifugal barrier of Newtonian theory becomes a barrier of finite height. For the Earth in orbit around the Sun, $\ell/M \sim 10^9$ and the differences between the Newtonian and relativistic potential over the orbit of the Earth are tiny but detectable in precise measurements, as we see in Chapter 10.

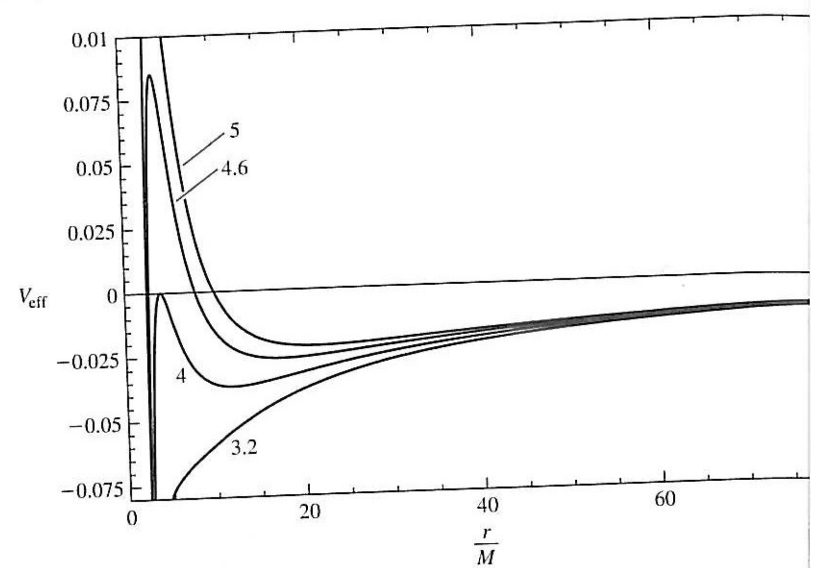

FIGURE 9.3 The effective potential $V_{\rm eff}(r)$ for radial motion for several different of ℓ. The values of ℓ/M label the curves.

Tipos de órbitas

- QQ Dibujamos Wef y buscamos extremos $\frac{dW_{ef}}{dr}=0 \Rightarrow r_{\substack{min\\max}}=\frac{\tilde{L}^2}{2M}\left[1\pm\sqrt{1-12\left(\frac{M}{\tilde{L}}\right)^2}\right]$
  - Para $\frac{\tilde{L}}{M}<\sqrt{12}$ deja de haber extremos y Wef siempre es negativo
  - Para $\frac{\tilde{L}}{M}>\sqrt{12}$ hay un un mínimo y un máximo; éste estará por encima de Wef=0 si $\frac{\tilde{L}}{M}>4$ y por debajo en caso contrario
- Puntos de retorno (PR, se desvanece la velocidad radial): $\epsilon=W_{ef}(r_{PR})$
- Tipos de órbitas para $\frac{\tilde{L}}{M}>\sqrt{12}$
  - Circulares
    - Inestable $r_{max}=\frac{\tilde{L}^2}{2M}\left[1-\sqrt{1-12\left(\frac{M}{\tilde{L}}\right)^2}\right]$
    - Estable → $r_{min}=\frac{\tilde{L}^2}{2M}\left[1+\sqrt{1-12\left(\frac{M}{\tilde{L}}\right)^2}\right]$
  - Ligadas $\epsilon<0$ entre dos puntos de retorno
  - Rebote $0<\epsilon<W_{ef}^{max}$
  - Caída hacia el centro $\epsilon>W_{ef}^{max}$

Tipos de órbitas de partículas

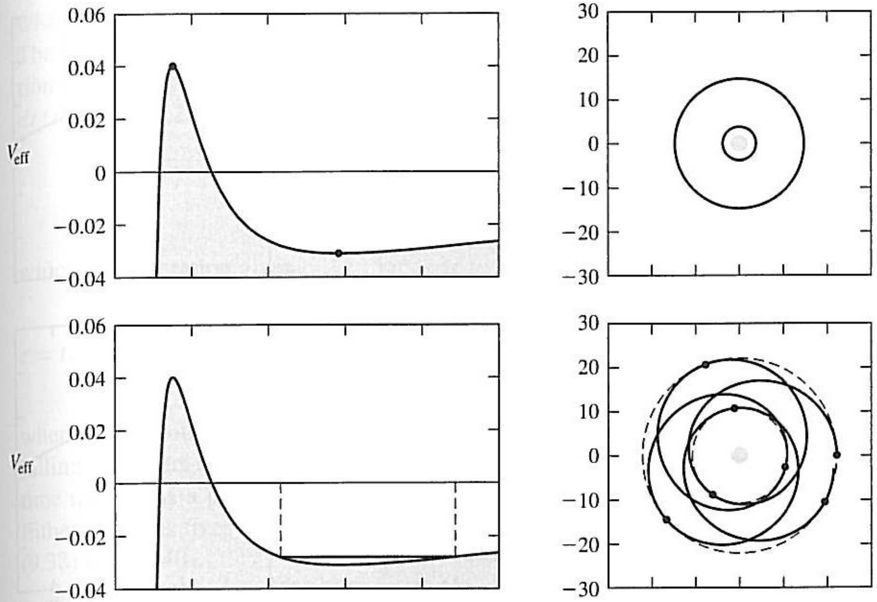

FIGURE 9.4 Four kinds of orbits in the Schwarzschild geometry. The pairs of figures on this page and the next show four orbits corresponding to different values of $\mathcal{E}$ for the illustrative value $\ell/M = 4.3$. The potential and its relationship to $\mathcal{E}$ are shown at left. The horizontal axis in these plots is r/M. The vertical axis is $V_{\text{eff}}(r)$. Horizontal lines indicate the values of $\mathcal{E}$. The vertical dashed lines are at turning points. The dots denote the possible locations of circular orbits. The shapes of the corresponding orbits are shown in the figures at right where Schwarzschild r and ϕ are plotted as polar coordinates in the plane. The shaded region at the center of each plot corresponds to $r < 2M$. The top figure on this page shows two circular orbits—the outer one is stable the inner one is unstable. The next figure shows a bound orbit in which the particle moves between two turning points marked by the dotted circles. The positions of closest approach (perihelion) and furthest excursion (aphelion) are indicated by dots. The precession of the perihelion is large for this relativistic orbit. (*Continued on next page.*)

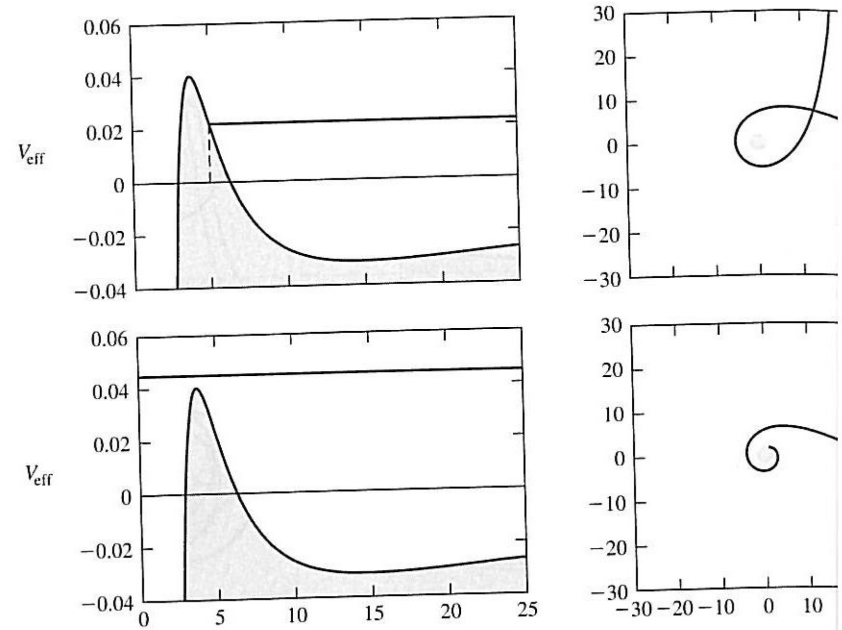

FIGURE 9.4 *continued.* The first figure on this page shows a scattering orbit. Tl
ticle comes in from infinity, passes around the center of attraction and moves out to
ity again. This is a highly relativistic orbit which differs significantly from a New
parabola. The last pair of figures shows a plunge orbit in which the particle comes i
infinity, moves part way around the central mass and then plunges into the cente
kind of orbit is not possible in Newtonian mechanics for a particle moving in a $1/r$
potential.

Órbitas de caída libre radial

- Caída radial $\tilde{L}=0$, desde el infinito, donde la energía inicial (cinética) $\epsilon=\epsilon_K>0$ $\epsilon=\epsilon_K=0$

- Caso $\epsilon=0$

$\epsilon=0\Rightarrow\tilde{E}=1\Rightarrow\quad dt=d\tau$ en el infinito el espacio es plano

$$\epsilon=\frac{1}{2}\left(\frac{d\,r}{d\tau}\right)^2+W_{ef}\Rightarrow\quad 0=\frac{1}{2}\left(\frac{d\,r}{d\tau}\right)^2-\frac{M}{r}\Rightarrow\quad \frac{d\,r}{d\tau}=\pm\sqrt{\frac{2M}{r}}\Rightarrow\quad \sqrt{r}\,dr=-\sqrt{2M}\,d\tau$$

geodésica hacia dentro

$$\tilde{E}=1\Rightarrow\quad \frac{d\,t}{d\tau}=\left(1-\frac{2M}{r}\right)^{-1}\qquad u^\alpha=\left[\left(1-\frac{2M}{r}\right)^{-1},-\sqrt{\frac{2M}{r}},\ 0\ ,\ 0\right]$$

- Límites de integración: $\tau=\tau_*\ ,\quad r=0\Rightarrow\quad r(\tau)=\left(\frac{3}{2}\right)^{2/3}(2M)^{1/3}(\tau_*-\tau)^{2/3}$
- Generalmente buscaremos primero $t=t(r)$ y luego usaremos $r(\tau)$ para obtener $t=t(\tau)$

$$1\equiv\tilde{E}=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}\Rightarrow\quad \frac{dt}{dr}=\frac{dt}{d\tau}\frac{d\tau}{dr}=\left(1-\frac{2M}{r}\right)^{-1}\left(-\sqrt{\frac{r}{2M}}\right)\Rightarrow\quad \int dt=\int-\left(\frac{2M}{r}\right)^{-1/2}\left(1-\frac{2M}{r}\right)^{-1}$$

- QQ Integrar $t=t_*+2M\left[\frac{-2}{3}\left(\frac{r}{2M}\right)^{3/2}-2\left(\frac{r}{2M}\right)^{1/2}+\log\left[\frac{(r/2M)^{1/2}+1}{(r/2M)^{1/2}-1}\right]\right]$

- Tiempo entre r=2M y r=0 es infinito para t pero finito para tau (malas coordenadas)

Órbitas circulares estables

- Son órbitas con $r = r_{min}$ donde r_{min} disminuye con L/M hasta el límite ISCO
- Innermost stable circular orbit (ISCO)

$$r_{min} = \frac{\tilde{L}^2}{2M}\left[1 + \sqrt{1 - 12\left(\frac{M}{\tilde{L}}\right)^2}\right] \qquad \frac{\tilde{L}}{M} = \sqrt{12} \Rightarrow r_{min}^{ISCO} = 6M$$

- Velocidad angular con t medido por observador estacionario en infinito

$$\Omega \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\frac{\tilde{L}}{\tilde{E}}$$

- Para órbitas circulares de r dado, L y E determinados por dos condiciones:
  - Potencial efectivo es mínimo en $r = r_{min}$

$$\epsilon = W_{ef}(r_{min}) \qquad \epsilon \equiv \frac{\tilde{E}^2 - 1}{2} = \frac{-M}{r_{min}} + \frac{\tilde{L}^2}{2r_{min}^2} - \frac{M\tilde{L}^2}{r_{min}^3}$$

- QQ Encontrar cociente $\frac{\tilde{L}}{\tilde{E}} = (Mr)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1}$

- 3ª ley de Kepler $\Omega = \frac{\sqrt{Mr}}{r^2} \Rightarrow \Omega^2 = \frac{M}{r^3}$ con período $P = 2\pi/\Omega$
- Cuadrivelocidad

$$u^\alpha = u^t(1,0,0,\Omega) \Rightarrow u^t = \left(1 - \frac{2M}{r} - r^2\Omega^2\right)^{-1/2} = \left(1 - \frac{3M}{r}\right)^{-1/2}$$

Órbitas ligadas: precesión

- Buscamos $\phi(r)$: $\epsilon = \frac{1}{2}\left(\frac{d\ r}{d\tau}\right)^2 + W_{ef}(r) \Rightarrow \frac{d\ r}{d\tau} = \pm\sqrt{2(\epsilon - W_{ef})}$

$$W_{ef} = \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)-1\right] \Rightarrow 2(\epsilon - W_{ef}) = \tilde{E}^2 - 1 - \left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)-1\right] = \tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)$$

$$\frac{d\phi}{dr} = \frac{d\phi}{d\tau}\frac{d\tau}{dr} = \pm\frac{\tilde{L}}{r^2}\frac{1}{\sqrt{\tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)}}$$

- ¿Se cierran las órbitas al volver a pasar por el mismo PR?
  - cerrada si ángulo barrido entre pasos sucesivos por PR: $\Delta\phi = 2\pi$
  - Precesión $\delta\phi_{pr} = \Delta\phi - 2\pi$

Que se anule $dr/d\tau$ significa que se anula la raíz en denominador y bastará integrar entre los radios para los que se anula éste.

$$\Delta\phi = 2\phi_{12} = 2\tilde{L}\int\frac{dr}{r^2}\frac{1}{\sqrt{c^2(\tilde{E}^2-1)+\frac{2GM}{r}-\frac{\tilde{L}^2}{r^2}+\frac{2GM\tilde{L}^2}{c^2r^3}}}$$

Órbitas ligadas: precesión

- QQ Comprobar que para $1/r^3 \sim 0$ $\quad \tilde{E} = 1 + \frac{2E_{Newt}}{mc^2} + \dots \;\; 1$ y se obtiene $\quad \Delta\phi = 2\pi$

  Cambiando de variable la integral se puede escribir:

$$u = 1/r \qquad \Delta\phi = 2\int \frac{du}{\sqrt{\left[(u_1 - u)(u - u_2)\right]}} = 2\pi$$

- QQ Encontrar $\delta\phi_{pr} = \frac{6\pi G}{c^2}\frac{M}{a(1-e^2)} \simeq 43\,''/siglo$ para Mercurio

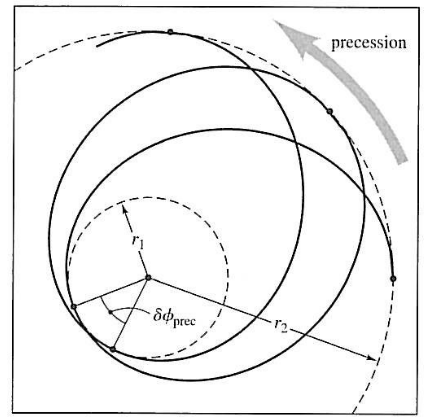

FIGURE 9.6 The shape of a bound orbit outside a spherical star. This is a picture of the orbital plane of the bound orbit whose radial motion is of the kind illustrated in the second pair of plots of Figure 9.4 The planet moves from a minimum radius r_1 out to a maximum radius r_2 and back to the same minimum radius. However, unlike the Keplerian ellipse of Newtonian gravitational theory, the orbit does not close. Rather, the angular position of the closest approach advances slightly on each return by an angle called the precession of the perihelion for a planet around the Sun. The figure shows a little over two orbits of a test mass that starts from the 3 o'clock position. The two positions of closest approach at the inner turning radius are indicated by dots. The angle between them is the precession of the perihelion per orbit.

Manteniendo el término relativista:

$$\Delta\phi = 2\tilde{L}\int \frac{dr}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \frac{1}{\sqrt{c^2\tilde{E}^2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} - \left(c^2 + \frac{\tilde{L}^2}{r^2}\right)}}$$

factor común — cuadrático en u

Desarrollando $1 - \frac{2GM}{c^2 r}$ hasta orden $\frac{1}{c^2}$

$$\Delta\phi = \left[1 + 2\left(\frac{GM}{c\tilde{L}}\right)^2\right] 2\int \frac{du}{\sqrt{\left[(u_1 - u)(u - u_2)\right]}} + 2\frac{GM}{c^2}\int \frac{u\,du}{\sqrt{\left[(u_1 - u)(u - u_2)\right]}} + \dots$$

$$\frac{\pi}{2}(u_1 + u_2) \simeq \pi \frac{GM}{\tilde{L}^2}$$

Se llega hasta $\delta\phi_{pr} \simeq 6\pi\left(\frac{GM}{c\tilde{L}}\right)^2$ y en órbitas de Newton $L^2 = \left(r^2\frac{d\phi}{d\tau}\right)^2 \approx \left(r^2\frac{d\phi}{dt}\right)^2 = GMa(1 - e^2)$

Órbitas de fotones. Conservación.

- Parametrizando trayectorias lumínicas con λ : $ds^2=0 \Rightarrow \vec{u}\cdot\vec{u}=\frac{g_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta}}{d\lambda\ d\lambda}=0$
- Leyes de conservación

$$-\xi_E\cdot\vec{u}=cte\equiv\tilde{E}=\left(1-\frac{2M}{r}\right)\frac{d\,t}{d\lambda}$$

$$\xi_L\cdot\vec{u}=cte\equiv\tilde{L}=r^2\,sen^2\,\theta\,\frac{d\phi}{d\lambda}$$

$$\vec{u}\cdot\vec{u}=g_{\alpha\beta}u^{\alpha}u^{\beta}=0\Rightarrow-\left(1-\frac{2M}{r}\right)\left(\frac{d\,t}{d\lambda}\right)^2+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{d\,r}{d\lambda}\right)^2+r^2\left(\frac{d\phi}{d\lambda}\right)^2=0 \qquad \text{(fotones)}$$

$$-\left(1-\frac{2M}{r}\right)^{-1}\tilde{E}^2+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{d\,r}{d\lambda}\right)^2+\frac{\tilde{L}^2}{r^2}=0$$

Multiplicando por $\frac{1}{c^2}\left(1-\frac{2M}{r}\right)$ llegamos a: $\frac{1}{b^2}\equiv\frac{\tilde{E}^2}{\tilde{L}^2}=\frac{1}{\tilde{L}^2}\left(\frac{d\,r}{d\lambda}\right)^2+W_{ef}^{fot}(r)$

$$W_{ef}^{fot}(r)\equiv\frac{1}{r^2}\left(1-\frac{2M}{r}\right)$$

Con partículas podíamos jugar con diferentes valores de L y de E.
Con fotones sólo podemos jugar con el cociente $b\equiv\left(\frac{\tilde{L}}{\tilde{E}}\right)$, que determina las propiedades de las órbitas.

Órbitas de rayos de luz

- *b* es parámetro de impacto

si órbita en infinito $(r \gg 2M)$:

$$\left(\frac{\tilde{L}}{\tilde{E}}\right) \approx \frac{r^2 \, d\phi/d\lambda}{dt/d\lambda} = r^2 \frac{d\phi}{dt} \quad = b$$

$$\tan\phi \simeq \phi \simeq d/r \qquad \frac{d\phi}{dt} = \frac{d\phi}{dr}\frac{dr}{dt} = \frac{b}{r^2}$$

$$dr/dt \approx -1$$

- Máximo del potencial en r=3M

$$W_{ef,max}^{fot}(r=3M) = \frac{1}{27M^2}$$

- Tipos de órbitas
  - Circular (inestable) $r=3M \qquad b^2 = 27M^2$
  - Rebote en PR $b^{-2} < \frac{1}{27M^2}$
  - Caída al centro $b^{-2} > \frac{1}{27M^2}$

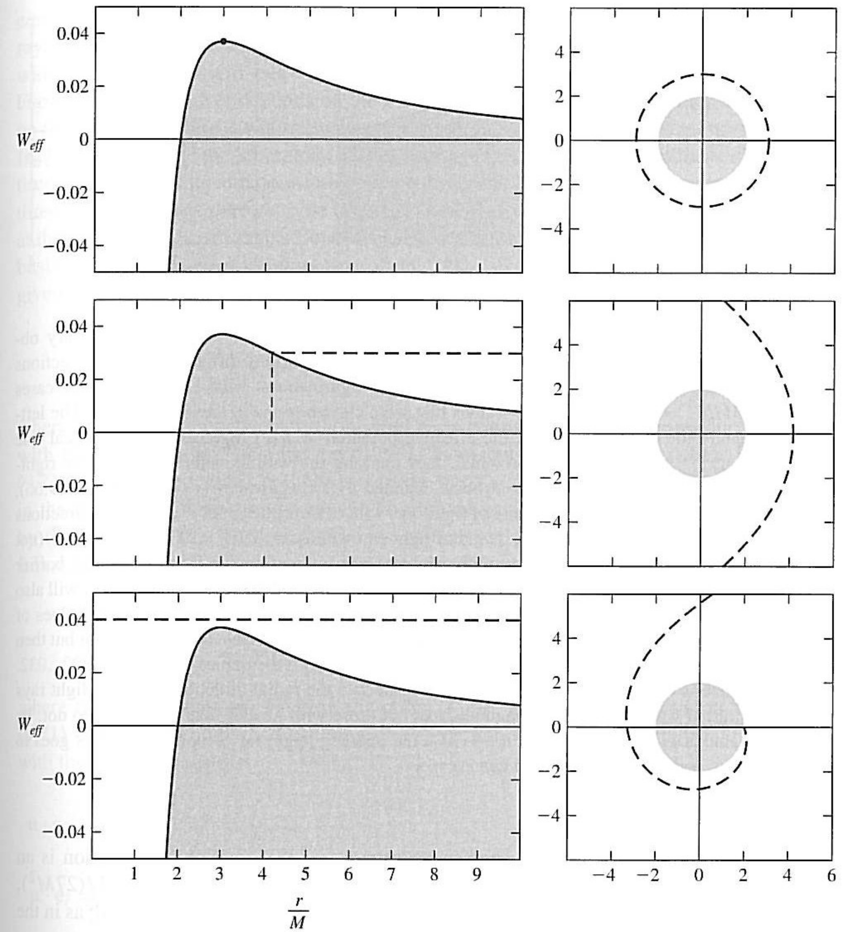

FIGURE 9.8 Three kinds of light ray orbits in the Schwarzschild geometry. The figure shows three orbits corresponding to different values of *b*. The potential and its relationship to $1/b^2$ are shown at left. The horizontal axis is r/M. The vertical axis is $W_{\text{eff}}(r)$. The heavy dotted lines are the values of $1/b^2$. The shape of the orbit at right. From the top down there are a circular orbit, a scattering orbit, and a plunge orbit.

Órbitas entre 2M y 3M

- Trayectorias que salen de radios entre 2M y 3M:
- Si $b^{-2} > \dfrac{1}{27M^2}$ el rayo escapa
- Si $b^{-2} < \dfrac{1}{27M^2}$ hay PR y el rayo cae hacia el centro
- Como $b^2 = \tilde{L}^2 / \tilde{E}^2$, si el rayo sale con momento angular suficientemente pequeño (apuntando cerca de la dirección radial), entonces escapará. Si no, vuelve hacia la fuente.

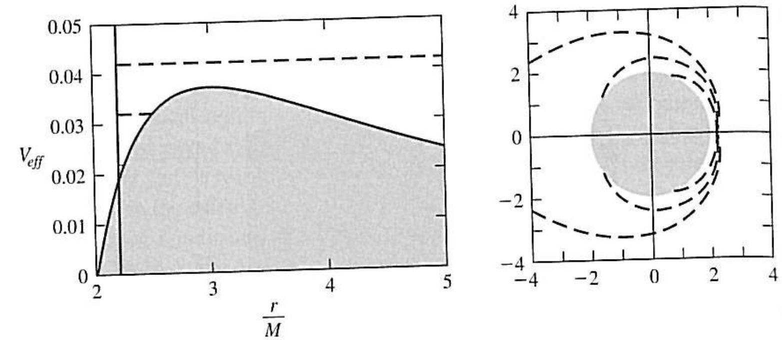

FIGURE 9.9 Light rays emitted between $r = 2M$ and $r = 3M$. A stationary observer at a radius $r = R = 2.2M$ emits light rays in various different outward directions corresponding to different values of b^2. This figure shows what happens to three cases $(M/b)^2 = .022, .032, .042$—values that were chosen to make intelligible plots. The left-hand plot shows a detail of the effective potential $W_{\text{eff}}(r)$ together with a vertical line marking $r = 2.2M$ and horizontal lines marking the various values of b^{-2}. The right-hand plot shows the equatorial plane spanned by the Cartesian (x, y) defined in (9.66), together with the orbits of pairs of light rays with these values of b^{-2} emitted in directions above and below the x-axis. A radial light ray with $b \equiv |\ell/e| = 0$ or infinite b^{-2} (not shown) will escape. Light rays with values of b^{-2} higher than the maximum of the barrier $1/(27M^2) = .037/M^2$, making sufficiently small angles with the radial direction, will also escape like the pair with the value $(M/b)^2 = .042$ illustrated. Light rays with values of b^{-2} less than the height of the barrier will not escape. They move outward for a bit but then fall back through the radius $r = 2M$ like the pairs with the values $(M/b)^2 = .022, .032$. There is thus a critical angle ψ_{crit} with respect to the radial direction such that light rays emitted with less than this angle escape, but those with greater than this angle do not. Its value is given in (9.74). As $R \to 2M$, the opening angle for escaping light rays goes to zero, and essentially no light can escape.

Desviación y retraso de la luz

QQ Ejercicios prácticos y experimentos de prueba de la RG

Desviación de la luz $\delta\phi = 4\frac{GM}{c^2 b}$

Retraso de señales $\Delta t \approx \frac{4GM}{c^3}\left[\log\left(\frac{4 r_R r_T}{r_1^2}\right) + 1\right]$

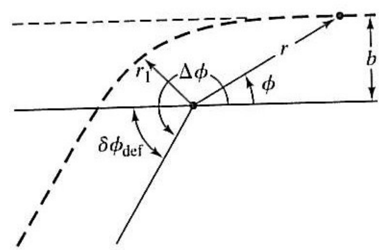

FIGURE 9.10 Quantities needed for calculating the deflection of light $\delta\phi_{\text{def}}$ by a spherical star. In this schematic diagram a light ray enters at right with an impact parameter b corresponding to a scattering orbit as in the second pass of plots in Figure 9.8. It approaches the center of attraction until the turning point at $r = r_1$, after which it moves out to infinity, emerging deflected by an angle $\delta\phi_{\text{def}}$. That deflection angle is the total angle swept out in the orbit $\Delta\phi$ less π.