EDPs y series de Fourier. 3º Matemáticas/4º PES **Test 3, 17/4/2018**

Nombre: SOLUCIONES ....

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1. Considera la función $f(x) = 1 - 2|x|$, con $x \in [-\frac{1}{2}, \frac{1}{2}]$.

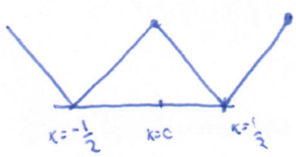
   a) Encuentra la serie de Fourier **real** de $f(x)$

   b) Justifica si la serie converge y a qué valor, en cada punto $x \in \mathbb{R}$

   c) Utiliza lo anterior para calcular $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

   d) Determina a qué función corresponde la serie $\displaystyle\sum_{n=1}^{\infty} \frac{\operatorname{sen}(2\pi(2n-1)x)}{(2n-1)^3}$, si $x \in [-\frac{1}{2}, \frac{1}{2}]$.

Gráficamente, la serie es:



$$f(x) = 1 - 2|x| = \begin{cases} 1+2x & \text{si } x \in [-\frac{1}{2}, 0] \\ 1-2x & \text{si } x \in [0, \frac{1}{2}] \end{cases}$$

. Observamos que es 1-periódica. Y par.

Los coeficientes de la serie de Fourier vienen dados por:

$$\hat{f}(n) = \int_{\mathbb{T}} f(x) e^{-2\pi i n x} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) e^{-2\pi i n x} = \int_{-\frac{1}{2}}^{0} (1+2x) e^{-2\pi i n x} dx + \int_0^{\frac{1}{2}} (1-2x) e^{-2\pi i n x} dx =$$

$$= \int_0^{\frac{1}{2}} (1-2x) e^{2\pi i n x} dx + \int_0^{\frac{1}{2}} (1-2x) e^{-2\pi i n x} dx = \int_0^{\frac{1}{2}} (1-2x) \big(\underbrace{e^{2\pi i n x} + e^{-2\pi i n x}}_{\cos(2\pi n x) + i\cancel{\operatorname{sen}(2\pi n x)} + \cos(-2\pi n x) + i \operatorname{sen}(-2\pi n x)}\big) dx =$$

$$= \int_0^{\frac{1}{2}} (1-2x) \cdot \underbrace{2 \cdot \cos(2\pi n x)}_{\left(\frac{2 \cdot \operatorname{sen}(2\pi n x)}{2\pi n}\right)'} dx \underset{\text{Integrando por partes}}{=} \underbrace{\left[(1-2x) \cdot \frac{\operatorname{sen}(2\pi n x)}{\pi n}\right]_{x=0}^{x=\frac{1}{2}}}_{0} - \int_0^{\frac{1}{2}} -2 \cdot \frac{\operatorname{sen}(2\pi n x)}{\pi n} dx =$$

$n \neq 0$

$$= \frac{2}{\pi n} \int_{x=0}^{\frac{1}{2}} \operatorname{sen}(2\pi n x) dx = \frac{2}{\pi n} \left[\frac{-\cos(2\pi n x)}{2\pi n}\right]_{x=0}^{x=\frac{1}{2}} = \frac{2}{\pi n} \left[\frac{-\overbrace{\cos(\pi n)}^{(-1)^n}}{2\pi n} - \frac{-\cos(0)}{2\pi n}\right] =$$

$$= \frac{2}{\pi n} \cdot \frac{1}{2\pi n} \left[(-1)^{n+1} + 1\right] = \frac{1}{(\pi n)^2} \left((-1)^{n+1} + 1\right) \quad \text{para } n \neq 0.$$

$$\hat{f}(n) = \frac{1}{(\pi n)^2} \cdot \left((-1)^{n+1} + 1\right) \quad n \neq 0.$$

~~X~~ separar n par / n impar

Para $n=0$ tenemos:

$$\hat{f}(0) = \int_{\mathbb{T}} f(x)dx = \int_{-\frac{1}{2}}^{0} 1+2x \, dx + \int_{0}^{\frac{1}{2}} 1-2x dx = \left[x+x^2\right]_{-\frac{1}{2}}^{0} + \left[x-x^2\right]_{0}^{\frac{1}{2}} =$$

$$= -\left(\frac{-1}{2}+\frac{1}{4}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) = \frac{1}{2} - \frac{1}{4} + \frac{1}{2} - \frac{1}{4} = \frac{1}{2}$$ ✓

Por tanto:

$$\hat{f}(n) = \begin{cases} \frac{1}{(\pi n)^2} \underbrace{\left((-1)^{n+1}+1\right)}_{=2} & \text{si } n \text{ impar} \\ 0 & \text{si } n \text{ par} \end{cases} \quad \text{y} \quad \hat{f}(0) = \frac{1}{2}$$

Y la serie de Fourier queda de la forma:

$$f(x) \sim \frac{1}{2} + \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x} = \sum_{n=1}^{\infty} \frac{1}{(\pi n)^2} \left((-1)^{n+1}+1\right) e^{2\pi i n x} + \sum_{m=-1}^{-\infty} \frac{1}{(\pi m)^2} \left((-1)^{m+1}+1\right) e^{2\pi i n x} + \underbrace{\frac{1}{2}}_{\hat{f}(0)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\left(\pi(2n-1)\right)^2} \cdot 2 \cdot e^{2\pi i (2n-1)x} + \sum_{m=-1}^{-\infty} \frac{1}{\left(\pi(2m+1)\right)^2} \cdot 2 \cdot e^{2\pi i (2m+1)x} + \underbrace{\frac{1}{2}}_{\hat{f}(0)}$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi^2(2n-1)^2} e^{2\pi i (2n-1)x} + \sum_{n=1}^{\infty} \frac{1}{\pi^2(-2n+1)^2} \cdot 2 \cdot e^{2\pi i (-2n+1)x} + \underbrace{\frac{1}{2}}_{\hat{f}(0)}$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi^2(2n-1)^2} \left(e^{2\pi i (2n-1)x} + e^{-2\pi i (2n-1)x}\right) + \underbrace{\frac{1}{2}}_{\hat{f}(0)}$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi^2(2n-1)^2} 2 \cdot \cos\left((2n-1) \cdot 2\pi x\right) + \underbrace{\frac{1}{2}}_{\hat{f}(0)}$$

✓

SOLUCIONES

b) Como $g(x) = 1 - 2|x|$ es de clase C^1 en $(\frac{-1}{2}, 0) \cup (0, \frac{1}{2})$, en estos intervalos, por Dini 1, tenemos:

$$1 - 2|x| = g(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 (2n-1)^2} \cdot \cos((2n-1) \cdot 2\pi x) + \frac{1}{2}$$

Además, como $g(0)^{\pm} = 1$, $g'(0)^{\pm} < +\infty$

$$g(\tfrac{1}{2})^{\pm} = g(\tfrac{-1}{2})^{\pm} = 0, \quad g'(\tfrac{1}{2})^{\pm}, g'(\tfrac{-1}{2})^{\pm} < +\infty$$

bien ✓

Por Dini 2, tenemos que $1 - 2|x| = g(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 (2n-1)^2} \cos((2n-1) 2\pi x) + \frac{1}{2}$ para $x \in [-\frac{1}{2}, \frac{1}{2}]$

c) Tomando $x = 0$, tenemos:

$$1 - \frac{1}{2} = \frac{1}{2} = \sum_{n=1}^{\infty} \frac{4}{\pi^2 (2n-1)^2} \cdot \overset{1}{\overset{\shortparallel}{\cos(0)}} \iff \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

✓

d) Si integramos nuestra serie:

$$\int_0^x \sum_{n=1}^{\infty} \frac{1}{2} + \frac{4}{\pi^2 (2n-1)^2} \cos((2n-1) 2\pi t) \, dt = \sum_{n=1}^{\infty} \int_0^x \frac{1}{2} + \frac{4}{\pi^2 (2n-1)^2} \cdot \cos((2n-1) 2\pi t) \, dt =$$

$$\overset{\wedge}{\sum_{n=1}^{\infty} \frac{K}{n^2}} \text{ absolutamente convergente.}$$

✓

$$\frac{x}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 (2n-1)^2} \cdot \left[\frac{\operatorname{sen}((2n-1) 2\pi t)}{(2n-1) 2\pi} \right]_0^x = \sum_{n=1}^{\infty} \frac{4}{\pi^2 (2n-1)^3} \cdot \frac{1}{2\pi} \cdot \operatorname{sen}((2n-1) 2\pi x) + \frac{x}{2} =$$

$$= \int_0^x g(t) \, dt$$

Luego:

$$\sum_{n=1}^{\infty} \frac{\operatorname{sen}(2\pi(2n-1)x)}{(2n-1)^3} = \frac{\pi^3}{2} \cdot \int_0^x f(t)\,dt - \frac{x}{2} = \frac{\pi^3}{2} \int_0^x (1-2|t|)\,dt - \frac{x}{2} = \begin{cases} \frac{\pi^3}{2}\left[(x+x^2) - \frac{x}{2}\right] \text{ si } x \in \left[\frac{-1}{2}, 0\right] \\ \frac{\pi^3}{2}\left[(x-x^2) - \frac{x}{2}\right] \text{ si } x \in \left[0, \frac{1}{2}\right) \end{cases}$$

Si $x \in (0, \frac{1}{2})$:

$$\frac{\pi^3}{2} \int_0^x (1-2|t|)\,dt = \frac{\pi^3}{2}(x-x^2)$$

Si $x \in (-\frac{1}{2}, 0)$

$$-\frac{\pi^3}{2} \int_x^0 1+2t\,dt = \frac{-\pi^3}{2}(-x-x^2) = \frac{\pi^3}{2}(x+x^2)$$