MAYO 2020

2 a) Usando $\operatorname{sen} A \operatorname{sen} B = \dfrac{\cos(A-B) - \cos(A+B)}{2}$

$\ominus \to c(A+B) = cAcB - sAsB$
$c(A-B) = cAcB + sAsB$
$c(A-B) - c(A+B) = 2\, sAsB$

$$\Rightarrow \int_0^{1/2} \operatorname{sen}(2\pi n x) \operatorname{sen}(2\pi m x)\, dx =$$

$$= \int_0^{1/2} \frac{\cos(2\pi(n-m)x) - \cos(2\pi(n+m)x)}{2}\, dx$$

$$\overset{n\neq m}{=} \frac{1}{2} \left[\frac{\operatorname{sen}(2\pi(n-m)x)}{2\pi(n-m)} - \frac{\operatorname{sen}(2\pi(n+m)x)}{2\pi(n+m)} \right]_0^{1/2} = 0$$

$(n=m)$

$$= \frac{1}{2} \left[x - \frac{\operatorname{sen}(2\pi(2n)x)}{2\pi\, 2n} \right]_0^{1/2} = 1/4. \to \|\varphi_n\|_{L^2} = \frac{1}{2}$$

b) Si $f(x) \in L^2(0,1/2)$ considero su extensión IMPAR

$$\tilde{f}(x) = \begin{cases} f(x) & , 0<x<1/2 \\ -f(-x) & , -\frac{1}{2}<x<0 \end{cases}$$

- $\displaystyle\int_0^{1/2} f(x) \operatorname{sen}(2\pi n x)\, dx = 0 \quad \forall n \geq 1 \Rightarrow \int_{-1/2}^{1/2} \tilde{f}(x) \operatorname{sen}(2\pi n x)\, dx = \int_0^{1/2} + \int_{-1/2}^{0} = -\int_0^{1/2} f(-x) \operatorname{sen}(2\pi n x)\, dx = \int_0^{1/2} f(x) \operatorname{sen}(2\pi n x)\, dx = 0$

- $\displaystyle\int_{-1/2}^{1/2} \tilde{f}(x) \cos(2\pi n x)\, dx = 0 \quad \forall n = 0, 1, 2, \dots$ por ser $\tilde{f}(x)\cos(2\pi n x)$ IMPAR

$$\Rightarrow \left.\begin{matrix}\hat{f}(n) = 0 \\ \forall n \in \mathbb{Z}\end{matrix}\right\} \Rightarrow \left.\begin{matrix}\tilde{f}(x) \equiv 0 \\ x \in (-\frac{1}{2}, \frac{1}{2})\end{matrix}\right\} \Rightarrow f(x) \equiv 0, \quad x \in (0, 1/2).$$

c) El sistema $\{2 \operatorname{sen}(2\pi m x)\}_{m=1}^{\infty}$ es BON $L^2(0,1/2)$

Por tanto, toda $f \in L^2(0,1/2)$ puede escribirse como

$$f(x) = L^2\text{-}\sum_{m=1}^{\infty} \langle f, \varphi_m \rangle \frac{\varphi_m(x)}{\|\varphi_m\|_{L^2}^2} = L^2\text{-}\sum_{m=1}^{\infty} 4 \int_0^{1/2} f(y) \operatorname{sen}(2\pi m y)\, dy \; \varphi_m(x)$$

$$\Rightarrow \boxed{A_m = 4 \int_0^{1/2} f(y) \operatorname{sen}(2\pi m y)\, dy}$$

2d) $\int_0^1 |f(x)|^2\,dx = \sum_{m=1}^{\infty} \left| \frac{\langle f, \varphi_m \rangle}{\|\varphi_m\|_2} \right|^2 = \sum_{m=1}^{\infty} \left| \frac{A_m/4}{1/2} \right|^2$

$= \frac{1}{4} \sum_{m=1}^{\infty} |A_m|^2 .$

$s(A+B) = sA\,cB + cA\,sB$

$s(A-B) = sA\,cB - cA\,sB$

$s(A+B) + s(A-B) = 2\,sA\,cB$

$\Downarrow$

$\boxed{sA \cdot cB = \frac{s(A+B) + s(A-B)}{2}}$

2c) $f(x) = \cos(2\pi x) \rightarrow$

$A_m = 4 \int_0^{1/2} \operatorname{sen}(2\pi m x) \cos(2\pi x)\,dx$

$= 2 \int_0^{1/2} \left(\operatorname{sen}(2\pi(m+1)x) + \operatorname{sen}(2\pi(m-1)x) \right) dx$

$m \neq 1$ $= -2 \left[\frac{\cos(2\pi(m+1)x)}{2\pi(m+1)} + \frac{\cos(2\pi(m-1)x)}{2\pi(m-1)} \right]_0^{1/2} = -2 \left[\frac{(-1)^{m+1}}{2\pi(m+1)} + \frac{(-1)^{m-1}}{2\pi(m-1)} - \frac{1}{2\pi(m+1)} - \frac{1}{2\pi(m-1)} \right]$

$= \frac{1-(-1)^m}{2\pi} \cdot \frac{m-1+m+1}{m^2-1} + \frac{1}{2\pi} \frac{m-1+m+1}{m^2-1} = \frac{(-1)^m+1}{\pi} \cdot \frac{2m}{m^2-1}$ //.

$m = 1$ $= 2 \int_0^{1/2} \operatorname{sen}(2\pi \cdot 2x)\,dx = -2 \left. \frac{\cos(4\pi x)}{4\pi} \right]_0^{1/2} = -\frac{2}{4\pi}(1-1) = 0$ //

<u>NOTA</u> $A_1 = 0$, $A_1 = \frac{4}{3\pi}$

3 $\begin{cases} u_t = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = \Delta u \\ u_r\big|_{r=1} = 0 \\ u(0,\cdot) = 1 \end{cases}$ ← es como en 1D con extremos aislados //

Busco las soluciones que se escriben de la forma

b $u(t,r,\theta) = U(t,r)\, \mathrm{sen}(2\theta)$

$$U_t\, \cancel{\mathrm{sen}(2\theta)} = U_{rr}\, \cancel{\mathrm{sen}(2\theta)} + \frac{U_r}{r}\, \cancel{\mathrm{sen}(2\theta)} + \frac{U}{r^2}\left(-4\, \cancel{\mathrm{sen}(2\theta)}\right)$$

$$\Rightarrow U_t = U_{rr} + \frac{U_r}{r} - \frac{4U}{r^2}$$

Busco $U(t,r) = T(t)\, R(r)$

$$\Rightarrow T' R = T R'' + \frac{T R'}{r} - 4\frac{TR}{r^2}$$

$$\Rightarrow \frac{T'}{T} = \frac{R'' + R'/r - 4R/r^2}{R} \equiv \mathrm{cte} = -\lambda^2$$

$$\Rightarrow \begin{cases} T' = -\lambda^2 T \longrightarrow T(t) = e^{-\lambda^2 t} \\ r^2 R'' + r R' - 4R = -r^2 \lambda^2 R \\ R'(1) = 0 \end{cases}$$

$$\rightarrow \begin{cases} r^2 R''(r) + r R'(r) + (r^2\lambda^2 - 4) R(r) = 0 \\ R'(1) = 0 \quad \exists\, R(0^+) \end{cases}$$

$$\Rightarrow R(r) = A \cdot J_2(\lambda r) + \cancel{B\, Y_2(\lambda r)} \quad B = 0$$

$$\Rightarrow R'(1) = 0 = A\, J_2'(\lambda 1)\, \lambda \Rightarrow \left.\begin{matrix} \lambda = 0 \text{ or} \\ \lambda \in Z(J_2') \end{matrix}\right\}$$

Sea $Z(J_2') = \{\lambda_0 = 0, \lambda_1, \lambda_2, \dots\}$

$$\Rightarrow \boxed{U(t,r) = A_0 + \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \cdot A_n \cdot J_2(\lambda_n r)}$$

c) $f(r,\theta) = 5r^2 \operatorname{sen}(2\theta) = u(0,r,\theta) = U(0,r)\operatorname{sen}(2\theta)$

$$\Rightarrow U(0,r) = 5r^2 = A_0 + \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} A_n J_2(\lambda_n r)$$

$$\Rightarrow A_n = \frac{\int_0^1 f(r)\, J_2(\lambda_n r)\, r\,dr}{\|J_2(\lambda_n r)\|_r^2}, \quad \boxed{n \geq 1}$$

A_0 - [illegible] por la [illegible] de Neumann

Aquí $$\|J_2(\lambda_n r)\|_1^2 = \int_0^1 J_2(\lambda_n r)^2\, r\,dr = \frac{1}{2}\left[\left(1-\frac{4}{\lambda_n^2}\right) J_2(\lambda_n)^2 + \cancel{J_2'(\lambda_n)^2}\right]$$

$$= \left(1-\frac{4}{\lambda_n^2}\right) J_2(\lambda_n)^2/2$$

$$\int_0^1 5r^2 J_2(\lambda_n r)\, r\,dr = 5\int_0^1 J_2(\lambda_n r)\, r^3\,dr = \frac{5}{\lambda_n} J_3(\lambda_n)$$

$$\Rightarrow A_n = \frac{\frac{5}{\lambda_n} J_3(\lambda_n)}{\left(1-\frac{4}{\lambda_n^2}\right)\frac{J_2(\lambda_n)^2}{2}} = \boxed{\frac{10\, J_3(\lambda_n)}{\left(\lambda_n-\frac{4}{\lambda_n}\right) J_2(\lambda_n)^2}}, \quad n \geq 1$$

①

$$\mathfrak{I}_N(x) = a_N \cdot (N F_N(x))^2 \quad // \quad \int_{\Pi} \mathfrak{I}_N(x)\, dx = 1$$

ⓐ Sabiendo que $F_N(x) = \sum_{|n|\leq N} \left(1-\frac{|n|}{N}\right) e^{2\pi i n x} \in \mathcal{T}_{N-1}$

$$\Rightarrow F_N(x)^2 \in \mathcal{T}_{2N-2} \Rightarrow \mathfrak{I}_N \in \mathcal{T}_{2N-2}.$$

$$\Rightarrow \widehat{\mathfrak{I}}_N(N) = 0 \quad y \quad \widehat{\mathfrak{I}}_N(0) = 1$$

ⓑ $1 = \int_{\Pi} \mathfrak{I}_N = a_N \int_{\Pi} |N F_N(x)|^2 = a_N \cdot N^2 \int F_N^2(x)\, dx \sim a_N N^2 \cdot N$

Hoja 5 § 7

$$\Rightarrow \boxed{a_N \sim 1/N^3}$$

De forma más precisa, usando

$$\int F_N(x)^2\, dx = \sum_{|n|\leq N} \left|1-\frac{|n|}{N}\right|^2 = 1 + 2\sum_{n=1}^{N} \left(1-\frac{n}{N}\right)^2$$

$$= 1 + 2\sum_{m=1}^{N-1} \frac{m^2}{N^2} = 1 + \frac{2}{N^2} \frac{(N-1)\cdot N\cdot(2N-1)}{6}$$

$N-n=m$

$$= \frac{3N + 2N^2 - 3N + 1}{3N} = \frac{2N^2+1}{3N} \Rightarrow a_N \cdot N^2 \cdot \frac{2N^2+1}{3N} = 1$$

$$\Rightarrow a_N = \frac{3}{(2N^2+1)N} \quad //$$

ⓒ Usando la Prop 5 del Núcleo de Fejér,

$$N \cdot F_N(x) = \left(\frac{\operatorname{sen} N\pi x}{\operatorname{sen} \pi x}\right)^2 \leq \min\{N^2, \tfrac{1}{|x|^2}\}$$

Por tanto

$$\mathfrak{I}_N(x) = a_N (N F_N(x))^2 \lesssim \frac{1}{N^3} \cdot \min\{N^4, \tfrac{1}{|x|^4}\}$$

Entonces

$$\int_{\Pi} |x\, \mathfrak{I}_N(x)| = 2\int_0^{1/2} x\, \mathfrak{I}_N(x)\, dx \lesssim \int_{|x|\leq \frac{1}{N}} x \cdot \frac{N^4}{N^3} + \int_{\frac{1}{N}\leq |x| \leq \frac{1}{2}} x \cdot \frac{1}{N^3} \cdot \frac{1}{x^4}\, dx$$

$$= N \cdot \left[\frac{x^2}{2}\right]_0^{1/N} + \frac{1}{N^3}\left[\frac{x^{-2}}{-2}\right]_{1/N}^{1/2} \leq \frac{1}{N} + \frac{1}{2N^3}(N^2 - 4) \leq C/N.$$

② Sea $\omega(\delta, f) := \sup_{\substack{|h| \le \delta \\ x \in T}} |f(x+h) - f(x)|$.

Probar por inducción

$$\omega(N\delta, f) \le N\,\omega(\delta, f)$$

$\boxed{N=1}$ obvio

$\boxed{N=2}$ $\omega(2\delta, f) = \sup_{\substack{|h| \le 2\delta \\ x \in T}} |f(x+h) \pm f(x+\tfrac{h}{2}) - f(x)|$

$$\le \sup_{\substack{|h| \le 2\delta \\ x \in T}} |f(\underbrace{x+h}_{y+\frac{h}{2}}) - f(\underbrace{x+\tfrac{h}{2}}_{y})| + |f(x+\tfrac{h}{2}) - f(x)|$$

$$\le 2\,\omega(\delta, f)$$

Suponer para $N-1$ y probamos N

$$\omega(N\delta, f) = \sup_{\substack{|h| \le N\delta \\ x \in T}} |f(x+h) \pm f(x+\tfrac{h}{N}) - f(x)|$$

$$\le \sup_{\substack{|h| \le N\delta \\ x \in T}} \underbrace{|f(\underbrace{x+h}_{y+\frac{N-1}{N}h}) - f(\underbrace{x+\tfrac{h}{N}}_{y})|}_{\omega((N-1)\delta, f)} + \underbrace{|f(x+\tfrac{h}{N}) - f(x)|}_{\omega(\delta, f)}$$

$$\overset{H.I.}{\le} (N-1)\,\omega(\delta, f) + \omega(\delta, f) = \omega(\delta, f)$$

Si $R > 0$ es un nº real,

$$\omega(R\delta, f) \le \omega((\lfloor R \rfloor + 1)\delta, f) \le (\lfloor R \rfloor + 1)\,\omega(\delta, f)$$

$$\le (R+1)\,\omega(\delta, f)$$

(c)

$$|f * J_N(x) - f(x)| \overset{\int J_N = 1}{=} \left| \int J_N(y)\,[f(x-y) - f(x)]\,dy \right|$$

$$\le \int_{|y|\le \frac{1}{N}} \underbrace{|f(x-y)-f(x)|}_{\le\ \omega(\frac{1}{N}, f)} J_N(y)\,dy + \int_{\frac{1}{N} \le |y| \le \frac{1}{2}} \underbrace{|f(x-y) - f(x)|}_{\omega(|y|, f) = \omega(\frac{N|y|}{N}, f) \overset{(b)}{\le} (N|y|+1)\,\omega(\frac{1}{N}, f)} J_N(y)\,dy$$

$$\le \omega(\tfrac{1}{N}, f)\left(\int_{|y|\le \frac{1}{N}} J_N(y)\,dy + \int_{\frac{1}{N}\le|y|\le \frac{1}{2}} N|y|\, J_N(y)\,dy + \int_{\frac{1}{N} \le |y| \le \frac{1}{2}} J_N(y)\,dy\right)$$

$$\le \omega(\tfrac{1}{N}, f)\cdot \left(1 + \int_{\mathbb{T}} N|y|\, J_N(y)\right)$$

(d) Si $f \in Lip_1(\mathbb{T})$, entonces $|f(x+h) - f(x)| \le M\cdot|h|$ $\forall h, x \in \mathbb{T}$

Por tanto, $\omega(\frac{1}{N}, f) \le M\cdot \frac{1}{N}$.

Usando (c),

$$\sup_{x\in\mathbb{T}} |f(x) - f * J_N(x)| \le M\cdot\frac{1}{N}\cdot\left(1+\int N|y|\,J_N(y)\right)$$

$$\overset{(c)}{\le} \frac{M}{N}\cdot\left(1 + N\frac{C}{N}\right) = M(1+C)\cdot\frac{1}{N} \quad //$$