Variable Compleja, 3º Grado en Matemáticas/4 PES   Test 2, 21/10/2014

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1. Si $f(z) = e^{z^2}$, $z \in \mathbb{C}$

 a) Calcula la parte real y el módulo de $f(z)$

 b) Demuestra que, para cada $x \in \mathbb{R}$ fijo, se tiene $\lim_{y\to\pm\infty} |f(x+iy)| = 0$

 c) Determina cuándo es $f(z) = -1$, y esboza los puntos en $\mathbb{C}$.

a) $f(z) = e^{(x+iy)^2} = e^{x^2-y^2+2xiy} = e^{x^2-y^2} \cdot e^{2xyi} = e^{x^2-y^2}(\cos(2xy) + i\,\mathrm{sen}(2xy))$

$\mathrm{Re}(f) = e^{x^2-y^2}\cos(2xy)$ ✓

$|f(z)| = e^{\mathrm{Re}\, z^2} = e^{x^2-y^2}$. ✓

b) Fijamos x, $\Rightarrow \lim_{y\to\pm\infty} |f(x+iy)| = \lim_{y\to\pm\infty} e^{x^2} \cdot e^{-y^2} = e^{x^2} \cdot \lim_{y\to\pm\infty} \frac{1}{e^{y^2}} = 0$

c) $f(z) = -1 \Leftrightarrow e^{z^2} = -1 = e^{i\pi} \Leftrightarrow e^{z^2 - i\pi} = 1$

$\overset{\text{Teo. 2.5}}{\Leftrightarrow} z^2 - i\pi \in 2\pi i\mathbb{Z} \Leftrightarrow z^2 \in \overset{\text{mal}}{\cancel{3\pi i\mathbb{Z}}}$ $i\pi(2\mathbb{Z}+1)$

$\forall k \in \mathbb{Z} \Rightarrow z^2 = 3\pi i k.$

$z^2 = x^2 - y^2 + 2xyi = 3\pi i k$ → $x^2 - y^2 = 0$
→ $2xy = 3\pi k$ $(2k+1)\pi$ $\Rightarrow x = \frac{3\pi k}{2y}$ (*)

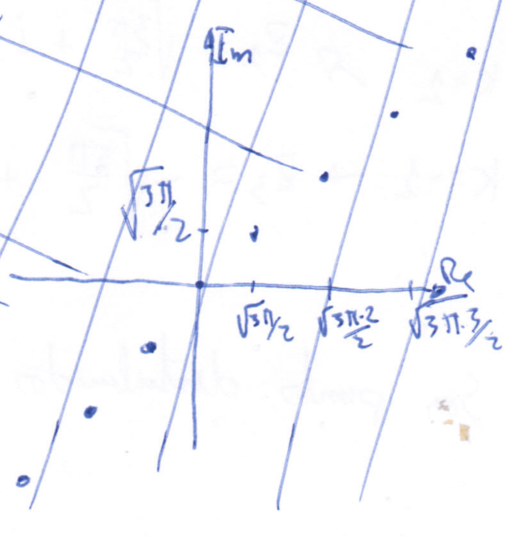
Detrás

~~$\frac{(3\pi k)^2}{4y^2} - y^2 = 0 \Leftrightarrow \frac{(3\pi k)^2}{4} = y^4 \Leftrightarrow y = \pm\sqrt[4]{\frac{(3\pi k)^2}{4}} = \pm\sqrt{\frac{3\pi k}{2}}$; $k \geq 0$~~

~~Si $y = +\sqrt{\frac{3\pi k}{2}} \Rightarrow x = \frac{3\pi k}{2\sqrt{3\pi k/2}} = \sqrt{\frac{3\pi k}{2}}$~~

~~Si $y = -\sqrt{\frac{3\pi k}{2}} \Rightarrow x = -\sqrt{\frac{3\pi k}{2}}$~~

~~$z = \pm\sqrt{\frac{3\pi k}{2}} \pm i\sqrt{\frac{3\pi k}{2}}$, $k \in \mathbb{Z}^+$~~



(*) $\begin{cases} x^2-y^2=0 \\ x=\dfrac{3\pi k}{2y} \end{cases} \Rightarrow \left(\dfrac{3\pi k}{2}\right)^2 = y^4 \Rightarrow y=\pm\sqrt[4]{\left(\dfrac{3\pi k}{2}\right)^2} = \pm\sqrt{\dfrac{3\pi}{2}\cdot|k|}$

necesitas $y \neq 0$

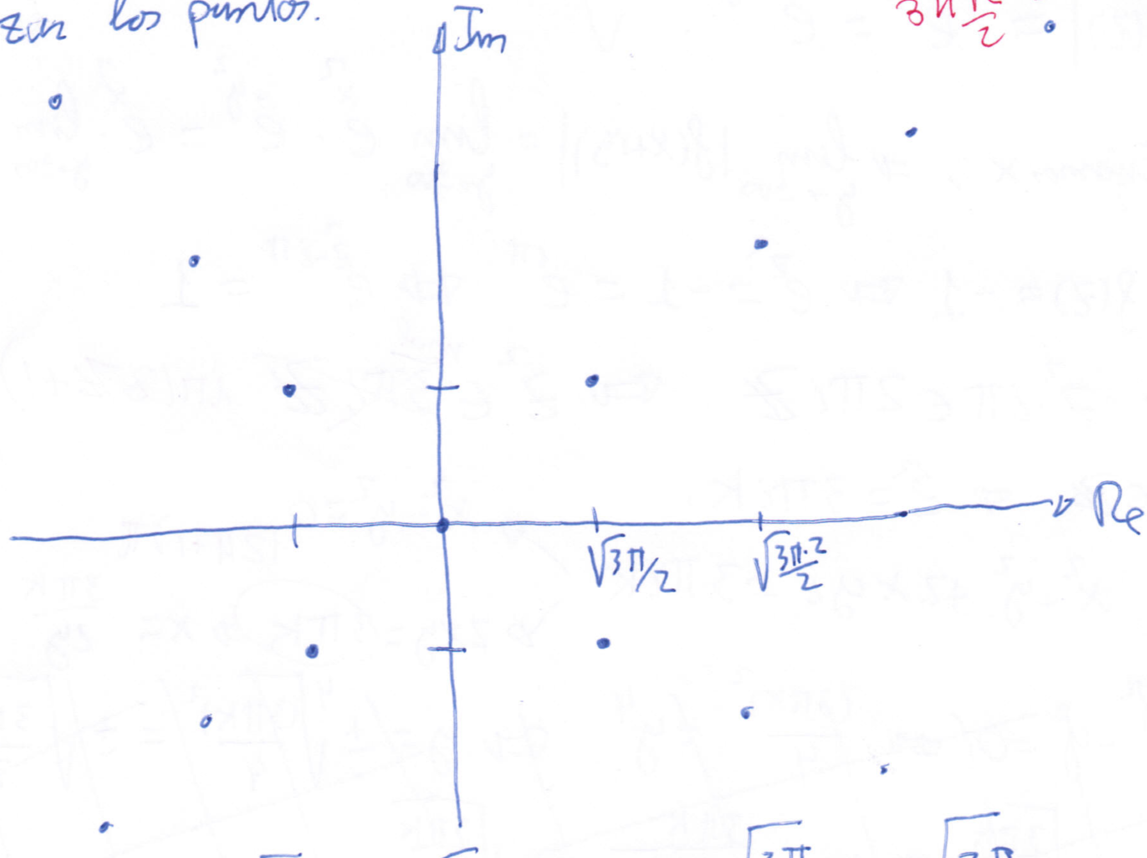
Si $y=+\sqrt{\dfrac{3\pi}{2}\cdot|k|} \Rightarrow x=\dfrac{3\pi k}{2\sqrt{\frac{3\pi}{2}\cdot|k|}} = \text{Signo}(k)\cdot\sqrt{\dfrac{3\pi}{2}\cdot|k|}$

Si $y=-\sqrt{\dfrac{3\pi}{2}\cdot|k|} \Rightarrow x=-\dfrac{3\pi k}{2\sqrt{\frac{3\pi}{2}\cdot|k|}} = -\text{Signo}(k)\cdot\sqrt{\dfrac{3\pi}{2}\cdot|k|}$

Esbozar los puntos:

bien salvo error

$3k\frac{\pi}{2} \to (2k+1)\frac{\pi}{2}$



$k=1 \to z_1=\sqrt{\frac{3\pi}{2}}+i\sqrt{\frac{3\pi}{2}}, \quad z_2=-\sqrt{\frac{3\pi}{2}}-i\sqrt{\frac{3\pi}{2}}$

$k=-1 \to z_3=-\sqrt{\frac{3\pi}{2}}+i\sqrt{\frac{3\pi}{2}}, \quad z_4=\sqrt{\frac{3\pi}{2}}-i\sqrt{\frac{3\pi}{2}}$

$\vdots$

Son puntos distribuidos en forma de aspa.