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Wavelet decompositions of anisotropic Besov spaces

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Abstract. In this paper we develop the natural multiresolution analysis framework related to anisotropic Besov spaces $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$. We prove two new Jackson and Bernstein type inequalities for these spaces, and obtain from well-known techniques [12, 7] new norm equivalences in terms of weighted sums of the wavelet coefficients. This provides a characterization for $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ (and for its dual) by means of compactly supported wavelets, which may be applied to the numerical resolution of semi-elliptic differential equations.

1. Introduction

Wavelet bases are increasingly used in the numerical resolution of elliptic PDE's. Their key role is the characterization of Sobolev and Besov norms by means of weighted sums of the basis coefficients. It is known that such characterizations provide optimal preconditioning of the Galerkin systems and fast numerical methods for the approximate solutions (see, e.g., [24, 3, 9]). In the case of semielliptic differential operators, their symbols have different degrees of smoothness in each direction, and their study requires the use of anisotropic Sobolev and Besov spaces (see [23, 37, 22]). In order to improve the numerical methods involving such operators, one is naturally led to develop the wavelet theory related to the anisotropic spaces $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$.

Since the work of Triebel, the spaces $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ are known to have Littlewood-Paley characterizations and Schauder bases [35, 28]. Other type of wavelet-like or atomic decompositions of theoretical interest have been found more recently (see [4, 26, 21, 18, 17]). The goal of this paper is to provide a new characterization of $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ using the general framework of multiresolution analyses and compactly supported wavelets that will make it more suitable for future applications.

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This will require a review of the machinery of biorthogonal multiresolution analyses in $L^p(\mathbb{R}^n)$ which involves some delicate points. For instance, the existence of compactly supported wavelets in higher dimensions is not yet completely understood, being avoided by most authors with tensor products of 1-dimensional wavelets. Our results are presented in such generality that hold for any compactly supported couple of biorthogonal wavelets with minimal smoothness. We also present the most relevant properties of MRA's in a general setting that we could not find in the current literature, and we think has interest in its own.

Postponing precise definitions to subsequent sections, the main result in this paper can be stated as follows. From now on, $\boldsymbol{a} = (a_1, \dots, a_n)$ will be a fixed *anisotropy* in $\mathbb{R}^n_+$, with "harmonic mean" 1:

$$\frac{1}{a_1} + \ldots + \frac{1}{a_n} = n. \tag{1.1}$$

A matrix M *compatible with* $\boldsymbol{a}$ will be of the form $M = \operatorname{diag}(\lambda^{\frac{1}{a_1}}, \dots, \lambda^{\frac{1}{a_n}})$, for some $\lambda > 1$. We also denote $m = |\det M| = \lambda^n$.

Theorem 1.1. *Let $1 \le p, q < \infty$ and $\boldsymbol{a}$ be as above. Let M be a dilation matrix compatible with $\boldsymbol{a}$, and let $\{\varphi_\gamma, \psi_{\ell;j,\gamma}\}_{\ell,j,\gamma}$, $\{\widetilde{\varphi}_\gamma, \widetilde{\psi}_{\ell;j,\gamma}\}_{\ell,j,\gamma}$ be a pair of biorthogonal admissible M-wavelet bases. Suppose that $\varphi, \widetilde{\varphi} \in B^{\alpha_0 \boldsymbol{a}}_{p,q}(\mathbb{R}^n) \cap H^{L\boldsymbol{a}}(\mathbb{R}^n)$, for some integer $L > \frac{n}{2}$, and $\alpha_0 > 0$. Then, for all $0 < \alpha < \alpha^* := \min\{\alpha_0, \frac{L}{a_1}, \dots, \frac{L}{a_n}\}$, a function $f \in L^p(\mathbb{R}^n)$ belongs to $B^{\alpha\boldsymbol{a}}_{p,q}(\mathbb{R}^n)$ if and only if*

$$\left(\sum_{\gamma\in\mathbb{Z}^n} |\langle f, \varphi_\gamma\rangle|^p\right)^{\frac{1}{p}} + \left(\sum_{j=0}^{\infty}\left[|\det M|^{\frac{\alpha j}{n}} \Big(\sum_{\ell=1}^{m-1}\sum_{\gamma\in\mathbb{Z}^n} |\langle f, \psi_{\ell;j,\gamma}\rangle|^p\Big)^{\frac{1}{p}}\right]^q\right)^{\frac{1}{q}} < \infty. \tag{1.2}$$

In this case, one can write

$$f = \sum_{\gamma\in\mathbb{Z}^n} \langle f, \varphi_\gamma\rangle \widetilde{\varphi}_\gamma + \sum_{j=0}^{\infty}\left[\sum_{\ell=1}^{m-1}\sum_{\gamma\in\mathbb{Z}^n} \langle f, \psi_{\ell;j,\gamma}\rangle \widetilde{\psi}_{\ell;j,\gamma}\right], \tag{1.3}$$

and the norm $\|f\|_{B^{\alpha\boldsymbol{a}}_{p,q}(\mathbb{R}^n)}$ is equivalent to the expression in (1.2).

The techniques we use to prove our result are perhaps known, and amount to the fundamental work of DeVore-Popov [16] (see also [13, 7]). There, it was given an algorithm to characterize the norm of a real interpolation space $Z_{\theta,q} = (V, Z)_{\theta,q}$ when, say $Z \subset V$ and $\{V_j\}_{j\in\mathcal{J}}$ is a *multilevel decomposition* of V. This relies essentially on two inequalities of Bernstein and Jackson type: for some $\rho > 1$ there must hold

$$|v|_Z \le C\rho^j \|v\|_V, \quad \forall v \in V_j, \qquad \text{and} \qquad \|v - P_j v\|_V \le C\rho^{-j}|v|_Z, \quad \forall v \in Z.$$

In this paper, we shall prove these inequalities for general anisotropic Besov spaces $Z = B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$, and with respect to multilevel decompositions arising from compactly supported wavelets. We mention that other types of direct and inverse inequalities related to different approximation spaces do exist in the literature (see Chapter 5 of [30]). However, the ones we give here seem to be new. For technical reasons, we have omitted from this paper the study of the cases $0 < p, q < 1$. These have been treated by R. Hochmuth ([21, 22]) in a slightly different setting which we knew of after the write up of this manuscript. A joint work in preparation using techniques from both papers will cover this situation as well [19].

Our paper will be structured as follows. In §2 we recall the definitions concerning anisotropic Besov spaces. In §3, we give a complete general presentation of biorthogonal multiresolution analyses in $L^p(\mathbb{R}^n)$ and state their main properties. We shall also discuss here the existence of compactly supported wavelets associated with a dilation matrix M compatible with $\boldsymbol{a}$. In §4 we prove the inequalities of Bernstein and Jackson for the spaces $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$, as well as a Whitney type estimate for the modulus of continuity. Section 5 will be devoted to the proof of Theorem 1.1 and its consequences, as well as to remarks concerning generalizations or hints to apply our results to the numerical study of PDE's. Finally, the appendix contains some technical results whose proofs have been postponed for the reader's convenience.

2. Anisotropic Besov spaces

We recall the definition and main properties of anisotropic Besov (and Sobolev) spaces in $\mathbb{R}^n$. Whenever f is a function in $\mathbb{R}^n$, we denote by $\Delta_h^k f$ the difference of order $k \geq 1$ and step $h \in \mathbb{R}^n$, defined as

$$\Delta_h f(x) = f(x+h) - f(x) \qquad \text{and} \qquad (\Delta_h^k f)(x) = \Delta_h(\Delta_h^{k-1} f)(x), \quad x \in \mathbb{R}^n \, .$$

When $\boldsymbol{k} = (k_1, \ldots, k_n)$ is a multi-index with $k_i \geq 0$ we define the *iterated difference of order* $\boldsymbol{k}$ by

$$\Delta_h^{\boldsymbol{k}} f(x) = (\Delta^{k_1}_{h_1 \boldsymbol{e}_1} \circ \ldots \circ \Delta^{k_n}_{h_n \boldsymbol{e}_n}) f(x),$$

where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ denotes the canonical basis of $\mathbb{R}^n$. If $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) > 0$ (that is, each $\alpha_i > 0$) and $1 \leq p, q < \infty$, we say that $f \in L^p(\mathbb{R}^n)$ belongs to the *anisotropic Besov space* $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ if the semi-norm

$$|f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} = \sum_{i=1}^n \Big[\int_0^\infty \Big(t^{-\alpha_i} \|\Delta^{k_i}_{t\boldsymbol{e}_i} f\|_p \Big)^q \frac{dt}{t} \Big]^{1/q}, \tag{2.1}$$

is finite (here $k_1, \ldots, k_n$ are integers such that $k_i > \alpha_i$, $i = 1, \ldots, n$). The norms $\|f\|_p + |f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)}$ are known to be equivalent in the space $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ for any choice of $k_i > \alpha_i$, and will be denoted by $\|f\|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)}$. In fact, if we define the *anisotropic pseudometric*

$$|x|_{\boldsymbol{\alpha}} = \Big(\sum_{i=1}^n |x_i|^{\alpha_i} \Big)^{1/\alpha}, \tag{2.2}$$

where α is the "mean smoothness"

$$\frac{n}{\alpha} = \frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_n}\,, \tag{2.3}$$

then we have the additional equivalence

$$(2.4) \qquad |f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} \approx \sum_{\boldsymbol{k}\in I_{\boldsymbol{\alpha}}} \Big[\int_{(0,\infty)^n} \Big(|h|_{\boldsymbol{\alpha}}^{-\alpha} \|\Delta_h^{\boldsymbol{k}} f\|_p \Big)^q \frac{dh}{|h|_{\boldsymbol{\alpha}}^n} \Big]^{1/q} .$$

In the above sum $I_{\boldsymbol{\alpha}}$ can be any finite set of multi-indices $\boldsymbol{k} = (k_1, \ldots, k_n)$ with the property $\frac{k_1}{\alpha_1} + \cdots + \frac{k_n}{\alpha_n} > 1$, and so that $I_{\boldsymbol{\alpha}}$ at least contains one coordinate element $(0, \ldots, k_i, \ldots, 0)$ for each $i \in \{1, \ldots, n\}$. These and other equivalent norms, as well as further properties of these spaces have been studied by Schmeisser and Triebel in [34, 33] and in §2.2.2 of [35].

As a special case, when $p = q = 2$, then $B^{\boldsymbol{\alpha}}_{2,2}(\mathbb{R}^n)$ coincides with the *anisotropic Bessel-potential space* $H^{\boldsymbol{\alpha}}(\mathbb{R}^n)$ of functions $f \in L^2(\mathbb{R}^n)$ such that

$$\|f\|_{H^{\alpha}(\mathbb{R}^n)} = \sum_{i=1}^{n} \|(1+|\xi_i|^2)^{\alpha_i/2} \hat{f}(\xi)\|_{L^2(\mathbb{R}^n)} \approx \|(1+|\xi|_{\boldsymbol{\alpha}}^2)^{\alpha/2} \hat{f}(\xi)\|_{L^2(\mathbb{R}^n)} < \infty$$

(see §4.2 in [35]). Furthermore, when $\boldsymbol{\alpha} = \boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$ is an integer-valued multi-index, one obtains the usual *anisotropic Sobolev spaces*

$$B^{\boldsymbol{\ell}}_{2,2}(\mathbb{R}^n) = H^{\boldsymbol{\ell}}(\mathbb{R}^n) = W^{\boldsymbol{\ell}}_2(\mathbb{R}^n) = \Big\{ f \in L^2(\mathbb{R}^n) : \sum_{i=1}^{n} \Big\| \frac{\partial^{\ell_i} f}{\partial x_i^{\ell_i}} \Big\|_2 < \infty \Big\} .$$

Finally, the isotropic Besov spaces $B^{\alpha}_{p,q}(\mathbb{R}^n)$ are nothing but $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ when $\boldsymbol{\alpha} = (\alpha, \ldots, \alpha)$.

The following real interpolation result of anisotropic Besov spaces will be crucial in the sequel: for all $0 < \theta < 1$

$$(2.5) \qquad \big(L^p(\mathbb{R}^n), B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n) \big)_{\theta,q} = B^{\theta\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n) .$$

For a proof see, e.g., Theorem 3 in [34].

3. Biorthogonal scaling functions and MRA's in $L^p(\mathbb{R}^n)$

In this section we introduce the notation and main properties of multiresolution analyses in $L^p(\mathbb{R}^n)$. Although most of the results are known in the wavelet folklore, we could not find a reference where these are presented all together in such generality. In what follows, the word "admissible" will mean in the sense it is defined in the next paragraph.

3.1. Admissible M-scaling functions

Let M be a *dilation matrix* in $\mathcal{M}_n(\mathbb{Z})$. That is, M has integer entries and all its eigenvalues lie outside the closed unit disk. We denote $m := |\det(M)|$.

Let φ belong to $H^s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$ (and hence $\varphi \in C(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $1 \le p \le \infty$). Then, φ is said to be an *admissible M-scaling function* if

(i) φ is compactly supported and $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$;

(ii) there exists a (finite) sequence of complex numbers $(h_k)_{k\in\mathbb{Z}^n}$ such that

$$\varphi(x) = \big|\det M\big|^{1/2} \sum_{k\in\mathbb{Z}^n} h_k \varphi(Mx-k)\,, \qquad x\in\mathbb{R}^n\,;$$

(iii) $\{\varphi(\cdot-k)\}_{k\in\mathbb{Z}^n}$ is a Riesz basis for the space it spans; i.e., for some $C>0$

$$\frac{1}{C}\sum_{k\in\mathbb{Z}^n}|a_k|^2 \le \|\sum_{k\in\mathbb{Z}^n} a_k\varphi(\cdot-k)\|^2_{L^2(\mathbb{R}^n)} \le C\sum_{k\in\mathbb{Z}^n}|a_k|^2\,.$$

We will often resume the above equivalence in the form

$$\Big(\sum_{k\in\mathbb{Z}^n}|a_k|^2\Big)^{1/2} \approx \|\sum_{k\in\mathbb{Z}^n} a_k\varphi(\cdot-k)\|_{L^2(\mathbb{R}^n)}\,.$$

Further, two such functions $\varphi,\widetilde{\varphi}$ are said to be an *admissible pair of biorthogonal M-scaling functions* if, in addition to (i)-(iii), the systems $\{\varphi(\cdot-k)\}_{k\in\mathbb{Z}^n}$ and $\{\widetilde{\varphi}(\cdot-k)\}_{k\in\mathbb{Z}^n}$ are biorthogonal, i.e.,

$$\int_{\mathbb{R}^n}\varphi(x)\overline{\widetilde{\varphi}(x-k)}\,dx = \delta_{0k}\,, \qquad \forall k\in\mathbb{Z}^n\,. \tag{3.1}$$

The existence of compactly supported scaling functions in $\mathbb{R}^n$ for an arbitrary dilation matrix M is a delicate matter subject of current research. For the moment, only some concrete examples when M has a relatively simple form are known (see *e.g.*, [2, 32, 20]). In this paper we shall only be interested in diagonal matrices $M=\operatorname{diag}(m_1,\dots,m_n)$, for suitable integers $m_i\ge 2$. Then, admissible M-scaling functions can be easily constructed from the 1-dimensional situation by just considering appropriate tensor products. Indeed, for each m_i, $i=1,\dots,n$, there exist admissible biorthogonal m_i-scaling functions $\varphi_i,\widetilde{\varphi}_i\in H^s(\mathbb{R})$ for real numbers s as large as desired. These constructions for $m_i=2$ are well known (see [10, 27]). The general case $m_i\ge 2$ can be found in [36]. Then we may define

$$\varphi(x)=\varphi_1(x_1)\cdots\varphi_n(x_n) \qquad\text{and}\qquad \widetilde{\varphi}(x)=\widetilde{\varphi}_1(x_1)\cdots\widetilde{\varphi}_n(x_n)\,, \tag{3.2}$$

which trivially satisfy conditions (i)-(iii), (3.1) above and, in addition, $\varphi,\widetilde{\varphi}\in H^s(\mathbb{R}^n)$. For the results we wish to present in this paper, any couple constructed as above will suffice. However, our results hold for more general scaling functions that fit into the definition of admissibility. The requirement will be that they satisfy the following additional properties (for a fixed integer $L>\frac{n}{2}$ to be determined later on):

(iv) The span $\{\varphi(\cdot-k)\}_{k\in\mathbb{Z}^n}$ contains locally the polynomials Π_{L-1} of degree $\le L-1$:

$$\Pi_{L-1}=\operatorname{span}\{x^{\boldsymbol{\ell}}=x_1^{\ell_1}\cdots x_n^{\ell_n} \;:\; |\boldsymbol{\ell}|=\ell_1+\ldots+\ell_n\le L-1\}.$$

(v) If $1\le p<\infty$, the sequence of spaces

$$V_j^{(p)}=\overline{\operatorname{span}}_{L^p(\mathbb{R}^n)}\{\varphi_{j,k}^{(p)}:=|\det M|^{j/p}\varphi(M^j\cdot-k) : k\in\mathbb{Z}^n\}\,, \qquad j\in\mathbb{Z}\,,$$

constitutes a *multiresolution analysis* (MRA) of $L^p(\mathbb{R}^n)$. That is,

(1) $V_j^{(p)} \subset V_{j+1}^{(p)}$, $j \in \mathbb{Z}$;

(2) $f(x) \in V_j^{(p)} \iff f(Mx) \in V_{j+1}^{(p)}$, $j \in \mathbb{Z}$;

(3) $\cup_{j\in\mathbb{Z}} V_j^{(p)}$ is dense in $L^p(\mathbb{R}^n)$ and $\cap_{j\in\mathbb{Z}} V_j^{(p)} = \{0\}$;

(4) $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}^n}$ is a *uniformly p-stable basis* of $V_0^{(p)}$, in the sense that

$$V_0^{(p)} = \Big\{ \sum_k \alpha_k \varphi(\cdot - k) : (\alpha_k) \in \ell^p(\mathbb{Z}^n) \Big\}$$

and

$$\| \sum_k \alpha_k \varphi(\cdot - k) \|_{L^p(\mathbb{R}^n)} \approx \Big(\sum_k |\alpha_k|^p \Big)^{1/p} .$$

(vi) The same properties as in (v) hold for $p = \infty$, where now $L^p(\mathbb{R}^n)$ is replaced by $C_0(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) \mid \lim_{|x|\to\infty} f(x) = 0\}$, and $\ell^p(\mathbb{Z}^n)$ by the corresponding sequence space $c_0(\mathbb{Z}^n) = \{(a_k)_k \in \ell^\infty(\mathbb{Z}^n) \mid \lim_{|k|\to\infty} a_k = 0\}$.

These properties are well-known in the general MRA theory (see Chapter 2 of [29]), and turn out to follow from the admissibilty and sufficient smoothness of the scaling functions φ and $\widetilde{\varphi}$. In fact, a standard argument shows that (v) and (vi) follow from the only assumptions $\varphi, \widetilde{\varphi} \in L_c^\infty(\mathbb{R}^n)$, as we illustrate in the appendix. In what concerns to (iv), the extension to higher dimensions requires some extra care. The next result is all we shall need, and it follows immediately from Theorem 2.1 and Proposition 2.1 in [11].

Proposition 3.1. : see [11].
Let $L > \frac{n}{2}$ and $M = \mathrm{diag}\,(\lambda^{\frac{1}{a_1}}, \dots, \lambda^{\frac{1}{a_n}})$, for some $\lambda > 1$. Let φ be an admissible compactly supported M-scaling function belonging to $H^{L\boldsymbol{a}}(\mathbb{R}^n)$. Then, property (iv) holds.

3.2. The multilevel decomposition and wavelets

We now recall how wavelet bases can be obtained from biorthogonal multiresolution analyses in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$. The general idea is well known and does not differ much from the 1-dimensional situation (see, e.g., Chapter 3 in [8]). We state it here for notational purposes, leaving the proofs, more or less routinary, to the reader. We point out, however, that deeper results are used when we quote the general existence of compactly supported biorthogonal wavelets. In what follows $\frac{1}{p} + \frac{1}{p'} = 1$, so that p' may take the value ∞.

Given a pair of biorthogonal MRA's $\{V_j^{(p)}\}$ and $\{\widetilde{V}_j^{(p')}\}$, as in the previous section, one may define the projectors $P_j = P_j^{(p)} : L^p(\mathbb{R}^n) \to V_j^{(p)}$ by

$$P_j f := \sum_{k\in\mathbb{Z}^n} \langle f, \widetilde{\varphi}_{j,k}^{(p')} \rangle \, \varphi_{j,k}^{(p)} , \qquad f \in L^p(\mathbb{R}^n) .$$

Note that $P_j = T^j P_0 T^{-j}$, where $Tf(x) = |\det M|^{1/p} f(Mx)$. In particular, we have the equality of the operator norms $\|P_j\| = \|P_0\|$, $j \in \mathbb{Z}$. From this, and properties (1)-(4) above, it is not difficult to establish the following for all $f \in L^p(\mathbb{R}^n)$:

(5) $\exists\, C > 0$ such that $\|P_j f - f\|_p \le C \inf_{g \in V_j^{(p)}} \|g - f\|_p$;

(6) $\|P_j f - f\|_p \to 0$, as $j \to +\infty$;

(7) when $1 < p < \infty$, then $\|P_j f\|_p \to 0$, as $j \to -\infty$.

We can also define the "detail spaces"

$$W_j^{(p)} := \left\{ f \in V_{j+1}^{(p)} : \langle f, \widetilde{f} \rangle = 0, \, \forall \widetilde{f} \in \widetilde{V}_j^{(p')} \right\}$$

and the corresponding projectors $Q_j = Q_j^{(p)} := P_{j+1} - P_j$, $j \in \mathbb{Z}$. Then one has

$$L^p(\mathbb{R}^n) = V_0^{(p)} \oplus \Big(\bigoplus_{j=0}^{\infty} W_j^{(p)} \Big),$$

which equals $\oplus_{j\in\mathbb{Z}} W_j^{(p)}$ when $p > 1$. That is, every $f \in L^p(\mathbb{R}^n)$ can be written as

$$f = P_0 f + \sum_{j=0}^{\infty} Q_j f \;\; \Big(= \sum_{j \in \mathbb{Z}} Q_j f\Big) \qquad \text{in } L^p(\mathbb{R}^n)\,. \tag{3.3}$$

This equality is often called a *multilevel decomposition* of the function $f \in L^p(\mathbb{R}^n)$ with respect to $\{V_j^{(p)}\}_j$. It is precisely the decomposition underlying equality (1.3) in Theorem 1.1.

Using more sophisticated machinery one can show the existence of compactly supported functions $\psi_1, \dots, \psi_{m-1}, \widetilde{\psi}_1, \dots, \widetilde{\psi}_{m-1} \in H^s(\mathbb{R}^n)$ such that, for each $j \in \mathbb{Z}$, the families $\{\psi_{a;j,k}^{(2)}\}$ and $\{\widetilde{\psi}_{a;j,k}^{(2)}\}$, $k \in \mathbb{Z}^n$, $a = 1, \dots, m-1$, form a pair of biorthogonal Riesz bases of $W_j^{(2)}$ and $\widetilde{W}_j^{(2)}$. This is shown, for instance, in Theorem 4 of Chapter 7 of [25], but the proof is by no means simple. In fact, the topological impediments that appear when $2 < |\det M| \le \frac{n+1}{2}$ can only be overcome in the compactly supported situation with deep algebraic geometry results. Still, these do not guarantee, for instance, that $\psi_a = \widetilde{\psi}_a$ whenever $\varphi = \widetilde{\varphi}$, which forces us to work with biorthogonal bases, rather than orthonormal ones. Once this existence is established, standard arguments of the type we present in the appendix tell us that the systems $\{\psi_{a;j,k}^{(p)}\}_{a,k}$ and $\{\widetilde{\psi}_{a;j,k}^{(p')}\}_{a,k}$ are uniformly p and p'-stable bases of $W_j^{(p)}$ and $\widetilde{W}_j^{(p')}$, respectively. This will let us write the projectors in terms of the basis:

$$Q_j f = \sum_{a=1}^{m-1} \sum_{k \in \mathbb{Z}^n} \langle f, \widetilde{\psi}_{a;j,k}^{(p')} \rangle \, \psi_{a;j,k}^{(p)}, \quad \text{in } L^p(\mathbb{R}^n), \tag{3.4}$$

and obtain

$$\|Q_j f\|_p \approx \Big(\sum_{a,k} |\langle f, \widetilde{\psi}_{a;j,k}^{(p')} \rangle|^p \Big)^{1/p}, \qquad \forall f \in L^p(\mathbb{R}^n)\,. \tag{3.5}$$

Two systems $\{\psi^{(p)}_{a;j,k}\}_{a,j,k}$ and $\{\widetilde{\psi}^{(p')}_{a;j,k}\}_{a,j,k}$ constructed as above are said to be a *pair of (admissible) biorthogonal wavelet bases*, and they may be used to obtain decompositions of many classical function spaces, whose norms can be written as weighted sums of the coefficients in (3.5).

As an example, when $M = 2I_n$ (I_n being the identity matrix in $\mathbb{R}^n$), and the scaling functions are smooth enough, it is a classical result of Y. Meyer (see §6.10 in [29]) that every $f \in B^{\alpha}_{p,q}(\mathbb{R}^n)$ can be written as

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}^n} \langle f, \widetilde{\varphi}^{(p')}_{0,k}\rangle \varphi^{(p)}_{0,k} + \sum_{j=0}^{\infty}\sum_{a=1}^{2^n-1}\sum_{k\in\mathbb{Z}^n} \langle f, \widetilde{\psi}^{(p')}_{a;j,k}\rangle \psi^{(p)}_{a;j,k} \\ &= \sum_{j\in\mathbb{Z}}\sum_{a=1}^{2^n-1}\sum_{k\in\mathbb{Z}^n} \langle f, \widetilde{\psi}^{(p')}_{a;j,k}\rangle \psi^{(p)}_{a;j,k}, \quad (\text{if } p > 1), \end{aligned}$$

and there is the equivalence of (semi)-norms:

$$|f|_{B^{\alpha}_{p,q}(\mathbb{R}^n)} \approx \Big(\sum_{j\in\mathbb{Z}}\sum_{a=1}^{2^n-1} \big(2^{\alpha j}\big(\sum_{k\in\mathbb{Z}^n} |\langle f, \widetilde{\psi}^{(p')}_{a;j,k}\rangle|^p\big)^{1/p}\big)^q\Big)^{1/q}. \tag{3.6}$$

A more complete treatment of this topic in isotropic spaces, with minimal assumptions on the wavelets (not necessarily of compact support) and details on the convergence of the series can be found in [5].

3.3. Matrices compatible with $\boldsymbol{a}$

The existence and properties of multiresolution analyses with dilation matrix M depend very much on the fact that $M \in \mathcal{M}_n(\mathbb{Z})$. Already in 1-dimension, when m is not an integer there exist no wavelets or scaling functions with compact support [1]. This will be a drawback in our construction since we will be limited to the study of anisotropies $\boldsymbol{a} = (a_1, \dots, a_n)$ for which at least one of the corresponding compatible matrices has integer entries. As we shall see, this happens precisely when

$$\left(\frac{1}{a_1}, \dots, \frac{1}{a_n}\right) \in \mu \log \mathbb{Z}^n_+, \quad \text{for some } \mu > 0. \tag{3.7}$$

Although this seems a restriction in the study of anistropic spaces by means of MRA's, in practical cases condition (3.7) is always satisfied (e.g., it holds for all $\boldsymbol{a} \in \mathbb{Q}^n_+$).

We begin by briefly justifying why the matrices $M = \mathrm{diag}\,(\lambda^{1/a_1}, \dots, \lambda^{1/a_n})$ are the only ones *compatible with* $\boldsymbol{a} = (a_1, \dots, a_n)$ for the kind of multilevel decomposition we wish to obtain in $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$. Here $\boldsymbol{\alpha} = \alpha\boldsymbol{a}$, for some $\alpha > 0$. Recall that the projection operators Q_j, from the previous section, can be written as $Q_j = T^j Q_0 T^{-j}$, where $Tf(x) = |\det M|^{1/p} f(Mx)$. In particular, $Q_j T = T Q_{j-1}$, $j \in \mathbb{Z}$. Now, we search for an equality (or equivalence) of seminorms of the form:

$$|f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} = \Big(\sum_{j\in\mathbb{Z}} \big(\lambda^{\alpha j}\|Q_j f\|_p\big)^q\Big)^{1/q}, \tag{3.8}$$

for some $\lambda > 1$. Thus,

$$
(3.9)\qquad
\begin{aligned}
|Tf|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} &= \Big(\sum_{j\in\mathbb{Z}} \big(\lambda^{\alpha j}\|TQ_{j-1}f\|_p\big)^q\Big)^{1/q} \\
&= \lambda^{\alpha}\Big(\sum_{j\in\mathbb{Z}} \big(\lambda^{\alpha(j-1)}\|Q_{j-1}f\|_p\big)^q\Big)^{1/q} = \lambda^{\alpha}|f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)}\,.
\end{aligned}
$$

In other words, the seminorm $|\cdot|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)}$ must satisfy a *homogeneity property* with respect to M. In terms of the pseudometric defined in (2.2), this is equivalent to have $|M\xi|_{\boldsymbol{\alpha}} = \lambda|\xi|_{\boldsymbol{\alpha}}$, $\forall\xi \in \mathbb{R}^n$. But this forces the matrix to be diagonal $M = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$ and $\lambda_1^{\alpha_1} = \dots = \lambda_n^{\alpha_n} = \lambda^{\alpha}$. This establishes our claim in what concerns to matrices compatible with $\boldsymbol{a}$. Note that, in this case, $|\det M|^{1/n} = \lambda$.

In the next lemma we characterize the n-tuples $\boldsymbol{a} = (a_1,\dots,a_n) > 0$ for which integer-valued matrices of the form specified above do exist.

Lemma 3.2. *Let $\boldsymbol{a} = (a_1,\dots,a_n) > 0$. Then, the following conditions are equivalent:*

a) *There exists a number $\lambda > 1$ such that $\lambda^{1/a_1},\dots,\lambda^{1/a_n} \in \mathbb{Z}_+$;*

b) *There exist $r_1,\dots,r_n \in \mathbb{Z}_+$ such that $\frac{a_i}{a_j} = \frac{\log r_j}{\log r_i}$, $\forall i,j = 1,\dots,n$;*

c) *There exists a number $\mu > 0$ such that $(\frac{1}{a_1},\dots,\frac{1}{a_n}) \in \mu\log\mathbb{Z}^n_+$.*

Proof. If a) holds, then $\lambda^{1/a_i} = r_i \in \mathbb{Z}_+$ and

$$\frac{a_i}{a_j} = \frac{\log\lambda^{1/a_j}}{\log\lambda^{1/a_i}} = \frac{\log r_j}{\log r_i}\,, \qquad \forall i,j = 1,\dots,n\,,$$

establishing b).
Suppose now that b) holds and choose $\mu = \frac{1}{a_1\log r_1}$. Then, we have

$$\frac{1}{a_j} = \frac{\log r_j}{a_1\log r_1} = \mu\log r_j\,, \qquad 1 \le j \le n\,,$$

which clearly implies c).
Finally, if c) is true and $r_1,\dots,r_n$ are positive integers such that $\frac{1}{a_i} = \mu\log r_i$, then, letting $\lambda = e^{1/\mu}$, we have

$$\lambda^{1/a_i} = e^{1/\mu a_i} = e^{\log r_i} = r_i \in \mathbb{Z}_+\,,$$

concluding the proof of the lemma. $\square$

4. Direct and inverse inequalities in $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$

In this section, we fix $1 \le p, q < \infty$ and $\alpha > 0$. We let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) = \alpha \boldsymbol{a}$, and consider a dilation matrix $M = \operatorname{diag}(\lambda^{1/a_1}, \ldots, \lambda^{1/a_n})$ compatible with $\boldsymbol{a}$. We take an integer $L > \max\{\alpha_1, \ldots, \alpha_n, \frac{n}{2}\}$, and assume $\varphi, \widetilde{\varphi}$ are a pair of admissible biorthogonal M-scaling functions in $H^{L\boldsymbol{a}}(\mathbb{R}^n)$, so that conditions (i)-(vi) in the previous section hold. Note that a priori these are only known to exist when $\boldsymbol{a}$ satisfies the conditions in Lemma 3.2, although our results hold in general. We shall, in addition, suppose that $\varphi, \widetilde{\varphi}$ belong to $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$, and show the following theorem.

Theorem 4.1. *With the above assumptions, there exists a constant $C > 0$ such that, for all $j \in \mathbb{Z}$, we have the following:*

a) Bernstein-type inequality

$$|f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} \le C\lambda^{\alpha j}\|f\|_p\,, \qquad \forall f \in V_j^{(p)}\,; \tag{4.1}$$

b) Jackson-type inequality

$$\|f - P_j f\|_p \le C\lambda^{-\alpha j}|f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)}\,, \qquad \forall f \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)\,. \tag{4.2}$$

We prove each of the inequalities separately in the next subsections. Note that, by the homogeneity property of the seminorm $|\cdot|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)}$ stated in (3.9), only the case $j = 0$ needs to be shown. For simplicity, we will denote $\varphi(\cdot - \gamma) = \varphi_\gamma, \gamma \in \mathbb{Z}^n$.

4.1. The Bernstein inequality

Let $I = [0,1]^n$ denote the unit cube in $\mathbb{R}^n$ and $I_\gamma = I + \gamma$ when $\gamma \in \mathbb{Z}^n$. Let

$$K_* = K_*(\varphi) = \{k \in \mathbb{Z}^n : \operatorname{supp}\varphi \cap \big[(L+1)I + k\big] \ne \emptyset\}\,. \tag{4.3}$$

The compact support of φ implies that this set must be finite. Now, let us consider $f = \sum_{\gamma \in \mathbb{Z}^n} a_\gamma \varphi_\gamma \in V_0^{(p)}$, so that $\|f\|_p \approx \Big(\sum_\gamma |a_\gamma|^p\Big)^{1/p}$. Then, we use the seminorm of $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ defined by (2.1) in order to show (4.1). We take a vector $h \in \mathbb{R}^n$ such that $|h| \le 1$. Then, we have

$$\begin{aligned}
\|\Delta_h^L f\|_p &= \Big(\sum_\gamma \int_{I+\gamma} |\Delta_h^L f(x)|^p dx\Big)^{1/p} \\
&\le C\Big(\sum_\gamma \sum_{m \in K_*} |a_{\gamma+m}|^p \int_I |\Delta_h^L \varphi_m(x)|^p dx\Big)^{1/p} \\
&\le C' \sup_{m \in K_*} \|\Delta_h^L \varphi_m\|_p \Big(\sum_\gamma |a_\gamma|^p\Big)^{1/p} \le C''\,\|\Delta_h^L\varphi\|_p\,\|f\|_p\,. \qquad (4.4)
\end{aligned}$$

We now compute the $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$-seminorm of f:

$$\begin{aligned}
|f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} &\le \sum_{i=1}^n \Big\{ \Big[\int_0^1 \big(t^{-\alpha_i} \|\Delta^L_{te_i} f\|_p\big)^q \frac{dt}{t} \Big]^{1/q} + \Big[\int_1^\infty \big(t^{-\alpha_i} \|\Delta^L_{te_i} f\|_p\big)^q \frac{dt}{t} \Big]^{1/q} \Big\} \\
&\le C\|f\|_p \sum_{i=1}^n \Big\{ \Big[\int_0^1 \big(t^{-\alpha_i} \|\Delta^L_{te_i} \varphi\|_p\big)^q \frac{dt}{t} \Big]^{1/q} + \Big[\int_1^\infty \frac{dt}{t^{1+\alpha_i q}} \Big]^{1/q} \Big\} \\
&\le C'\|f\|_p \,,
\end{aligned}$$

where in the second inequality we have used (4.4) and the fact that $\|\Delta^L_h f\|_p \le C\|f\|_p$. The Bernstein-type inequality has then been established.

4.2. The Jackson inequality

The heart of the proof is contained in the following two lemmata. We postpone their proofs to the next subsection.

Lemma 4.2. *There exist a constant $C > 0$ such that for all $f \in L^p(\mathbb{R}^n)$ and $\gamma \in \mathbb{Z}^n$, then*

$$\|P_0 f\|_{L^p(I_\gamma)} \le C\|f\|_{L^p(J_\gamma)} \,,$$

where $J_\gamma = \cup_{m\in K_\#}(I_{\gamma+m})$, and $K_\#$ is a finite set in $\mathbb{Z}^n$ independent of γ or f.

Roughly speaking this result says that P_0 is a "local" operator; that is, the size of $P_0 f$ at a given hypercube is controlled by the values of f in a slightly bigger hypercube.

The second lemma we are about to state, reminds Whitney's inequality from approximation theory.

Lemma 4.3. *Let $f \in L^p(\mathbb{R}^n)$ and $J = [a_1, b_1] \times \ldots \times [a_n, b_n]$ be a rectangle in $\mathbb{R}^n$. Then, for every $L \ge 1$ there exists a polynomial Q in $\mathbb{R}^n$ of degree $\le L-1$ such that*

$$\|f - Q\|_{L^p(J)} \le C \sum_{\ell_1+\ldots+\ell_n=L} \frac{1}{|J|} \int_0^{b_1-a_1} \cdots \int_0^{b_n-a_n} \|\Delta^{\ell_1}_{h_1} \circ \ldots \circ \Delta^{\ell_n}_{h_n} f\|_{L^p(\widetilde{J})} dh_1 \ldots dh_n \,, \tag{4.5}$$

where $\widetilde{J} = J + L \cdot [0, b_1 - a_1] \times \ldots \times [0, b_n - a_n]$, and $C = C(n, p, L) > 0$.

Assuming the lemmata, we prove now (4.2) for the case $j = 0$. Let $\gamma \in \mathbb{Z}^n$ be fixed and define J_γ as in Lemma 4.2. Associated with this cube and $f \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$, let Q be the polynomial of degree $\le L-1$ whose existence was guaranteed in Lemma 4.3. Now, using property (iv) from the previous section, we can find a function $g \in V_0$ such that $g_{|J_\gamma} = Q$. Then, we have

$$\begin{aligned}
\|f - P_0 f\|_{L^p(I_\gamma)} &\le \|f - Q\|_{L^p(I_\gamma)} + \|g - P_0 f\|_{L^p(I_\gamma)} \\
&\le \|f - Q\|_{L^p(I_\gamma)} + C\|g - f\|_{L^p(J_\gamma)} \\
&\le C\|f - Q\|_{L^p(J_\gamma)} \,,
\end{aligned}$$

where in the second inequality we have used Lemma 4.2. Let us denote by $|\boldsymbol{\ell}| = \ell_1 + \ldots + \ell_n$, the length of the multi-index $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$, and observe that the measure of J_γ is a constant (say c^n) independent of $\gamma \in \mathbb{Z}^n$. Then, the previous inequality and Lemma 4.3 give us

$$\begin{aligned}
\sum_{\gamma\in\mathbb{Z}^n} \|f - P_0 f\|^p_{L^p(I_\gamma)} &\leq C \sum_\gamma \sum_{|\boldsymbol{\ell}|=L} \Big[\frac{1}{c^n}\int_{[0,c]^n} \|\Delta_h^{\boldsymbol{\ell}} f\|_{L^p(\widetilde{J}_\gamma)} dh\Big]^p \\
&\leq C\Big[\frac{1}{c^n}\int_{[0,c]^n} \Big(\sum_{|\boldsymbol{\ell}|=L}\sum_{\gamma\in\mathbb{Z}^n} \|\Delta_h^{\boldsymbol{\ell}} f\|^p_{L^p(\widetilde{J}_\gamma)}\Big)^{1/p} dh\Big]^p \\
&\leq C'\Big[\sum_{|\boldsymbol{\ell}|=L} \frac{1}{c^n}\int_{[0,c]^n} \|\Delta_h^{\boldsymbol{\ell}} f\|_{L^p(\mathbb{R}^n)} dh\Big]^p .
\end{aligned}$$

Hence

$$\begin{aligned}
\|f - P_0 f\|_{L^p(\mathbb{R}^n)} &\leq C \sum_{|\boldsymbol{\ell}|=L} \frac{1}{c^n}\int_{[0,c]^n} \|\Delta_h^{\boldsymbol{\ell}} f\|_{L^p(\mathbb{R}^n)} dh \\
&\leq C \sum_{|\boldsymbol{\ell}|=L} \Big[\frac{1}{c^n}\int_{[0,c]^n} \|\Delta_h^{\boldsymbol{\ell}} f\|^q_{L^p(\mathbb{R}^n)} dh\Big]^{1/q} \\
&= C \sum_{|\boldsymbol{\ell}|=L} \Big[\frac{1}{c^n}\int_{[0,c]^n} \frac{|h|^{n+\alpha q}_{\boldsymbol{\alpha}}}{|h|^{n+\alpha q}_{\boldsymbol{\alpha}}} \|\Delta_h^{\boldsymbol{\ell}} f\|^q_{L^p(\mathbb{R}^n)} dh\Big]^{1/q} \\
&\leq C' \, |f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} .
\end{aligned}$$

In the last step we have used the definition of seminorm for $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ given in (2.4) and the fact that $\frac{\ell_1}{\alpha_1} + \ldots + \frac{\ell_n}{\alpha_n} \geq \frac{\ell_1+\ldots+\ell_n}{\alpha_{\max}} > 1$. This establishes Jackson's inequality modulo the lemmata, which will be shown in the next subsection.

4.3. The proof of the lemmata

We begin with the proof of Lemma 4.2, where we make use of $\widetilde{\varphi}$, the dual scaling function of φ, and the important fact that it has compact support as well.

Proof of Lemma 4.2:
Let $f \in L^p(\mathbb{R}^n)$ and write $P_0 f = \sum_m a_m \varphi_m$, where $a_m = \int_{\mathbb{R}^n} f(x)\overline{\widetilde{\varphi}_m(x)}\, dx$, $m \in \mathbb{Z}^n$. Then, recalling (4.3), we obtain

$$\begin{aligned}
\|P_0 f\|_{L^p(I_\gamma)} &= \|\sum_{m\in K_*} a_{\gamma+m}\varphi_{\gamma+m}\|_{L^p(I_\gamma)} \\
&\leq \sum_{m\in K_*} |a_{\gamma+m}| \|\varphi_m\|_{L^p(I)} \\
&\leq \|\varphi\|_p \sum_{m\in K_*}\sum_{\delta\in\mathbb{Z}^n} \int_{I_\delta} |f(x)||\widetilde{\varphi}_{\gamma+m}(x)| dx \\
&\leq C\|\widetilde{\varphi}\|_\infty \sum_{m'\in K_\#} \int_{I_\gamma+m'} |f(x)| dx \leq C' \|f\|_{L^p(J_\gamma)} ,
\end{aligned}$$

where we have taken $K_\# = K_*(\varphi) - K_*(\widetilde{\varphi})$.□

Proof of Lemma 4.3:
We will proceed in two steps. First, we show a "weak" version of (4.5) in which the right hand side is bounded by an L^∞-norm of the p-modulus of continuity. In the second step, we show that this L^∞-norm is actually equivalent to an L^1-norm, obtaining (4.5) and establishing the lemma. The ideas involved in the proof of these assertions come mostly from the 1-dimensional version that appears in various texts (see Theorem 7.1 and Lemma 7.2 in [31], or Theorem 6.2.4 in [15]). For completeness, we shall sketch the proof in the 2-dimensional case, since this constitutes a key step for all the results presented in this paper. However, we point out that inequality (4.6) below is actually a particular situation of the n-dimensional Whitney inequality proved by Brudnyi in [6].

Step 1. *In the conditions of Lemma* 4.3 *there exists a polynomial* Q *in* $\mathbb{R}^n$ *with* $\deg(Q) \leq L-1$ *and such that*

$$\|f - Q\|_{L^p(J)} \leq C \sup_{0<h_i<b_i-a_i} \sum_{\ell_1+\ldots+\ell_n=L} \|\Delta^{\ell_1}_{h_1 \mathbf{e}_1} \circ \ldots \circ \Delta^{\ell_n}_{h_n \mathbf{e}_n} f\|_{L^p(\tilde{J})}\,, \tag{4.6}$$

for a constant $C = C(n,p,L) > 0$.

To sketch the proof we shall assume $n = 2$ and $J = [0,1]^2$. The idea is to first approximate f with a *Steklov-type function*:

$$g_{L,t}(x,y) = f(x,y) - \frac{(-1)^L}{t^L} \int_0^t \cdots \int_0^t (\Delta^L_{(s_1+\ldots+s_L)\mathbf{e}_1} f)(x,y)\, ds_1 \ldots ds_L,$$

where $(x,y) \in [0,1]^2$ and $0 < t \leq \frac{1}{L}$. This function verifies:

$$\|f - g_{L,t}\|_{L^p(J)} \leq \sup_{0<h<1} \|\Delta^L_{h\mathbf{e}_1} f\|_{L^p(J)} \tag{4.7}$$

and

$$\frac{\partial^L g_{L,t}}{\partial x^L}(x,y) = -\sum_{\ell=1}^{L} \frac{(-1)^\ell \binom{L}{\ell}}{(t\ell)^L} (\Delta^L_{t\ell\mathbf{e}_1} f)(x,y)\,. \tag{4.8}$$

In particular, if we approximate $g_{L,t}$ by its Taylor polynomial in the first variable of degree $L-1$:

$$T(x,y) = g_{L,t}(\frac{1}{2},y) + \sum_{\ell=1}^{L-1} \frac{\partial^\ell g_{L,t}}{\partial x^L}(\frac{1}{2},y)\, \frac{(x-\frac{1}{2})^\ell}{\ell!}\,,$$

it follows using the integral remainder that:

$$\|f - T\|_{L^p(J)} \leq \|f - g_{L,t}\|_{L^p(J)} + \|g_{L,t} - T\|_{L^p(J)} \leq C \sup_{0<h<1} \|\Delta^L_{h\mathbf{e}_1} f\|_{L^p(2J)}. \tag{4.9}$$

To obtain from here a polynomial in two variables we write : $T(x,y) = \sum_{\ell=0}^{L-1} \beta_\ell(y)(x-\frac{1}{2})^\ell$, and observe that $\beta_\ell(y) = \frac{1}{\ell!}\frac{\partial^\ell g_{L,t}}{\partial x^\ell}(\frac{1}{2},y) \in L^p([0,1],dy)$. Thus, using the 1-dimensional version of Whitney's inequality (see Theorem 7.1 in [31]) for each $0 \leq$

$\ell \leq L-1$, we find a polynomial $Q_\ell(y)$ of degree $\leq L-\ell-1$ so that:

$$\|\beta_\ell - Q_\ell\|_{L^p([0,1],dy)} \leq C \sup_{0<s<\frac{1}{L-\ell}} \|\Delta_s^{L-\ell}\beta_\ell\|_{L^p([0,1],dy)}\,. \tag{4.10}$$

An elementary estimate on $\beta_\ell(y) = \frac{1}{\ell!}\frac{\partial^\ell g_{L,t}}{\partial x^\ell}(\frac{1}{2}, y)$, using the definition of $g_{L,t}$, leads easily to:

$$\|\Delta_s^{L-\ell}\beta_\ell\|_{L^p([0,1],dy)} \leq C \sup_{0<h_1,h_2<1} \|\Delta_{h_2\boldsymbol{e}_2}^{L-\ell} \circ \Delta_{h_1\boldsymbol{e}_1}^{\ell} f\|_{L^p((L+1)J)}\,.$$

Thus, the polynomial $Q(x,y) = \sum_{\ell=0}^{L-1} Q_\ell(y)(x-\frac{1}{2})^\ell$ will satisfy (4.6).

Remark 4.4. We point out that it is possible to approach f with other families of polynomials different from Π_{L-1} (see, e.g., [14] or [22]). In this case, the modulus of continuity in (4.6) has to be replaced accordingly, but always leading to a term that is part of an anisotropic Besov space seminorm. This sharper approach, however, has the difficulty that those families of polynomials may not be reproduced by the scaling functions in §3, which is a crucial step in the proof of the Jackson inequality in §4.2.

Step 2. *For every $\boldsymbol{k} = (k_1, \ldots k_n) \geq 1$, $\delta_1, \ldots, \delta_n > 0$ and every rectangle $J \subset \mathbb{R}^n$,*

$$\sup_{0<h_i<\delta_i} \|\Delta_h^{\boldsymbol{k}} f\|_{L^p(J)} \leq \frac{C}{\delta_1 \cdots \delta_n} \int_0^{\delta_1} \cdots \int_0^{\delta_n} \|\Delta_h^{\boldsymbol{k}} f\|_{L^p(J)} dh_1 \ldots dh_n\,, \tag{4.11}$$

where the constant $C = C(n, p, \boldsymbol{k}) > 0$.

The 1-dimensional analog of this result can be found in [31] (see Lemma 7.2). As before, we only show the case $n = 2$. The general case can be easily obtained by iteration.

For every $h = (h_1, h_2) \in [0, \delta_1] \times [0, \delta_2]$ and $y \in [a_2, b_2]$, we may apply the 1-dimensional version of (4.11) to obtain

$$\Big(\int_{a_1}^{b_1} |\Delta_{h_1}^{k_1}(\Delta_{h_2}^{k_2} f)(x,y)|^p dx\Big)^{1/p} \leq \frac{C}{\delta_1} \int_0^{\delta_1} \Big(\int_{a_1}^{b_1} |\Delta_s^{k_1}(\Delta_{h_2}^{k_2} f)(x,y)|^p dx\Big)^{1/p} ds\,.$$

Raising to the power p, integrating with respect to y, and using Minkowski's inequality, we obtain

$$\Big(\int_J |\Delta_h^{\boldsymbol{k}} f(x,y)|^p dxdy\Big)^{1/p} \leq \frac{C}{\delta_1} \int_0^{\delta_1} \Big(\int_J |\Delta_{(s,h_2)}^{\boldsymbol{k}} f(x,y)|^p dxdy\Big)^{1/p} ds\,. \tag{4.12}$$

Now, we look at the inner integral on the right hand side and we apply once again the 1-dimensional result to get, for all $x \in [a_1, b_1]$

$$\Big(\int_{a_2}^{b_2} |\Delta_{(s,h_2)}^{\boldsymbol{k}} f(x,y)|^p dy\Big)^{1/p} \leq \frac{C}{\delta_2} \int_0^{\delta_2} \Big(\int_{a_2}^{b_2} |\Delta_{(s,t)}^{\boldsymbol{k}} f(x,y)|^p dy\Big)^{1/p} dt\,. \tag{4.13}$$

Inserting (4.13) in (4.12), and using Minkowski once more, we obtain (4.11). $\square$

5. The main theorem and its corollaries

In this section we fix $1 \le p, q < \infty$, $\alpha_0 > 0$ and an integer $L > \frac{n}{2}$. As usual, $M = \text{diag}\,(\lambda^{\frac{1}{a_1}}, \dots, \lambda^{\frac{1}{a_n}})$, for some $\lambda > 1$. We assume that $\varphi, \widetilde{\varphi}$ are a pair of admissible biorthogonal M-scaling functions belonging to $B^{\alpha_0 \boldsymbol{a}}_{p,q}(\mathbb{R}^n) \cap H^{L\boldsymbol{a}}(\mathbb{R}^n)$, as in the statement of Theorem 1.1. We now address the representation of functions in $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$, for $\boldsymbol{\alpha} = \alpha \boldsymbol{a}$, $\alpha > 0$, by means of the MRA's generated by φ and $\widetilde{\varphi}$. Using the multilevel decomposition described in (3.3) we can write every $f \in L^p(\mathbb{R}^n)$ as

$$f = P_0 f + \sum_{j=0}^{\infty} Q_j f = \sum_{k \in \mathbb{Z}^n} \alpha_k \varphi^{(p)}_{0,k} + \sum_{j=0}^{\infty} \Big(\sum_{a=1}^{m-1} \sum_{k \in \mathbb{Z}^n} \beta_{ajk} \psi^{(p)}_{a;j,k} \Big) \tag{5.1}$$

where $\alpha_k = \alpha_k(f) = \langle f, \widetilde{\varphi}^{(p')}_{0,k} \rangle$ and $\beta_{ajk} = \beta_{ajk}(f) = \langle f, \widetilde{\psi}^{(p')}_{a;j,k} \rangle$, and the series converge in the $L^p(\mathbb{R}^n)$-norm. Moreover, when $p > 1$, the assertion (7) in §3.2 allows us to write as well

$$f = \sum_{j=-\infty}^{\infty} Q_j f = \sum_{j \in \mathbb{Z}} \Big(\sum_{a=1}^{m-1} \sum_{k \in \mathbb{Z}^n} \beta_{ajk} \psi^{(p)}_{a;j,k} \Big), \qquad \forall f \in L^p(\mathbb{R}^n), \tag{5.2}$$

also with convergence of the series in $L^p(\mathbb{R}^n)$. In particular, these decompositions hold whenever $f \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$.

We now pass to the main result in this paper, where we show that for certain values of α the norm $\|f\|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)}$ can be characterized with a weighted sum of the coefficients in (5.1) and (5.2). We state it in a form slightly more general than Theorem 1.1 above.

Theorem 5.1. *In the conditions at the beginning of this section, if* $0 < \alpha < \alpha^* = \min\{\alpha_0, \frac{L}{a_1}, \dots, \frac{L}{a_n}\}$, *then*

$$\begin{aligned} B^{\alpha \boldsymbol{a}}_{p,q}(\mathbb{R}^n) &= \Big\{ f \in L^p(\mathbb{R}^n) : \sum_{j \in \mathbb{Z}} |\det M|^{\frac{\alpha q}{n} j} \Big(\sum_{a,k} |\beta_{ajk}(f)|^p \Big)^{q/p} < \infty \Big\} \\ &= \Big\{ f \in L^p(\mathbb{R}^n) : \sum_k |\alpha_k(f)|^p + \sum_{j=0}^{\infty} |\det M|^{\frac{\alpha q}{n} j} \Big(\sum_{a,k} |\beta_{ajk}(f)|^p \Big)^{q/p} < \infty \Big\}. \end{aligned}$$

Moreover, we have the equivalence of seminorms

$$\begin{aligned} |f|_{B^{\alpha \boldsymbol{a}}_{p,q}(\mathbb{R}^n)} &\approx \Big(\sum_{j \in \mathbb{Z}} |\det M|^{\frac{\alpha q}{n} j} \Big(\sum_{a,k} |\beta_{ajk}(f)|^p \Big)^{q/p} \Big)^{1/q} \\ \|f\|_{B^{\alpha \boldsymbol{a}}_{p,q}(\mathbb{R}^n)} &\approx \Big(\sum_k |\alpha_k(f)|^p \Big)^{1/p} + \Big(\sum_{j=0}^{\infty} |\det M|^{\frac{\alpha q}{n} j} \Big(\sum_{a,k} |\beta_{ajk}(f)|^p \Big)^{q/p} \Big)^{1/q}. \end{aligned}$$

Proof. The proof is a direct application of Theorem 1 in [7]. Indeed, there it is shown that for a Banach space $Z \subset L^p(\mathbb{R}^n)$ with $\|\cdot\|_Z = \|\cdot\|_p + |\cdot|_Z$ and a multiresolution analysis $\{V_j\}$ satisfying the Bernstein and Jackson inequalities

$$|v|_Z \le C\, \lambda^{\alpha j} \|v\|_p, \quad \forall v \in V_j \qquad \text{and} \qquad \|v - P_j v\|_p \le C\, \lambda^{-\alpha j} |v|_Z, \quad \forall v \in Z,$$

then the real interpolation space $Z_{\theta,q} = (L^p(\mathbb{R}^n), Z)_{\theta,q}$ can be characterized as

$$\begin{aligned} Z_{\theta,q} &= \Big\{ v \in L^p(\mathbb{R}^n) : \sum_{j\in\mathbb{Z}} \left(\lambda^{\theta\alpha j}\|Q_j v\|_p\right)^q < \infty \Big\} \\ &= \Big\{ v \in L^p(\mathbb{R}^n) : \|P_0 v\|_p + \sum_{j=0}^{\infty} \left(\lambda^{\theta\alpha j}\|Q_j v\|_p\right)^q < \infty \Big\}. \end{aligned}$$

Moreover, one has the equivalence of seminorms

$$|v|_{Z_{\theta,q}} \approx \Big(\sum_{j\in\mathbb{Z}} \left(\lambda^{\theta\alpha j}\|Q_j v\|_p\right)^q \Big)^{1/q} \approx \|P_0 v\|_p + \Big(\sum_{j=0}^{\infty} \left(\lambda^{\theta\alpha j}\|Q_j v\|_p\right)^q \Big)^{1/q}.$$

Notice also that, from the boundedness of P_0 and (5.1):

$$\begin{aligned} \frac{1}{C}\|P_0 v\|_p &\le \|v\|_p \le \|P_0 v\|_p + \sum_{j=0}^{\infty} \|Q_j v\|_p \\ &\le \|P_0 v\|_p + \Big(\sum_{j=0}^{\infty} \left(\lambda^{\theta\alpha j}\|Q_j v\|_p\right)^q \Big)^{1/q} \Big(\sum_{j=0}^{\infty} \lambda^{-\theta\alpha q' j} \Big)^{1/q'}, \end{aligned}$$

where q' is the conjugate exponent of q. Then, it easily follows that

$$\|v\|_{Z_{\theta,q}} \approx \|P_0 v\|_p + \Big(\sum_{j=0}^{\infty} \left(\lambda^{\theta\alpha j}\|Q_j v\|_p\right)^q \Big)^{1/q}.$$

Now, with the hypotheses above, we have $L > \max\{\alpha a_1, \ldots, \alpha a_n\}$, so that the Bernstein and Jackson inequalities in Theorem 4.1 hold. Thus, when $0 < \alpha < \alpha^*$, an application of the previous results to the space $Z = B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$, together with the interpolation property (2.5), and the equivalences in (4) and (3.5), establish our theorem. □

Particularly interesting for applications are the anisotropic Bessel potential spaces.

Corollary 5.2. *Let $L > \frac{n}{2}$, $\lambda > 1$ and $M = \operatorname{diag}(\lambda^{\frac{1}{a_1}}, \ldots, \lambda^{\frac{1}{a_n}})$. Let $\varphi, \widetilde{\varphi} \in H^{L\boldsymbol{a}}(\mathbb{R}^n)$ be a pair of admissible biorthogonal M-scaling functions. Then, for all $0 < \alpha < L$, a function f belongs to $H^{\alpha\boldsymbol{a}}(\mathbb{R}^n)$ if and only if*

$$\Big(\sum_{j\in\mathbb{Z}} \big(1 + |\det M|^{\frac{2\alpha}{n} j}\big) \sum_{a,k} |\beta_{ajk}(f)|^2 \Big)^{1/2} < \infty.$$

The last expression is equivalent to the norm $\|f\|_{H^{\alpha\boldsymbol{a}}}$.

Proof. We apply Theorem 5.1 with $p = q = 2$, and we observe, by the Riesz basis assumption on the basis $\{\psi_{a;j,k}\}_{a,j,k}$, that $\|f\|_2 \approx \Big(\sum_{a,j,k} |\beta_{ajk}(f)|^2 \Big)^{1/2}$. □

We also obtain the following interesting corollary (compare with the results in [26, 17]).

Corollary 5.3. *Let $\boldsymbol{a} = (a_1, \dots, a_n)$ satisfy the conditions in Lemma 3.2. Let $1 \le p, q < \infty$, $\alpha > 0$ and $\boldsymbol{\alpha} = \alpha \boldsymbol{a}$. Then, there exists $\lambda = \lambda(\boldsymbol{a}) > 1$ and $m = m(\boldsymbol{a}) \ge 2$ such that the space of sequences*

$$b^{\boldsymbol{\alpha}}_{p,q} = \Big\{ s = (\alpha_k, \beta_{ajk}) : \|s\|_{b^{\boldsymbol{\alpha}}_{p,q}} = \Big(\sum_{k \in \mathbb{Z}^n} |\alpha_k|^p \Big)^{1/p} + \Big(\sum_{j=0}^{\infty} \lambda^{\alpha q j} \big(\sum_{a=1}^{m-1} \sum_{k \in \mathbb{Z}^n} |\beta_{ajk}|^p \big)^{q/p} \Big)^{1/q} < \infty \Big\}$$

and the Besov space $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ are isomorphic.

Proof. By Lemma 3.2 we can find an integer valued matrix M, compatible with $\boldsymbol{a}$ and so that $m = \det(M) = \lambda^n$. Then, after choosing appropriate scaling functions Theorem 5.1 tells us that the map $f \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n) \mapsto (\alpha_k(f), \beta_{ajk}(f)) \in b^{\boldsymbol{\alpha}}_{p,q}$ is bounded and one-to-one. We only need to see that it is surjective. But whenever $s = (\alpha_k, \beta_{ajk}) \in b^{\boldsymbol{\alpha}}_{p,q}$ then

$$f_j := \sum_{a=1}^{m-1} \sum_{k \in \mathbb{Z}^n} \beta_{ajk} \psi^{(p)}_{a;j,k} \in W^{(p)}_j \quad \text{and} \quad g = \sum_{k \in \mathbb{Z}^n} \alpha_k \varphi^{(p)}_{0,k} \in V^{(p)}_0 .$$

Moreover, the series $g + \sum_{j=0}^{\infty} f_j$ converges in $L^p(\mathbb{R}^n)$ to a function f, since

$$\begin{aligned} \sum_{j=j_0}^{j_1} \|f_j\|_p &\le C \sum_{j=j_0}^{j_1} \Big(\sum_{a,k} |\beta_{ajk}|^p \Big)^{1/p} \\ &\le C \Big(\sum_{j=j_0}^{j_1} \lambda^{\alpha j q} \big(\sum_{a,k} |\beta_{ajk}|^p \big)^{q/p} \Big)^{1/q} \to 0, \qquad \text{as } j_1 \ge j_0 \to \infty . \end{aligned}$$

It follows from the biorthogonality of the pairs $\varphi, \widetilde{\varphi}$ and $\psi_a, \widetilde{\psi}_a$ that the sequence s equals $(\alpha_k(f), \beta_{ajk}(f))$, and, from Theorem 5.1, that $f \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$. □

Finally, we give a characterization of the dual space $B^{-\boldsymbol{\alpha}}_{p',q'}(\mathbb{R}^n) := (B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n))'$ when $1 < p, q < \infty$. In terms of the biorthogonal MRA's $\{V_j, \widetilde{V}_j\}$, one can express the duality pairing $(L^p, L^{p'})$ as follows:

$$f \in (L^p(\mathbb{R}^n))' = L^{p'}(\mathbb{R}^n) \quad \Longrightarrow \quad \langle f, g \rangle = \sum_{j \in \mathbb{Z}} \langle \widetilde{Q}_j f, Q_j g \rangle, \quad \forall\ g \in L^p(\mathbb{R}^n)$$

(see [7, §5]). This idea extends to distributions $f \in B^{-\boldsymbol{\alpha}}_{p',q'}(\mathbb{R}^n)$. Indeed, at least when $0 < \alpha \le \alpha_0$, Bernstein's inequality implies $Q_j g \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$, $\forall\ g \in L^p$, and therefore we may define $\widetilde{Q}_j f \in L^{p'}$ by:

$$\langle \widetilde{Q}_j f, g \rangle := {}_{B^{-\boldsymbol{\alpha}}_{p',q'}} \langle f, Q_j g \rangle_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)}, \quad \forall\ g \in L^p(\mathbb{R}^n).$$

Thus, at least in a formal level we have :

$$(5.3) \quad f \in B^{-\boldsymbol{\alpha}}_{p',q'} \quad \Rightarrow \quad {}_{B^{-\boldsymbol{\alpha}}_{p',q'}} \langle f, g \rangle_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} = \sum_{j \in \mathbb{Z}} \langle \widetilde{Q}_j f, Q_j g \rangle, \qquad g \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n).$$

Corollary 5.4. *In the conditions at the beginning of this section, if* $1 < p, q < \infty$, *and* $0 < \alpha < \alpha^* = \min\{\alpha_0, \frac{L}{a_1}, \dots, \frac{L}{a_n}\}$, *then a distribution* f *belongs to* $B^{-\alpha\boldsymbol{a}}_{p',q'}(\mathbb{R}^n)$ *if and only if* $f \in B^{-\alpha_0\boldsymbol{a}}_{p',q'}(\mathbb{R}^n)$ *and*

$$\sum_{j=0}^{\infty} |\det M|^{-\frac{\alpha q'}{n}j} \Big(\sum_{a,k} |\widetilde{\beta}_{ajk}(f)|^{p'} \Big)^{q'/p'} < \infty\,,$$

where $\widetilde{\beta}_{ajk}(f) = \langle f, \psi_{a;j,k}\rangle$. *In this case, the duality pairing has the form* (5.3) *and we have the equivalences of norms:*

$$\begin{aligned} \|f\|_{B^{-\alpha\boldsymbol{a}}_{p',q'}(\mathbb{R}^n)} &\approx \|f\|_{B^{-\alpha_0\boldsymbol{a}}_{p',q'}(\mathbb{R}^n)} + \Big(\sum_{j=0}^{\infty} |\det M|^{-\frac{\alpha q'}{n}j} \big(\sum_{a,k} |\widetilde{\beta}_{ajk}(f)|^{p'} \big)^{q'/p'} \Big)^{1/q'} \\ &\approx \big(\sum_{k} |\widetilde{\alpha}_k(f)|^{p'} \big)^{\frac{1}{p'}} + \Big(\sum_{j=0}^{\infty} |\det M|^{-\frac{\alpha q'}{n}j} \big(\sum_{a,k} |\widetilde{\beta}_{ajk}(f)|^{p'} \big)^{q'/p'} \Big)^{1/q'}, \end{aligned}$$

where $\widetilde{\alpha}_k(f) = \langle f, \varphi_k\rangle$.

Proof. The assumption $f \in B^{-\alpha_0\boldsymbol{a}}_{p',q'}(\mathbb{R}^n)$ is made so that $\tilde{\beta}_{ajk}(f) = \langle f, \psi_{a;j,k}\rangle$ and $\widetilde{\alpha}_k(f) = \langle f, \varphi_k\rangle$ are well-defined. The rest is a consequence of Theorem 5.1 above and Theorem 2 in [7], which characterizes the dual of the interpolation space $Z_{\theta,q}$ in terms of a multiresolution analysis. □

5.1. Extensions and examples

There is a particularly simple extension of our results that may be of interest in practice. It involves MRA's and wavelet bases translated over a general lattice, other than $\mathbb{Z}^n$. Recall that a *lattice* is a set of points $\Gamma = A\mathbb{Z}^n$, for some $A \in GL_n(\mathbb{R})$. As it is well-known, one may define scaling functions, MRA's or wavelets exactly as in §3 but replacing $k \in \mathbb{Z}^n$ by $\gamma \in \Gamma$. The advantage is that the dilation matrix $M \in \mathcal{M}_n(\mathbb{R})$ may have *real* entries, as long as it preserves the lattice: $M\Gamma \subset \Gamma$. Note that φ is an M- scaling function relative to $\Gamma = A\mathbb{Z}^n$ if and only if $\varphi_A = |\det(A)|\varphi(A\,\cdot)$ is an M_A-scaling function relative to $\mathbb{Z}^n$, where $M_A = A^{-1}MA$.

This more general setting has two interesting consequences. On the one hand, the restriction on the anisotropy $\boldsymbol{a} = (a_1, \dots, a_n)$ given by Lemma 3.2 can be slightly relaxed by requiring only that

$$A^{-1}\mathrm{diag}\,(\lambda^{\frac{1}{a_1}}, \dots, \lambda^{\frac{1}{a_n}})A \in \mathcal{M}_n(\mathbb{Z}),$$

for some $\lambda > 1$ and $A \in GL_n(\mathbb{R})$. That is, it suffices that $M = \mathrm{diag}\,(\lambda^{\frac{1}{a_1}}, \dots, \lambda^{\frac{1}{a_n}})$ is *similar* to an integer-valued matrix. This is however not enough to guarantee the existence of admissible bases for all anisotropies $\boldsymbol{a}$.

The second application of MRA's over lattices is to the spaces $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n; \{\boldsymbol{f}_i\})$, where $\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\}$ is a basis of $\mathbb{R}^n$. These spaces are defined exactly as in §2 but replacing $\boldsymbol{e}_j$ by $\boldsymbol{f}_j$ in (2.1) (see Remark 8 in [33]). To obtain multilevel decompositions for $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n; \{\boldsymbol{f}_i\})$ it seems natural to use the lattice generated by $\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_n\}$, and given by the matrix A with columns the elements of the basis. Indeed, using the canonical isometry

$$g \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n;\{\boldsymbol{f}_i\}) \longmapsto g_A = |\det(A)|^{\frac{1}{p}} g(A\,\cdot) \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n),$$

one obtains an equivalence between the results we proved above for $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ and those for the spaces $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n;\{\boldsymbol{f}_i\})$.

Finally, we briefly sketch with an example how one would use Theorem 1.1 in the numerical resolution of PDE's. The techniques are well-known [24, 13, 8] and we do not develop them here. Consider the semielliptic differential operator

$$\mathcal{L}u = -\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^4 u}{\partial x_2^4} + g(x_1,x_2)u$$

where $0 < c_1 \le g(x_1,x_2) \le c_2$. Such an operator has been studied in [37] in the context of anisotropic spaces. The existence and uniqueness of weak solutions to $\mathcal{L}u = f$ follows from the Hilbert space theory of $H = H^{(1,2)}(\mathbb{R}^2)$. Indeed, $\mathcal{L}$ maps H into H' boundedly, and the bilinear form $a(u,v) := \langle \mathcal{L}u, v\rangle$ is bounded and coercive in $H \times H$. Thus, for all $f \in H'$ there exists $u \in H$ such that

$$\langle \mathcal{L}u, v\rangle = \langle f, v\rangle\,, \qquad \forall v \in H\,. \tag{5.4}$$

For the numerical resolution of (5.4) one uses an admissible multiresolution analysis $\{V_j\}_{j\in\mathcal{J}}$ with dilation matrix $M = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$. The associated Galerkin system is then

$$u_J \in V_J\,, \qquad \langle \mathcal{L}u_J, v_J\rangle = \langle f, v_J\rangle\,, \qquad \forall v_J \in V_J,\ J \ge 0\,.$$

Using the basis $\{\varphi_{0,k}\} \cup \{\psi_{a;j,k}\}_{a,k,0\le j<J}$ of V_J, this system can be rewritten as:

$$\hat{A}_J \hat{u}_J = \hat{f}_J\,, \tag{5.5}$$

where $\hat{u}_J$ and $\hat{f}_J$ are the column vectors with entries the coefficients of u_J and $\widetilde{P}_J f$ in this and the dual basis, respectively, and the symmetric matrix $\hat{A}_J$ is given by the blocks

$$\hat{A}_j = \begin{pmatrix} \langle \mathcal{L}\varphi_{0,k}, \varphi_{0,k'}\rangle & \langle \mathcal{L}\psi_{a;j,k}, \varphi_{0,k'}\rangle \\ \\ \langle \mathcal{L}\varphi_{0,k}, \psi_{a';j',k'}\rangle & \langle \mathcal{L}\psi_{a;j,k}, \psi_{a';j',k'}\rangle \end{pmatrix}_{\substack{k,k'\in\mathbb{Z} \\ a,a'=1,\dots,7 \\ 0\le j,j'<J}}\,.$$

Now, for a column vector of entries $\hat{v} = (\breve{v}_k, \hat{v}_{ajk})_{a,k,0\le j<J}$, corresponding to the function $v_J = \sum_k \breve{v}_k \varphi_{0,k} + \sum_{a,k,j} \hat{v}_{ajk}\psi_{a;j,k}$, Theorem 5.1 tells us that:

$$\begin{aligned} \hat{v}^T \hat{A}_J \hat{v} &= \langle \mathcal{L}v_J, v_J\rangle \approx \|v_J\|_H^2 \\ &\approx \|P_0 v_J\|_2^2 + \sum_{j=0}^{J-1} 4^{2j} \|Q_j v_J\|_2^2 \\ &\approx \sum_{k\in\mathbb{Z}} |\breve{v}_k|^2 + \sum_{j=0}^{J-1} 4^{2j} \sum_{a=1}^{7} \sum_{k\in\mathbb{Z}} |\hat{v}_{ajk}|^2\,. \end{aligned}$$

Therefore, the matrix $\hat{A}_J$ is spectrally equivalent to a diagonal matrix. This allows optimal preconditioning of (5.5) and fast numerical methods for matrix inversion and approximation of solutions. Of course, in practice one should truncate the infinite matrix $\hat{A}_J$, or consider bounded domains where the spaces V_j are finite dimensional.

Another observation is that the previous example makes use of a particularly simple space

$$H^{(1,2)}(\mathbb{R}^2) = H^1(\mathbb{R}) \otimes L^2(\mathbb{R}) \cap L^2(\mathbb{R}) \otimes H^2(\mathbb{R}),$$

for which the wavelet characterization may be obtained easily in the case of tensor product wavelets. Our results let us use as well other type of compactly supported wavelets that may fit better in the numerical algorithm. This can be particularly interesting for less trivial examples of semi-elliptic equations. The monograph [22] contains a more complete approach to this kind of problems with other examples of semi-elliptic and even non-hypoelliptic linear and non-linear equations.

6. Appendix

We sketch the proofs of some of the L^p-properties of multiresolution analyses announced in §3. These are minor modifications of those contained in Chapter 2 of [29] and Chapter 3 of [8]. We just show (3), (4) and (7), since the others are either trivial or follow easily from these ones. We start with (4), which is an elementary consequence of the following lemma (see Lemma 8 in Chapter 2 of [29]).

Lemma 6.1. *If $\varphi, \widetilde{\varphi} \in L^\infty_c(\mathbb{R}^n)$ and $\langle \varphi_k, \widetilde{\varphi}_{k'} \rangle = \delta_{k,k'}$, $k, k' \in \mathbb{Z}^n$, then for all $1 \le p \le \infty$ we have*

$$\Big\| \sum_{k \in \mathbb{Z}^n} a_k \varphi(\cdot - k) \Big\|_p \simeq \Big(\sum_{k \in \mathbb{Z}^n} |a_k|^p \Big)^{\frac{1}{p}}.$$

Proof. The proof for $p = 1, \infty$ is elementary. The other cases follow from interpolation and the boundedness of the operators:

$$\{a_k\}_k \in \ell^p(\mathbb{Z}^n) \mapsto \sum_k a_k \varphi_k \in L^p(\mathbb{R}^n) \quad \text{and} \quad f \in L^p(\mathbb{R}^n) \mapsto \{\langle f, \widetilde{\varphi}_k \rangle\}_k \in \ell^p(\mathbb{Z}^n).$$

□

We pass now to (3), which is actually a consequence of the Jackson and Bernstein inequalities in §4 (note that there is no risk of circularity since (3) was not used in Theorem 4.1).

Lemma 6.2. *Let $\varphi, \widetilde{\varphi}$ be admissible M-scaling functions in $H^{L\mathbf{a}}(\mathbb{R}^n)$ for some $L > \frac{n}{2}$. Then, for all $1 \le p < \infty$, we have*

$$\cup_{j \in \mathbb{Z}} V_j^{(p)} \text{ is dense in } L^p(\mathbb{R}^n) \quad \text{and} \quad \cap_{j \in \mathbb{Z}} V_j^{(p)} = \{0\}.$$

Proof. In order to use Theorem 4.1, we take α small enough so that we have $L > \max\{\alpha a_1, \dots, \alpha a_n\}$ and $\varphi, \widetilde{\varphi} \in B^{\alpha\mathbf{a}}_{p,p}(\mathbb{R}^n)$. Then, for $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$ there exists $g \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $\|f - g\|_p < \varepsilon$. Further, applying (4.2), we obtain

$$\|g - P_j g\|_p \le C\lambda^{-j\alpha} |g|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} < \varepsilon\,,$$

provided j is large enough. We conclude that

$$\lim_{j\to+\infty} \|f - P_j f\|_p = 0\,.$$

For the other assertion, if $f \in V^{(p)}_j$, $\forall j \in \mathbb{Z}$, then inequality (4.1) when $j \to -\infty$ implies that $|f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)} = 0$. Therefore, since $f \in L^p(\mathbb{R}^n)$ we must have $f \equiv 0$. □

Remark 6.3. *Although we shall not use it in this paper, note that the previous lemma admits a version for $p = \infty$, by just replacing $L^\infty(\mathbb{R}^n)$ by $C_0(\mathbb{R}^n)$. Indeed, this amounts to a verification of the corresponding Bernstein and Jackson inequalities for $p = \infty$. Note further that we required $\varphi, \widetilde{\varphi} \in H^{L\mathbf{a}}$, for some $L > \frac{n}{2}$, in order to use Proposition* 3.1*. However, just the property $\varphi \in L^\infty_c$ implies that V_0 contains locally the constants (see, e.g., §2.7 in [7]). Since this suffices to have a Jackson inequality, it follows that Lemma* 6.2 *holds under the minimal assumptions $\varphi, \widetilde{\varphi} \in L^\infty_c$.*

Lemma 6.4. *Let $\varphi, \widetilde{\varphi}$ be admissible M-scaling functions in $L^\infty_c(\mathbb{R}^n)$. Then, for all $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$ we have*

$$\lim_{j\to-\infty} \|P_j f\|_p = 0. \tag{6.1}$$

Proof. By density, it suffices to show (6.1) for $f \in C^\infty_c(\mathbb{R}^n)$. By Lemma 6.1

$$\|P_j f\|_p \simeq \left(\sum_{k\in\mathbb{Z}^n} |\langle f, \widetilde{\varphi}^{(p')}_{j,k}\rangle|^p \right)^{\frac{1}{p}}.$$

Now, for any $1 < \overline{p} < p$ we have

$$\begin{aligned} |\langle f, \widetilde{\varphi}^{(p')}_{j,k}\rangle| &\le |\det(M)|^{j/p'-j} \int_{\mathbb{R}^n} |f(M^{-j}x)||\widetilde{\varphi}(x-k)|\,dx \\ &\le |\det(M)|^{-j/p} \left(\int_{\mathbb{R}^n} |f(M^{-j}x)|^{\overline{p}} |\widetilde{\varphi}(x-k)|\,dx \right)^{1/\overline{p}} \|\widetilde{\varphi}\|_1^{1/\overline{p}'}. \end{aligned}$$

Thus, since $p/\overline{p} > 1$, we may use Minkowski's inequality in the following to obtain

$$\begin{aligned} \|P_j f\|_p &\le C\,|\det M|^{-j/p}\, \|\widetilde{\varphi}\|_1^{1/\overline{p}'} \left[\int_{\mathbb{R}^n} |f(M^{-j}x)|^{\overline{p}} \left(\sum_k |\widetilde{\varphi}(x-k)|^{p/\overline{p}} \right)^{\overline{p}/p} dx \right]^{1/\overline{p}} \\ &\le C\,|\det M|^{j(\frac{1}{\overline{p}}-\frac{1}{p})}\, \|\widetilde{\varphi}\|_1^{1/\overline{p}'} \left\| \sum_k |\widetilde{\varphi}(x-k)| \right\|_\infty \|f\|_{\overline{p}}, \end{aligned}$$

which goes to 0 as $j \to -\infty$. □

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