MEAN VALUE FORMULAS FOR ORNSTEIN-UHLENBECK AND HERMITE TEMPERATURES

GUILLERMO FLORES AND GUSTAVO GARRIGÓS

Abstract. We obtain explicit mean value formulas for the solutions of the diffusion equations associated with the Ornstein-Uhlenbeck and Hermite operators. From these, we derive various useful properties, such as maximum principles, uniqueness theorems and Harnack-type inequalities.

1. Introduction

Given an open set $E \subset \mathbb{R}^n \times \mathbb{R}$, we denote by $C^{2,1}(E)$ the set of real-valued functions $u(x,t)$ on E such that the partial derivatives $\partial_t u$ and $\partial^2_{x_i,x_j} u$, $1 \le i,j \le n$, all exist and are continuous on E.

We say that $u \in C^{2,1}(E)$ is a *temperature in* E, denoted $u \in \mathscr{T}(E)$, if

$$\partial_t u = \Delta u = \sum_{i=1}^n \partial^2_{x_i,x_i} u, \qquad (t,x) \in E,$$

that is, if $u(x,t)$ solves the classical heat equation in the domain E.

In this paper we shall be interested in two variants of the above PDE, namely the *Ornstein-Uhlenbeck heat equation*, given by

$$\partial_t U = \Big(\Delta - 2x \cdot \nabla\Big) U, \tag{1.1}$$

and the *Hermite heat equation*, given by

$$\partial_t U = \Big(\Delta - |x|^2\Big) U. \tag{1.2}$$

Functions $U \in C^{2,1}(E)$ satisfying (1.1) or (1.2) will be called OU-temperatures or H-temperatures, and the corresponding classes will be denoted by $\mathscr{T}_{\mathrm{OU}}(E)$ and $\mathscr{T}_{\mathrm{H}}(E)$, respectively.

The goal of this paper is to provide *explicit* mean value formulas for functions in $\mathscr{T}_{\mathrm{OU}}$ and $\mathscr{T}_{\mathrm{H}}$, which are similar to the mean value formulas

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for classical temperatures in $\mathscr{T}$ due to Watson; see [15, 16] or [4, p. 53]. Namely, we shall prove the following theorem, which seems to be new in these settings.

Theorem 1.1. *There exists a family of bounded open sets $\Xi(x,t;r)$, for $(x,t)\in\mathbb{R}^{n+1}$ and $r>0$, and positive kernels $K^{\mathrm{OU}}_{x,t}(y,s)$ and $K^{\mathrm{H}}_{x,t}(y,s)$ in $C^\infty\big(\mathbb{R}^n\times(-\infty,t)\big)$, such that the following properties are equivalent:*

(a) $U\in\mathscr{T}_{\mathrm{OU}}(E)$

(b) *for every* $\overline{\Xi}(x,t;r)\subset E$ *it holds*
$$U(x,t)=\frac{1}{(4\pi r)^{\frac{n}{2}}}\iint_{\Xi(x,t;r)}U(y,s)\,K^{\mathrm{OU}}_{x,t}(y,s)\,dyds.$$

The same equivalence holds for $U\in\mathscr{T}_{\mathrm{H}}(E)$ if $K^{\mathrm{OU}}_{x,t}$ is replaced by $K^{\mathrm{H}}_{x,t}$.

The explicit expressions for the kernels and balls are given below; see (2.15), (2.17) and Figure 1. This result may be used as a starting point to establish several classical properties, such as strong maximum principles, uniqueness theorems or Harnack inequalities, for functions in $\mathscr{T}_{\mathrm{OU}}$ and $\mathscr{T}_{\mathrm{H}}$.

In particular, we shall prove below the following uniqueness theorem with an optimal unilateral growth condition, which seems new in this generality.

Theorem 1.2. *Let $0<T_0\le\infty$, and let $U\in\mathscr{T}_{\mathrm{OU}}(\mathbb{R}^n\times(0,T_0))$ be continuous in $\mathbb{R}^n\times[0,T_0)$ and with $U(x,0)\equiv0$. Suppose additionally that for some $A>0$ it holds*
$$U(x,t)\le A\,e^{|x|\,p(|x|)},\quad\forall\,|x|\ge1,\ \forall\,t\in(0,T_0)\tag{1.3}$$
where $p(r)$ is a positive continuous function, such that $r^\gamma p(r)$ is non-decreasing for some $\gamma\ge0$, and
$$\int_1^\infty\frac{dr}{p(r)}=\infty.$$
Then, necessarily $U\equiv0$ in $\mathbb{R}^n\times[0,T_0)$.

Conversely, for every such $p(r)$ with $\int_1^\infty\frac{dr}{p(r)}<\infty$, there exists a function $U\in\mathscr{T}_{\mathrm{OU}}(\mathbb{R}^n\times\mathbb{R})$ with $U(x,0)\equiv0$, $U\not\equiv0$ and
$$|U(x,t)|\le e^{|x|p(|x|\vee1)},\quad\forall\,x\in\mathbb{R}^n,\ t\in(0,\infty).$$
Finally, both statements hold as well when $\mathscr{T}_{\mathrm{OU}}$ is replaced by $\mathscr{T}_{\mathrm{H}}$.

2. Notation and main results

2.1. **A transference principle.** Our aproach will be based on known transference formulas between the classes $\mathscr{T}$, $\mathscr{T}_{\mathrm{OU}}$ and $\mathscr{T}_{\mathrm{H}}$ which we describe next. Given $U\in C^{2,1}(E)$, we consider the transformation
$$V(x,t):=e^{-nt}e^{-|x|^2/2}U(x,t).\tag{2.1}$$
It is easily verified that
$$\Big(\partial_t-\Delta+|x|^2\Big)[V(x,t)]\;=\;e^{-nt}e^{\frac{-|x|^2}{2}}\Big(\partial_t-\Delta+2x\cdot\nabla\Big)[U(x,t)],\tag{2.2}$$

from which we conclude that

$$U \in \mathscr{T}_{\mathrm{OU}}(E) \quad \Longleftrightarrow \quad V \in \mathscr{T}_{\mathrm{H}}(E). \tag{2.3}$$

This elementary transformation has recently been used in various contexts regarding Hermite operators; see e.g. [1, 2, 8].

We next give a transformation which relates the classes $\mathscr{T}$ and $\mathscr{T}_{\mathrm{OU}}$. It is suggested by the Mehler formula expression, and it is more or less implicit in the early works in the topic [10, 11, 13].

Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^n \times (-\infty, 1/4)$ be the C^∞-diffeomorphism given by

$$\varphi(x,t) = \Big(\frac{x}{e^{2t}}, \frac{1-e^{-4t}}{4}\Big), \tag{2.4}$$

whose inverse takes the form

$$\varphi^{-1}(y,s) = \Big(\frac{y}{\sqrt{1-4s}}\, , \, \tfrac14 \log \tfrac{1}{1-4s} \Big).$$

Observe that φ is increasing and preserves the positivity in the t-variable. Next, we define the transformation $T : C(\varphi(E)) \to C(E)$ given by

$$U(x,t) := Tu(x,t) = \; u\big(\varphi(x,t)\big). \tag{2.5}$$

Then we have the following elementary relation.

Lemma 2.1. *If $U = Tu \in C^{2,1}(E)$, then*

$$\Big(\partial_t - \Delta + 2x\cdot\nabla_x\Big)[U(x,t)] \;=\; e^{-4t}\, T\Big[\big(\partial_s - \Delta_y\big)u\Big]. \tag{2.6}$$

Proof. From $U(x,t) = u\big(xe^{-2t}, (1-e^{-4t})/4\big)$ one easily obtains

$$\begin{aligned}
\partial_t U(x,t) &= e^{-4t} u_s\big(\varphi(x,t)\big) - 2e^{-2t}\, x\cdot(\nabla_y u)\big(\varphi(x,t)\big)\\
\partial_{x_i} U(x,t) &= e^{-2t}\, u_{y_i}\big(\varphi(x,t)\big)\\
\partial^2_{x_i x_i} U(x,t) &= e^{-4t} u_{y_i y_i}\big(\varphi(x,t)\big).
\end{aligned}$$

From here the expression (2.6) follows immediately. □

As a consequence we conclude that,

$$u \in \mathscr{T}(\varphi(E)) \quad \Longleftrightarrow \quad U = Tu \in \mathscr{T}_{\mathrm{OU}}(E). \tag{2.7}$$

2.2. **Classical mean value formulas.** We recall the mean value formula for classical temperatures in $\mathscr{T}$. More generally, we state the result for *subtemperatures*, that is for functions $u \in C^{2,1}(E)$ such that

$$u_t \le \Delta u, \quad \text{in } E,$$

which we shall denote by $u \in \mathscr{T}^{\mathrm{sub}}(E)$. For each $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, we define the *heat ball* $\Omega(x,t;r)$, with "center" (x,t) and "radius" r, by

$$\Omega(x,t;r) = \Big\{(y,s) \in \mathbb{R}^n \times \mathbb{R} : \, \Phi(x-y, t-s) > \tfrac{1}{(4\pi r)^{\frac n2}}\Big\},$$

where

$$\Phi(x,t) = \begin{cases} \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}} & \text{if } x \in \mathbb{R}^n,\ t > 0 \\ 0 & \text{if } x \in \mathbb{R}^n,\ t \le 0 \end{cases}$$

is the usual fundamental solution of the heat equation. Observe that $(y,s) \in \Omega(x,t;r)$ iff

$$|y-x|^2 < 2n(t-s)\ln\tfrac{r}{t-s}, \quad s \in (t-r,t),$$

so $\Omega(x,t;r)$ is an open convex set, axially symmetric about the line $\{x\} \times \mathbb{R}$, and with (x,t) lying at the top boundary and $(x,t-r)$ at the bottom boundary; see e.g. [4, p. 52] or [16, p. 2]. Heat balls are also translation invariant, in the sense that

$$\Omega(x,t;r) = (x,t) + \Omega(0,0;r).$$

The next theorem, due to N. Watson [15], is known as mean value property for the heat equation; see also [16, Theorem 1.16].

Theorem 2.2. *Let $u \in C^{2,1}(E)$. If $u \in \mathscr{T}^{\rm sub}(E)$ then*

$$u(x,t) \le \frac{1}{(4\pi r)^{\frac{n}{2}}} \iint_{\Omega(x,t;r)} u(y,s)\,\frac{|x-y|^2}{4(t-s)^2}\,dyds \tag{2.8}$$

provided $\overline{\Omega}(x,t;r) \subset E$. Conversely if for every $(x,t) \in E$ and every $\varepsilon > 0$ there exists some $r < \varepsilon$ such that (2.8) holds, then $u \in \mathscr{T}^{\rm sub}(E)$.

In particular, the following characterization holds.

Corollary 2.3. *Let $u \in C^{2,1}(E)$. Then $u \in \mathscr{T}(E)$ if and only if*

$$u(x,t) = \frac{1}{(4\pi r)^{\frac{n}{2}}} \iint_{\Omega(x,t;r)} u(y,s)\,\frac{|x-y|^2}{4(t-s)^2}\,dyds \tag{2.9}$$

for all heat balls $\overline{\Omega}(x,t;r) \subset E$.

2.3. **Mean value formulas in $\mathscr{T}^{\rm sub}_{\rm OU}$ and $\mathscr{T}^{\rm sub}_{\rm H}$.** We first define the corresponding notion of "heat ball", by using the transformation φ in (2.4). Namely, given $(x_0,t_0) \in \mathbb{R}^{n+1}$ and $r > 0$ we let

$$\Xi(x_0,t_0;r) := \varphi^{-1}\Big(\Omega\big(\varphi(x_0,t_0);r\big)\Big). \tag{2.10}$$

Equivalently, $(x,t) \in \Xi(x_0,t_0;r)$ iff

$$\Big|\frac{x}{e^{2t}} - \frac{x_0}{e^{2t_0}}\Big|^2 < \tfrac{n}{2}\left(e^{-4t} - e^{-4t_0}\right)\ln\tfrac{4r}{e^{-4t}-e^{-4t_0}},$$

so this set is now axially symmetric with respect to the curve $\Gamma(x_0,t_0) = \{(x_0\exp\big(2(t-t_0)\big),t) : t \in \mathbb{R}\}$, with the point (x_0,t_0) lying at the top boundary. Also, unlike the classical heat balls, these Ξ-balls are no longer translation invariant. See Figure 1 below for drawings in various situations.

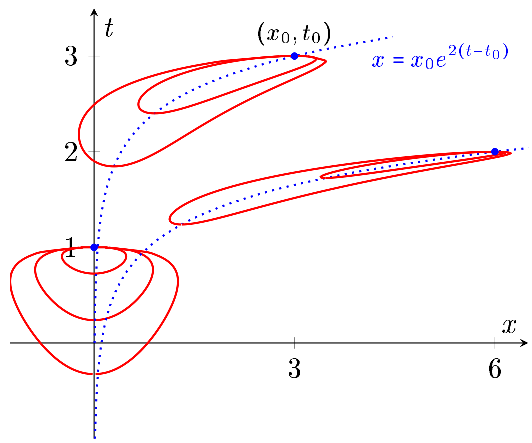

FIGURE 1. Hermite heat balls $\Xi(x_0, t_0; r)$ of various centers and radii.

We now state a slightly more general version than Theorem 1.1, which is also valid for subtemperatures. As before, we say that $U \in C^{2,1}(E)$ belongs to $\mathscr{T}_{\mathrm{OU}}^{\mathrm{sub}}(E)$ if

$$\partial_t U \le \big(\Delta - 2x \cdot \nabla\big) U \quad \text{in } E, \tag{2.11}$$

and that $U \in \mathscr{T}_{\mathrm{H}}^{\mathrm{sub}}(E)$ if

$$\partial_t U \le \big(\Delta - |x|^2\big) U \quad \text{in } E. \tag{2.12}$$

With the notation in (2.1) and (2.5), it follows from (2.2) and (2.6) that

$$u \in \mathscr{T}^{\mathrm{sub}}(\varphi(E)) \iff U \in \mathscr{T}_{\mathrm{OU}}^{\mathrm{sub}}(E) \iff V \in \mathscr{T}_{\mathrm{H}}^{\mathrm{sub}}(E). \tag{2.13}$$

Theorem 2.4. *Let $U \in C^{2,1}(E)$. If $U \in \mathscr{T}_{\mathrm{OU}}^{\mathrm{sub}}(E)$ and $(x,t) \in E$ then*

$$U(x,t) \le \frac{1}{(4\pi r)^{\frac{n}{2}}} \iint_{\Xi(x,t;r)} U(y,s)\, K_{x,t}^{\mathrm{OU}}(y,s) dy ds \tag{2.14}$$

provided $\overline{\Xi}(x,t;r) \subset E$, where

$$K_{x,t}^{\mathrm{OU}}(y,s) = 8e^{-2(n+2)s} \frac{|xe^{-2t} - ye^{-2s}|^2}{|e^{-4t} - e^{-4s}|^2}. \tag{2.15}$$

Conversely if for every $(x,t) \in E$ and every $\varepsilon > 0$ there exists some $r < \varepsilon$ such that (2.14) *holds, then $U \in \mathscr{T}_{\mathrm{OU}}^{\mathrm{sub}}(E)$.*

Remark 2.5. When $U \in \mathscr{T}_{\mathrm{OU}}(E)$, then the above theorem, applied to the functions U and $-U$, easily implies Theorem 1.1.

The version of Hermite subtemperatures takes the following form.

Theorem 2.6. *Let $V \in C^{2,1}(E)$. If $V \in \mathscr{T}_{\mathrm{H}}^{\mathrm{sub}}(E)$ and $(x,t) \in E$ then*

$$V(x,t) \le \frac{1}{(4\pi r)^{\frac{n}{2}}} \iint_{\Xi(x,t;r)} V(y,s)\, K_{x,t}^{\mathrm{H}}(y,s) dy ds \tag{2.16}$$

provided $\overline{\Xi}(x,t;r)\subset E$*, where*

$$K^{\mathrm{H}}_{x,t}(y,s) = e^{(s-t)n}\, e^{\frac{|y|^2-|x|^2}{2}}\, K^{\mathrm{OU}}_{x,t}(y,s). \tag{2.17}$$

Conversely if for every $(x,t)\in E$ *and every* $\varepsilon>0$ *there exists some* $r<\varepsilon$ *such that* (2.16) *holds, then* $V\in\mathscr{T}^{\mathrm{sub}}_{\mathrm{H}}(E)$*.*

2.4. **Proof of Theorems 2.4 and 2.6.**

Proof of Theorem 2.4. Assume that $U\in\mathscr{T}^{\mathrm{sub}}_{\mathrm{OU}}(E)$. By (2.13) we know that $u=T^{-1}U\in\mathscr{T}^{\mathrm{sub}}(\varphi(E))$, so we can apply Theorem 2.2 at the point $(\tilde x,\tilde t)=\varphi(x,t)$ to obtain

$$u(\tilde x,\tilde t)\le \frac{1}{(4\pi r)^{\frac n2}}\iint_{\Omega(\tilde x,\tilde t;r)} u(\tilde y,\tilde s)K_{\tilde x,\tilde t}(\tilde y,\tilde s)\,d\tilde y d\tilde s, \tag{2.18}$$

with $K_{\tilde x,\tilde t}(\tilde y,\tilde s)=2^{-1}|\tilde x-\tilde y|^2/(\tilde t-\tilde s)^2$ and provided $\overline{\Omega}(\tilde x,\tilde t;r)\subset\varphi(E)$. Now, making the change of variables

$$(\tilde y,\tilde s)=\Big(\frac{y}{e^{2s}},\frac{1-e^{-4s}}{4}\Big)=\varphi(y,s),$$

whose jacobian is given by

$$\left|\frac{\partial(\tilde y,\tilde s)}{\partial(y,s)}\right| = e^{-2(n+2)s},$$

and using the identity $U=u\circ\varphi$, one easily obtains the formula in (2.14) with

$$K^{\mathrm{OU}}_{x,t}(y,s)=e^{-2(n+2)s}\,K_{\varphi(x,t)}\big(\varphi(y,s)\big), \tag{2.19}$$

which agrees with (2.15). The converse follows by a completely similar argument. $\square$

Proof of Theorem 2.6. Again, (2.13) implies that $U(x,t)=e^{tn}e^{\frac{|x|^2}{2}}V(x,t)$ belongs to $\mathscr{T}^{\mathrm{sub}}_{\mathrm{OU}}(E)$, and therefore we can use the formula (2.14). From here, elementary operations lead to (2.16). $\square$

3. Consequences: Maximum principles

As it is standard in potential theory, we shall employ the mean value formulas in (2.14) and (2.16) to easily derive maximum principles (in strong form) for functions in $\mathscr{T}^{\mathrm{sub}}_{\mathrm{OU}}$ and $\mathscr{T}^{\mathrm{sub}}_{\mathrm{H}}$. These results are known in the literature for more general parabolic pdes, but typically with different proofs; see e.g. [12, ch 3.3] or [7, ch 2.2]. Here we follow the approach in [4, ch 2.3].

We shall use the general set terminology in [7, 12]. Given an open set E and a point $(x_0,t_0)\in E$, we denote by $E_{t_0}=\{(x,t)\in E : t<t_0\}$. Also, we denote by $\Lambda(x_0,t_0,E)$ the set of points (x,t) that are lower than (x_0,t_0) relative to E, in the sense that there is a polygonal path $\gamma\subset E$ joining (x_0,t_0) to (x,t), along which the temporal variable is *strictly decreasing.* By a polygonal path we mean a path which is a union of finitely many line

segments. So, if $E = \Omega \times (0,\infty)$, for some connected open set $\Omega \subset \mathbb{R}^n$, then $\Lambda(x_0,t_0,E) = E_{t_0} = \Omega \times (0,t_0)$, for all $x_0 \in \Omega$ and $t_0 > 0$. For drawings in other situations, see Figure 2 below or [12, p. 169].

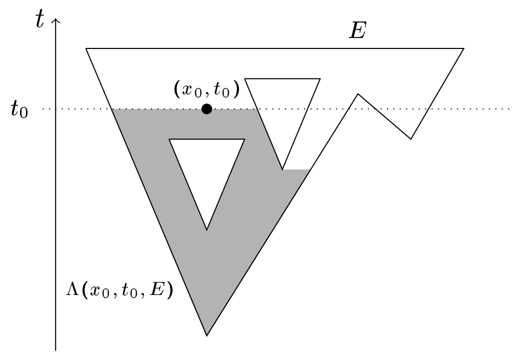

FIGURE 2. The subset $\Lambda(x_0,t_0,E)$ (in grey) of a set E.

Remark 3.1. Observe that $\varphi\big(\Lambda(x_0,t_0,E)\big) = \Lambda\big(\varphi(x_0,t_0),\varphi(E)\big)$, since the mapping φ is monotone in the t-variable. In particular, this implies that $\Xi(x_0,t_0;r) \subset \Lambda(x_0,t_0,E)$, for all sufficiently small $r > 0$, since the same property holds for the classical heat balls.

Remark 3.2. Both $\Omega(x_0,t_0;r)$ and $\Xi(x_0,t_0;r)$, have the same tangent plane at the point (x_0,t_0), which is normal to the direction $(0,1)$. In particular, every poligonal path $\gamma(s)$, with $\gamma(0) = (x_0,t_0)$ and with strictly decreasing temporal variable, has a portion $\gamma\big((0,\varepsilon]\big) \subset \Xi(x_0,t_0;r)$, for some $\varepsilon > 0$.

We can now derive a *strong maximum principle* for Hermite and Ornstein-Uhlenbeck subtemperatures.

Corollary 3.3. (Strong maximum principle) *Assume that $U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{OU}}(E)$ and there exists $(x_0,t_0) \in E$ such that*

$$\sup_{\Lambda(x_0,t_0,E)} U = U(x_0,t_0).$$

Then U is constant in the set $\Lambda(x_0,t_0,E)$. The same holds when $U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{H}}(E)$ provided that $U(x_0,t_0) \geq 0$.

Proof. Let $M := U(x_0,t_0)$. Assume first that $U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{H}}(E)$ and hence that $M \geq 0$. Observe from (2.12) that the constant $-M$ also belongs to $\mathscr{T}^{\mathrm{sub}}_{\mathrm{H}}$. If $r > 0$ is sufficiently small, then using twice the mean value property (2.16) and the positivity of the involved kernel one obtains

$$\begin{aligned} U(x_0,t_0) &\leq \frac{1}{(4\pi r)^{\frac{n}{2}}} \iint_{\Xi(x_0,t_0;r)} U(y,s) K^{\mathrm{H}}_{x_0,t_0}(y,s)\,dyds \\ &\leq \frac{1}{(4\pi r)^{\frac{n}{2}}} \iint_{\Xi(x_0,t_0;r)} M \cdot K^{\mathrm{H}}_{x_0,t_0}(y,s)\,dyds \leq M. \end{aligned} \tag{3.1}$$

Thus equality holds in the middle expressions, and hence

$$\iint_{\Xi(x_0,t_0;r)} \big[M - U(y,s)\big]\, K^{\mathrm{H}}_{x_0,t_0}(y,s)\, dyds = 0,$$

which implies that $U \equiv M$ within $\Xi(x_0,t_0;r)$.

To prove constancy in all $\Lambda(x_0,t_0,E)$ we follow a standard argument, see e.g. [4, p.56]. Let $(y_0,s_0) \in \Lambda(x_0,t_0,E)$, i.e. $s_0 < t_0$, and let γ be a polygonal path joining (x_0,t_0) to (y_0,s_0) for which the temporal variable is strictly decreasing. Consider

$$s_1 := \min\Big\{ s \in [s_0,t_0] \;:\; U|_{\{\gamma\}\cap\mathbb{R}^n\times[s,t_0]} \equiv M \Big\},$$

where $\{\gamma\}$ is the image set of γ. The set in the brackets is non-empty, and since U is continuous, the minimum is attained. Assume $s_1 > s_0$. Then $U(z_1,s_1) = M$ for some point $(z_1,s_1) \in \{\gamma\}$ and so, as before, U is identically equal to M within $\Xi(z_1,s_1;r)$ for all sufficiently small $r > 0$. But, by Remark 3.2, the ball $\Xi(z_1,s_1;r)$ contains $\{\gamma\} \cap \mathbb{R}^n \times [s_1-\varepsilon, s_1)$ for some small ε, which is a contradiction. Therefore, $s_1 = s_0$ and then $U \equiv M$ on $\{\gamma\}$. This completes the proof when $U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{H}}$ and $M \geq 0$.

When $U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{OU}}$ one applies exactly the same argument, this time using that the constant $M \in \mathscr{T}_{\mathrm{OU}}$, and hence the last step in (3.1) is an equality. □

From the previous result one can obtain versions of the *weak maximum principle*. We state one which is valid for any open set $E \subset \mathbb{R}^{n+1}$ (possibily unbounded).

Corollary 3.4. (Weak maximum principle) *Let $U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{OU}}(E)$ and $t_0 \in \mathbb{R}$. Assume that for some $A \in \mathbb{R}$ it holds*

$$\limsup_{\substack{(x,t)\to(x_1,t_1)\\(x,t)\in E_{t_0}}} U(x,t) \;\leq\; A, \quad \forall\ (x_1,t_1) \in \partial_{\mathrm{P}}\big(E_{t_0}\big) \cup \{\infty\}^1 \tag{3.2}$$

where $\partial_{\mathrm{P}}\big(E_{t_0}\big) := \partial\big(E_{t_0}\big) \setminus \big[\{t=t_0\}\cap E\big]$. Then

$$U(x,t) \leq A, \quad \forall\ (x,t) \in E_{t_0}. \tag{3.3}$$

The same holds for $U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{H}}(E)$ when, additionally, $\sup_{E_{t_0}} U \geq 0$.

Proof. Let $M := \sup_{E_{t_0}} U$. We must show that $M \leq A$. Pick a sequence of points $\{P_n\}_{n\geq1} \subset E_{t_0}$ such that $U(P_n) \nearrow M$. There are several cases:
(i) If $\{P_n\}_{n\geq1}$ is not bounded, then

$$M = \lim_{n\to\infty} U(P_n) \leq \limsup_{\substack{(x,t)\to\infty\\(x,t)\in E_{t_0}}} U(x,t) \leq A. \tag{3.4}$$

[1] In this expression the infinity point only plays a role if E_{t_0} is unbounded.

(ii) If $\{P_n\}_{n\geq 1}$ is bounded, then passing to a subsequence we may assume that P_n converges to some $P \in \overline{E_{t_0}}$. Assume first that $P \in \partial_{\mathrm{P}} E_{t_0}$. In this case we directly use (3.2) to obtain

$$M = \lim_{n\to\infty} U(P_n) \leq \limsup_{\substack{(x,t)\to P \\ (x,t)\in E_{t_0}}} U(x,t) \leq A. \tag{3.5}$$

(iii) Finally, assume that the point P in case (ii) belongs to the remaining set, namely $P \in E_{t_0}$ or $P \in E \cap \{t = t_0\}$. In either case $U(P) = M$ and the strong maximum principle in Corollary 3.3 shows that[2] $U \equiv M$ in $\Lambda(P, E)$. If $\Lambda(P, E)$ is unbounded, then we can find a sequence $Q_n \in \Lambda(P, E)$ going to ∞, so that (3.4) holds (with Q_n in place of P_n), and we would be done. If $\Lambda(P, E)$ is bounded, then we pick any straight line $\gamma(s)$ with $\gamma(0) = P$ and strictly decreasing in the t-variable. Consider the finite positive number

$$s^* = \min\{s > 0 \,:\, \gamma(s) \notin \Lambda(P, E)\},$$

and the point $Q = \gamma(s^*)$. By construction, Q must belong to $\partial_P(E_{t_0})$. Then if $s_n \nearrow s^*$ we have $Q_n = \gamma(s_n) \in \Lambda(P, E)$, so (3.5) holds with P_n, P replaced by Q_n, Q. This completes the proof of $M \leq A$ in all cases. □

In the special case of bounded domains E and continuous functions up to the boundary one recovers the following classical statement.

Corollary 3.5. *Let $E \subset \mathbb{R}^{n+1}$ be open and bounded, and let $U \in C(\overline{E}) \cap \mathscr{T}^{\mathrm{sub}}_{\mathrm{OU}}(E)$. Then, for every $t_0 \in \mathbb{R}$*

$$\max_{\overline{E_{t_0}}} U \;=\; \max_{\partial_{\mathrm{P}} E_{t_0}} U. \tag{3.6}$$

The same holds when $U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{H}}(E)$ if, additionally, the left hand-side of (3.6) *is non-negative.*

Proof. Use the previous corollary with $A = \max_{\partial_{\mathrm{P}} E_{t_0}} U$. □

Remark 3.6. For band domains $E = \Omega \times (0, T_0)$, with $\Omega \subset \mathbb{R}^n$ a bounded open set, (3.6) takes the form

$$\max_{\overline{\Omega}\times[0,t_0]} U \;=\; \max_{(\overline{\Omega}\times\{0\})\cup(\partial\Omega\times[0,t_0])} U, \quad \text{if } t_0 \in (0, T_0].$$

That is, the maximum of U in $\overline{\Omega} \times [0, t_0]$ is always attained at some point of its *parabolic boundary*; [4, p. 52].

Finally, as in [4, p. 56], we deduce a corollary about infinite propagation speed. Here we say that U is a *supertemperature*, denoted $U \in \mathscr{T}^{\mathrm{sup}}_{\mathrm{OU}}$, if $-U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{OU}}$ (and likewise for $\mathscr{T}^{\mathrm{sup}}_{\mathrm{H}}$).

[2] In the case $U \in \mathscr{T}^{\mathrm{sub}}_{\mathrm{H}}(E)$ this step uses the assumption $M \geq 0$ stated after (3.3).

Corollary 3.7. (Infinite propagation). *Let $E = \Omega \times (0, T_0)$, with $\Omega \subset \mathbb{R}^n$ bounded, open and connected. Let $U \in C(\bar{E})$ belong to the class $\mathscr{T}^{\sup}_{\mathrm{OU}}(E)$ or $\mathscr{T}^{\sup}_{\mathrm{H}}(E)$. Assume that*

$$U \geq 0 \text{ in } \partial_{\mathrm{P}} E, \quad \text{and} \quad \exists\, x_0 \in \Omega \text{ such that } U(x_0, 0) > 0. \tag{3.7}$$

Then $U(x,t) > 0$ at all $(x,t) \in E$.

Proof. Suppose, for contradiction, that there exists $P = (x_1, t_1) \in E$ such that $U(P) \leq 0$. Call $V = -U$, which is a subtemperature. Then $\sup_E V \geq V(P) \geq 0$, so we can use Corollary 3.5 (in the special case of Remark 3.6) to deduce that

$$\sup_E V = \sup_{\partial_{\mathrm{P}} E} V \leq 0.$$

But then

$$V(P) = \sup_E V = 0,$$

which in turn, by Corollary 3.3 and the connectivity of Ω, implies that $V(x,t) \equiv 0$ if $t \in (0, t_1)$, $x \in \Omega$. This contradicts the second condition in (3.7). □

4. Uniqueness of solutions

We next derive some uniqueness results for the Cauchy problem when $E = \mathbb{R}^n \times (0, T_0)$ with $0 < T_0 \leq \infty$. Let L denote one of the operators

$$\Delta, \quad \Delta - 2x \cdot \nabla \quad \text{or} \quad \Delta - |x|^2. \tag{4.1}$$

Given $f \in C(\mathbb{R}^n \times (0, T_0))$ and $g \in C(\mathbb{R}^n)$, we say that $u(x,t)$ is a *classical solution* of

$$\begin{cases} u_t = Lu + f & \text{in } \mathbb{R}^n \times (0, T_0) \\ u(x,0) = g & \text{on } \mathbb{R}^n, \end{cases} \tag{4.2}$$

whenever u belongs to $C\big(\mathbb{R}^n \times [0, T_0)\big) \cap C^{2,1}\big(\mathbb{R}^n \times (0, T_0)\big)$ and satisfies (4.2). When $L = \Delta$, a well-known condition to ensure uniqueness is

$$\sup_{0<t<T_0} |u(x,t)| \leq A e^{c|x|^2}, \quad x \in \mathbb{R}^n, \tag{4.3}$$

for some $A, c > 0$; see e.g. [4, Thm 2.3.7]. On the other hand, for every $\varepsilon > 0$ there exist infinitely many solutions of (4.2) with $f = g = 0$ and

$$\sup_{0<t<T_0} |u(x,t)| \leq e^{|x|^{2+\varepsilon}}, \quad x \in \mathbb{R}^n. \tag{4.4}$$

When L is one of the last two operators in (4.1), the condition (4.3) is also sufficient for uniqueness. This could be proved from the above maximum principles, but is also a special case of results for general parabolic equations; see e.g. [7, Theorem 2.4.10]. Here we attempt to replace (4.3) by an *optimal* growth condition, which goes back to the work of Täcklind [14].

Definition 4.1. Let $p(r)$ be a positive and continuous function for $r \geq 1$. We define the class $\mathscr{C}(p)$ as the set of all $u \in C(\mathbb{R}^n \times [0, T_0))$ such that

$$\sup_{0<t<T_0} |u(x,t)| \leq A e^{|x| p(|x|)}, \quad \forall\, |x| \geq 1, \tag{4.5}$$

for some constant $A > 0$ (which may depend on u and T_0).

Below we also consider the function

$$\bar{p}(r) := \inf_{s \geq r} p(s),$$

that is, the largest non-decreasing minorant of $p(r)$. Note that $\bar{p}(r)$ is also continuous, and that $\bar{p}(r) = p(r)$ when p is non-decreasing.

Theorem 4.2 (Täcklind [14])**.** *Let $p(r)$ be positive, continuous for $r \geq 1$. Then, the following are equivalent for the operator $L = \Delta$*

(a) the equation (4.2) *has at most one classical solution u in the class $\mathscr{C}(p)$*

(b) the function p satifies

$$\int_1^\infty \frac{dr}{\bar{p}(r)} = \infty. \tag{4.6}$$

In practice, one would often use (4.5) with functions $p(r)$ which are eventually increasing, such as $p(r) = r(\log r)(\log\log r)\cdots(\log^{(k)} r)$, for $r \geq e^{\cdot^{\cdot^{e}}}$. In that case, the condition (4.6) is equivalent to

$$\int_\alpha^\infty \frac{dr}{p(r)} = \infty, \quad \text{for some } \alpha \geq 1. \tag{4.7}$$

In general, however, when p oscillates, (4.6) may not necessarily imply (4.7). We now give a simple criterion which guarantees the equivalence of both conditions. The proof is a slight modification of Täcklind original argument [14, p. 16]; see also [9, p. 396].

Lemma 4.3. *Let $p(r)$ be positive and continuous for $r \geq 1$. Assume also that for some $\gamma \geq 0$ the function $r^\gamma p(r)$ is non-decreasing. Then* (4.6) *is equivalent to* (4.7)*.*

Proof. Since $\bar{p}(r) \leq p(r)$, we only need to show that (4.6) $\Rightarrow$ (4.7). We first notice that, if $\liminf_{r\to\infty} p(r) < \infty$ then (4.7) always holds. Indeed, in such case we can find a sequence $r_j \nearrow \infty$ such that

$$r_j \geq 2 r_{j-1} \quad \text{and} \quad \sup p(r_j) \leq C. \tag{4.8}$$

Then, for $r \in (r_{j-1}, r_j)$ we have $r^\gamma p(r) \leq r_j^\gamma p(r_j)$ and hence

$$\int_{r_{j-1}}^{r_j} \frac{dr}{p(r)} \geq \int_{r_{j-1}}^{r_j} \frac{r^\gamma\, dr}{r_j^\gamma p(r_j)} \geq c_\gamma \frac{r_j^{\gamma+1} - r_{j-1}^{\gamma+1}}{C\, r_j^\gamma} \geq c'_\gamma r_j \to \infty. \tag{4.9}$$

Thus, we may assume that $\lim_{r\to\infty} p(r) = \infty$. This implies that the set $\{r : \bar{p}(r) = p(r)\}$ is necessarily unbounded (because $\min_{[j,\infty)} p$ is always attained at some $r_j \geq j$, and then $p(r_j) = \bar{p}(r_j)$). We now follow the construction in [14, p. 16]. Let $\ell_0 = 1$, and let ℓ_j be the smallest real number such that

$$\ell_j \geq 2\ell_{j-1} \quad \text{and} \quad \bar{p}(\ell_j) = p(\ell_j). \tag{4.10}$$

Clearly,

$$\int_{\ell_{j-1}}^{\ell_j} \frac{dr}{\bar{p}(r)} \geq \frac{\ell_j - \ell_{j-1}}{\bar{p}(\ell_j)} \geq \frac{\ell_j/2}{p(\ell_j)}.$$

On the other hand, since by construction $\bar{p}(r) = \bar{p}(\ell_j)$ when $r \in (2\ell_{j-1}, \ell_j)$, we also have

$$\int_{\ell_{j-1}}^{\ell_j} \frac{dr}{\bar{p}(r)} = \int_{\ell_{j-1}}^{2\ell_{j-1}} + \int_{2\ell_{j-1}}^{\ell_j} \ldots \leq \frac{\ell_{j-1}}{\bar{p}(\ell_{j-1})} + \frac{\ell_j - 2\ell_{j-1}}{\bar{p}(\ell_j)} \leq \frac{\ell_{j-1}}{p(\ell_{j-1})} + \frac{\ell_j}{p(\ell_j)}.$$

Thus, we conclude that

$$\int_1^\infty \frac{dr}{\bar{p}(r)} = \infty \quad \Longleftrightarrow \quad \sum_{j=1}^\infty \frac{\ell_j}{p(\ell_j)} = \infty. \tag{4.11}$$

Finally, if we further assume that $r^\gamma p(r)$ is non-decreasing, then

$$\int_{\ell_{j-1}}^{\ell_j} \frac{dr}{p(r)} = \int_{\ell_{j-1}}^{\ell_j} \frac{r^\gamma dr}{r^\gamma p(r)} \geq \frac{(\ell_j^{\gamma+1} - \ell_{j-1}^{\gamma+1})/(\gamma+1)}{\ell_j^\gamma p(\ell_j)} \geq c_\gamma \frac{\ell_j}{p(\ell_j)}, \tag{4.12}$$

so from (4.11) we obtain (4.7). □

For our proof below we need one more auxiliary result.

Lemma 4.4. *Let $p(r)$ be as in the statement of Lemma 4.3, and for fixed $\lambda > 0$ define $p_1(r) = p(r) + \lambda r$. Then*

$$\int_1^\infty \frac{dr}{\bar{p}_1(r)} = \infty \quad \Longleftrightarrow \quad \int_1^\infty \frac{dr}{\bar{p}(r)} = \infty. \tag{4.13}$$

Proof. Note that $p_1(r)$ also satisfies the conditions of Lemma 4.3, so it suffices to show that

$$\int_1^\infty \frac{dr}{p_1(r)} = \infty \quad \Longleftrightarrow \quad \int_1^\infty \frac{dr}{p(r)} = \infty.$$

The implication "$\Rightarrow$" is trivial since $p_1(r) \geq p(r)$. We now show "$\Leftarrow$". If $\liminf p(r) < \infty$, then picking a sequence r_j as in (4.8) and arguing as in (4.9) we obtain

$$\int_{r_{j-1}}^{r_j} \frac{dr}{p_1(r)} \geq c_\gamma \frac{r_j^{\gamma+1}}{r_j^\gamma (p(r_j) + \lambda r_j)} \geq c_\gamma \frac{r_j}{C + \lambda r_j} \geq c'_\gamma,$$

which implies the result. So we may assume that $\lim p(r) = \infty$, and using the same numbers ℓ_j as in the proof of Lemma 4.3 we have

$$\int_1^\infty \frac{dr}{p_1(r)} = \sum_{j=1}^\infty \int_{\ell_{j-1}}^{\ell_j} \frac{r^\gamma dr}{r^\gamma (p(r) + \lambda r)} \geq c_\gamma \sum_{j=1}^\infty \frac{\ell_j}{p(\ell_j) + \lambda \ell_j},$$

arguing in the right inequality as in (4.12). Consider the set of indices

$$J = \{j \in \mathbb{N} : p(\ell_j) \le \ell_j\}.$$

If J is an infinite set then

$$\int_1^\infty \frac{dr}{p_1(r)} \ge c_\gamma \sum_{j\in J} \frac{\ell_j}{p(\ell_j) + \lambda\ell_j} \ge c_\gamma \sum_{j\in J} \frac{\ell_j}{(1+\lambda)\ell_j} = \infty.$$

On the contrary, if J is finite, then there exists some j_0 such that $p(\ell_j) > \ell_j$ for all $j \ge j_0$. Thus

$$\int_1^\infty \frac{dr}{p_1(r)} \ge c_\gamma \sum_{j\ge j_0} \frac{\ell_j}{p(\ell_j) + \lambda\ell_j} \ge \tfrac{c_\gamma}{1+\lambda} \sum_{j\ge j_0} \frac{\ell_j}{p(\ell_j)} = \infty,$$

the last equality due to (4.11). □

We can now state a characterization result. The statements for $\mathscr{T}_{\mathrm{OU}}$ seem new in the literature, while the sufficient condition for $\mathscr{T}_{\mathrm{H}}$ is a special case of the more general setting in [9].

Theorem 4.5. *Let $p(r)$ be positive and continuous for $r \ge 1$, and assume that $r^\gamma p(r)$ is non-decreasing for some $\gamma \ge 0$. Then, the following assertions are equivalent for each of the operators $L = \Delta - 2x\cdot\nabla$ and $L = \Delta - |x|^2$.*
(a) the equation (4.2) has at most one classical solution u in the class $\mathscr{C}(p)$
(b) the function p satifies

$$\int_1^\infty \frac{dr}{p(r)} = \infty. \tag{4.14}$$

Proof. Case $L = \Delta - 2x\cdot\nabla$. Assume first that property (b) holds. We must show that every continuous OU-temperature $U(x,t)$ in $\mathbb{R}^n \times [0, T_0)$ such that $U(x,0) \equiv 0$ and which belongs to the class $\mathscr{C}(p)$ is identically zero. We may assume that $T_0 < \infty$, and we let $s_0 = (1 - e^{-4T_0})/4$. Using (2.5), we consider the function

$$u = T^{-1}U \in \mathscr{T}(\mathbb{R}^n \times (0, s_0))$$

which is continuous in $\mathbb{R}^n \times [0, s_0)$ and satisfies $u(y,0) \equiv 0$. We only need to show that $u \in \mathscr{C}(p_2)$, for a suitable p_2 with

$$\int_1^\infty \frac{dr}{\bar{p}_2(r)} = \infty, \tag{4.15}$$

since then Täcklind's Theorem 4.2 will imply that $u \equiv 0$ (and hence also $U \equiv 0$). Let $y = x/\exp(2t)$ and $s = (1 - e^{-4t})/4$ with $s \in (0, s_0)$ and $|y| \ge 1$. Then, in view of (2.5), if we set $\alpha = \exp(2T_0)$, we have

$$\begin{aligned} |u(y,s)| = |U(x,t)| &\le A\, e^{e^{2t}|y| p(e^{2t}|y|)} \\ &\le A\, e^{\alpha^\gamma |y| p(\alpha|y|)} = A\, e^{|y| p_2(|y|)}, \end{aligned}$$

with $p_2(r) = \alpha^\gamma p(\alpha r)$. Then $u \in \mathscr{C}(p_2)$, where $p_2(r)$ satisfies the usual conditions and by Lemma 4.3 also (4.15). This concludes the proof of this part.

We next show that (a) $\Rightarrow$ (b). Suppose by contradiction that $\int_1^\infty \frac{dr}{\bar{p}(r)} < \infty$. Then, following Täcklind's work [14, (35)] one may construct a function $u \in \mathscr{T}(\mathbb{R}^n \times \mathbb{R})$ such that $u(y,0) \equiv 0$, $u \not\equiv 0$ and

$$\sup_{s\in\mathbb{R}} |u(y,s)| \le e^{|y|\,\bar{p}(|y|\vee 1)}, \quad \forall\, y \in \mathbb{R}^n. \tag{4.16}$$

We again use (2.5) to define the function

$$U = Tu \in \mathscr{T}_{\rm OU}(\mathbb{R}^n \times \mathbb{R}), \tag{4.17}$$

which satisfies $U(x,0) \equiv 0$ and $U \not\equiv 0$. If we show that $U \in \mathscr{C}(p)$ we would reach a contradiction with (a). But this is immediate from

$$|U(x,t)| = \Big|u\Big(\tfrac{x}{e^{2t}}, \tfrac{1-e^{-4t}}{4}\Big)\Big| \le e^{\frac{|x|}{e^{2t}}\bar{p}(\frac{|x|}{e^{2t}}\vee 1)} \le e^{|x|\bar{p}(|x|\vee 1)}, \quad t>0,\ x\in\mathbb{R}^n. \tag{4.18}$$

Case $L = \Delta - |x|^2$. We show that (b) $\Rightarrow$ (a). Let $V \in \mathscr{T}_{\rm H} \cap \mathscr{C}(p)$ be continuous in $\mathbb{R}^n \times [0,T_0)$ and with $V(x,0) \equiv 0$. We must show that $V \equiv 0$ in $\mathbb{R}^n \times (0,T_0)$, for which we may assume that $T_0 < \infty$. By (2.1) and (2.3), the function

$$U(x,t) = e^{nt} e^{\frac{|x|^2}{2}} V(x,t) \in \mathscr{T}_{\rm OU} \cap C(\mathbb{R}^n \times [0,T_0)) \quad \text{and} \quad U(x,0) \equiv 0.$$

Using that $V \in \mathscr{C}(p)$ we see that, for $t \in (0,T_0)$ and $|x| \ge 1$,

$$|U(x,t)| \le e^{nT_0}\, e^{\frac{|x|^2}{2}}\, A\, e^{|x|\,p(|x|)} = A'\, e^{|x|\,p_1(|x|)},$$

with $p_1(r) = r/2 + p(r)$. Thus, $U \in \mathscr{C}(p_1)$ and by Lemma 4.4 we have $\int_1^\infty dr/\bar{p}_1(r) = \infty$, so the previous case gives that $U \equiv 0$ (hence also $V \equiv 0$).

We finally prove the converse implication (a) $\Rightarrow$ (b), assuming again for contradiction that (b) fails. Consider the function $U(x,t)$ constructed in (4.17) using the Täcklind example in (4.16). Arguing once again as in section 2.1, we define

$$V(x,t) = e^{-nt} e^{-\frac{|x|^2}{2}}\, U(x,t),$$

which belongs to $\mathscr{T}_{\rm H}(\mathbb{R}^n \times \mathbb{R})$ and satisfies $V(x,0) \equiv 0$, $V \not\equiv 0$ and

$$|V(x,t)| \le |U(x,t)| \le e^{|x|\bar{p}(|x|\vee 1)}, \quad t>0,\ x\in\mathbb{R}^n. \tag{4.19}$$

Thus $V \in \mathscr{C}(p)$, which is in contradiction with (a). $\square$

We now turn to Theorem 1.2, where we shall use an additional argument to replace the sufficient condition in (4.5) by a unilateral bound. We need the following maximum principle, shown in [9, Theorem III] in a general setting which includes the H-operator (but not the OU-operator).

Theorem 4.6 (Hayne). *Let L be one of the operators in* (4.1). *Let $U \in C^{2,1}(\mathbb{R}^n \times (0,T_0)) \cap C(\mathbb{R}^n \times [0,T_0))$ be such that*

(i) $U_t \le LU$ *in* $\mathbb{R}^n \times (0,T_0)$
(ii) $U(x,0) \le 0,\ x \in \mathbb{R}^n$
(iii) $U(x,t) \le Ae^{|x|p(|x|)},\ |x| \ge 1,\ t \in (0,T_0)$,

where $p(r)$ is positive continuous with $r^\gamma p(r)$ non-decreasing for some $\gamma \geq 0$, and

$$\int_1^\infty \frac{dr}{p(r)} = \infty.$$

Then, $U(x,t) \leq 0$ in $\mathbb{R}^n \times [0,T_0)$.

Proof. The cases $L = \Delta$ and $L = \Delta - |x|^2$ are contained in the settings of [9, Theorem III]. So we only show how to derive the result for $L = \Delta - 2x \cdot \nabla$. Suppose that $U \in \mathscr{T}_{\mathrm{OU}}^{\mathrm{sub}}$ satisfies (i)-(iii) above. As in §2.1, define

$$V(x,t) = e^{-nt} e^{-\frac{|x|^2}{2}} U(x,t),$$

which also belongs to $\mathscr{T}_{\mathrm{H}}^{\mathrm{sub}}$, and clearly satisfies (i)-(iii) (with U replaced by V). Then the H-case gives $V \leq 0$, and hence also $U \leq 0$ in $\mathbb{R}^n \times [0,T_0)$. □

Proof of Theorem 1.2. The statements of the converse implications are contained in the proof of Theorem 4.5; see (4.18) and (4.19), so we only explain how to derive the direct implications. Suppose that $U \in \mathscr{T}_{\mathrm{OU}}$ (or $U \in \mathscr{T}_{\mathrm{H}}$) is continuous in $\mathbb{R}^n \times [0,T_0)$ and satisfies $U(x,0) = 0$ and the upper condition in (1.3). Then, Hayne's Theorem 4.6 implies that $U(x,t) \leq 0$ in all $\mathbb{R}^n \times [0,T_0)$. That is we have

$$U \in \mathscr{T}_{\mathrm{OU}}, \quad U \leq 0 \text{ in } \mathbb{R}^n \times [0,T_0) \quad \text{and} \quad U(x,0) \equiv 0.$$

But these are the conditions of Widder uniqueness theorem for negative temperatures, which in the general version given by Aronson and Besala [3] will imply that $U \equiv 0$. This also applies if $U \in \mathscr{T}_{\mathrm{H}}$, as both H and OU temperatures are covered in the setting of [3]. □

5. Harnack inequalities

A well-known use of mean value formulas is to establish Harnack-type inequalities. The procedure to do so for classical temperatures $u \in \mathscr{T}$ in a set $E \subset \mathbb{R}^n \times \mathbb{R}$ is explained in detail in Watson's book; see [16, §1.7]. A crucial step is to replace the weight function $K_{x,t}(y,s) = 2^{-1}|x-y|^2/(t-s)^2$ in the mean value formula (2.9), by an expression which does not blow-up when $(y,s) \to (x,t)$ within the ball $\Omega(x,t;r)$. This can be done by regarding u as a temperature in $\mathbb{R}^{n+m} \times \mathbb{R}$ and deriving a formula by the method of descent; see [16, (1.23)]. The procedure will change slightly the shape of the balls, which are given by $(y,s) \in \Omega_m(x,t;r)$ iff

$$|y-x|^2 < 2(n+m)(t-s) \ln \tfrac{r}{t-s}, \quad s \in (t-r,t), \tag{5.1}$$

and it also leads to slightly more complicated kernels

$$K^m_{x,t}(y,s) = c_m \Big(\frac{|x-y|^2}{(t-s)^2} + a_{n,m} \frac{\ln \frac{r}{t-s}}{t-s} \Big) R^m, \tag{5.2}$$

with $R^2 = [2(n+m)(t-s)\ln\frac{r}{t-s} - |x-y|^2]/r$; see [16, p.19]. If $m \geq 3$, the multiplication by the factor R^m implies that

$$\sup_{r>0} \sup_{(x,t)\in\mathbb{R}^{n+1}} \sup_{(y,s)\in\Omega_m(x,t;r)} r\, K^m_{x,t}(y,s) < \infty. \tag{5.3}$$

Such constructions are also possible for temperatures in $\mathscr{T}_{\rm OU}$, and produce the following improvement over Theorem 1.1.

Theorem 5.1. *There exist sets $\Xi(x,t;r)$, for $(x,t)\in\mathbb{R}^{n+1}$ and $r>0$, and positive kernels $K^{\rm OU}_{x,t}(y,s)$ in $C^\infty\big(\mathbb{R}^n\times(-\infty,t)\big)$ satisfying*

$$A_n = \sup_{r>0} \sup_{(x,t)\in\mathbb{R}^{n+1}} \sup_{(y,s)\in\Xi(x,t;r)} r\, e^{2(n+2)s} K^{\rm OU}_{x,t}(y,s) < \infty, \tag{5.4}$$

and such that Theorem 2.4 holds for these kernels.

Proof. The same proof given in §2.4 works here replacing (2.18) by the mean value formula in [16, Theorem 1.25], based in the balls $\Omega_m(\tilde{x},\tilde{t};r)$ in (5.1) and the kernels $K^m_{\tilde{x},\tilde{t}}$ in (5.2). Taking $m=3$, and defining the kernels $K^{\rm OU}_{x,t}$ by the formula (2.19), all the assertions follow easily. □

From Theorem 5.1 and the arguments in [16, §1.7] (or by a direct change of variables), one can obtain the following Harnack inequality.

Corollary 5.2. *Let $E\subset\mathbb{R}^n\times\mathbb{R}$, μ a measure on E, and K a compact set such that*

$$K \subset \bigcup_{P\in\operatorname{Supp}\mu} \Lambda(P,E). \tag{5.5}$$

Then there exists a constant $\kappa=\kappa(E,\mu,K)>0$ such that

$$\max_K U \leq \kappa \int_S U\,d\mu, \quad \forall\, U\in\mathscr{T}_{\rm OU}(E) \quad \textit{with} \quad U\geq 0.$$

Remark 5.3. In the special case when $\mu=\delta_{(x_0,t_0)}$ and $K\subset\Lambda(x_0,t_0,E)$ one obtains

$$\max_K U \leq \kappa\, U(x_0,t_0), \quad \forall\, U\in\mathscr{T}_{\rm OU}(E) \quad \text{with} \quad U\geq 0.$$

As it is expected from a parabolic equation, this is an *interior* Harnack inequality, in the sense that the t-location of the set K must be *strictly* below the point (x_0,t_0).

We finally state a Harnack-type inequality with no time separation, but with a different right hand side. The result in (5.6) below, for parameters $q\geq 1$, could be obtained directly from Theorem 5.1. We give, however, a stronger formulation valid for all $q>0$. This formulation, for $u\in\mathscr{T}^{\rm sub}$ and for standard cylinders $C_r(x_0,t_0)=B_r(x_0)\times(t_0-r^2,t_0)$, is contained in the work of Ferretti and Safonov; see [6, Theorem 3.4], [5, Thm 3.1]. In our setting we shall consider the following "Hermite-cylindrical" sets

$$\Gamma_R(x_0,t_0) = \Big\{(x,t)\in\mathbb{R}^{n+1} \,:\, \big|e^{2(t_0-t)}x - x_0\big| < R, \quad t\in(t_0-R^2,t_0)\Big\},$$

and the measure $d\nu(x,t) = e^{-2(n+2)t}\,dx\,dt$.

Corollary 5.4. *For every $q > 0$, there exists a constant $\kappa = \kappa(q,n) > 0$ such that such that*

$$\max_{\Gamma_R(x_0,t_0)} U \le \kappa \left[\fint_{\Gamma_{4R}(x_0,t_0)} U^q \, d\nu \right]^{1/q}, \tag{5.6}$$

for all non-negative $U \in \mathscr{T}_{\mathrm{OU}}^{\mathrm{sub}}(E)$ and all $\overline{\Gamma}(x_0,t_0,4R) \subset E$ with $R \le 1$.

Proof. We begin with the following observation: if $(y_0,s_0) = \varphi(x_0,t_0)$ and $r = Re^{-2t_0}$ with $R \le 1$, then

$$C_r(y_0,s_0) \subset \varphi\big(\Gamma_R(x_0,t_0)\big) \subset C_{\lambda r}(y_0,s_0), \tag{5.7}$$

for some $\lambda < 4$. To see this, first note that, if $(y,s) = \varphi(x,t)$,

$$|y - y_0| < r \quad \iff \quad \left|e^{2(t_0-t)}x - x_0\right| < re^{2t_0} = R.$$

On the other hand

$$s_0 - s = e^{-4t_0}(e^{4(t_0-t)} - 1)/4 \ge e^{-4t_0}(t_0 - t), \tag{5.8}$$

So if $s_0 - s \le r^2$ then $t_0 - t \le R^2$, and the left inclusion in (5.7) holds, actually for all $R > 0$. For the right inclusion we use the convexity inequality

$$e^z - 1 \le \lambda^2 z, \quad z \in (0,4), \quad \text{with } \lambda^2 = (e^4-1)/4 < 14. \tag{5.9}$$

Then, if $t_0 - t \le R^2$ and $R \le 1$, from (5.8) we see that $s_0 - s \le (\lambda r)^2$. Thus, (5.7) holds, and from here it also follows that

$$\nu\big(\Gamma_R(x_0,t_0)\big) = \left|\varphi\big(\Gamma_R(x_0,t_0)\big)\right| \approx r^{n+2} \approx (Re^{-2t_0})^{n+2}.$$

We now use [6, Theorem 3.4] and the change of variables in (2.5), to obtain

$$\sup_{\Gamma_R} U = \sup_{\varphi(\Gamma_R)} u \le \sup_{C_{\lambda r}} u \lesssim \left[\fint_{C_{4r}} u^q \right]^{1/q} \lesssim \left[\fint_{\Gamma_{4R}(x_0,t_0)} U^q \, d\nu \right]^{1/q}.$$

□

As a corollary, we deduce a uniqueness criterion which is slightly less restrictive than Theorem 4.5 above. The proof is similar to [5, Thm 3.2].

Corollary 5.5. *Let $U \in \mathscr{T}_{\mathrm{OU}}(\mathbb{R}^n \times (0,T_0)) \cap C(\mathbb{R}^n \times [0,T_0))$ be such that $U(x,0) \equiv 0$. Suppose that for some $q > 0$ and for some $p(r)$ as in Theorem 4.5 with $\int_1^\infty dr/p(r) = \infty$ it holds*

$$\int_0^{T_0} \int_{\mathbb{R}^n} |U(x,t)|^q \, e^{-|x|p(|x| \vee 1)} \, dx \, dt < \infty.$$

Then $U \equiv 0$.

Proof. We may assume that $T_0 < \infty$. Observe that $U^2 \in \mathscr{T}_{\mathrm{OU}}^{\mathrm{sub}}$ (by direct computation, or from Theorem 2.4 and Hölder's inequality). Applying Corollary 5.4 to U^2 with $q/2$, $|x_0| \ge 2$ $t_0 \in (0,T_0)$ and $R = 1$, we obtain

$$\begin{aligned} |U(x_0,t_0)|^q &\lesssim \fint_{\Gamma_1(x_0,t_0)} |U(x,t)|^q \, e^{-|x|p(|x|\vee 1)} \, e^{|x|p(|x|\vee 1)} \, d\nu(x,t) \\ &\le C_{T_0} \int_0^{T_0} \int_{\mathbb{R}^n} |U(x,t)|^q \, e^{-|x|p(|x|\vee 1)} \, dx \, dt \, e^{c_2|x_0|p(2|x_0|)}, \end{aligned}$$

using that for $(x,t) \in \Gamma_R(x_0,t_0)$ it holds $c_1|x_0| \leq |x| \leq 2|x_0|$. Thus, $U \in \mathscr{C}(p_2)$ with $p_2(r) = c_3 p(2r)$. Since $\int_1^\infty dr/p_2(r) = \infty$, the result follows from Theorem 4.5. $\square$

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