A characterization of wavelet families arising from biorthogonal MRA's of multiplicity d

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Abstract

In this paper we give a necessary and sufficient condition for a pair of wavelet families $\Psi = \{\psi^1, \dots, \psi^L\}$, $\tilde{\Psi} = \{\tilde{\psi}^1, \dots, \tilde{\psi}^L\}$, in $L^2(\mathbb{R}^n)$, to arise from a pair of biorthogonal MRA's. The condition is given in terms of simple equations involving the functions ψ^ℓ and $\tilde{\psi}^\ell$. To work in greater generality, we allow multiresolution analyses of arbitrary multiplicity, based on lattice translations and matrix dilations. Our result extends the characterization theorem of G. Gripenberg and X. Wang for dyadic orthonormal wavelets in $L^2(\mathbb{R})$, and includes, as particular cases, the sufficient conditions of P. Auscher and P.G. Lemarié in the biorthogonal situation.

1 Introduction

An *orthonormal (dyadic) wavelet* in $L^2(\mathbb{R})$ is a function ψ such that the system

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}, \tag{1.1}$$

forms an orthonormal basis of $L^2(\mathbb{R})$. This type of wavelets has been studied by many authors, and their main properties can be found in the texts: [DAU], [HW], [KL], [MEY],... In this paper, the *Fourier transform* of a function $f \in L^1(\mathbb{R}^n)$ will be denoted by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n,$$

with the usual extension to $L^2(\mathbb{R}^n)$ via the Plancherel theorem.

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It is well-known (see, *e.g.*, chapter 7 of [HW]) that a function ψ in the unit sphere of $L^2(\mathbb{R})$ is a wavelet if and only if

$$(A1) \qquad \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1, \quad a.e.\, \xi \in \mathbb{R}$$

$$(A2) \qquad \sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\overline{\widehat{\psi}(2^j(\xi+2q\pi))} = 0, \quad a.e.\, \xi \in \mathbb{R}, \;\; \forall\, q \in 2\mathbb{Z}+1.$$

A related (although simpler) statement asserts that the system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ in (1.1), is orthonormal if and only if

$$(A3) \qquad \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi+2k\pi)|^2 = 1, \quad a.e.\, \xi \in \mathbb{R}$$

$$(A4) \qquad \sum_{k\in\mathbb{Z}} \widehat{\psi}(2^j(\xi+2k\pi))\overline{\widehat{\psi}(\xi+2k\pi)} = 0, \quad a.e.\, \xi \in \mathbb{R}, \;\; \forall\, j \in \mathbb{Z}^+.$$

These four equations have turned out to be very useful when constructing examples of wavelets, as well as for proving many of their interesting properties: [FW], [HWW1], [HWW2], [GS],...

A more sophisticated method to produce wavelets, as well as algorithms to work with them, is the notion of *multiresolution analysis (MRA)*. That is, a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying:

$$(i) \qquad V_j \subset V_{j+1}, \quad \overline{\cup V_j} = L^2(\mathbb{R})$$

$$(ii) \qquad f(x) \in V_j \;\; \text{if and only if} \;\; f(2x) \in V_{j+1}$$

$$(iii) \qquad \exists\, \varphi \in V_0 \;\; : \;\; \{\varphi(\cdot - k)\}_{k\in\mathbb{Z}} \;\; \text{is an orthonormal basis of } V_0.$$

The main property of these objects is the existence of a function $\psi \in L^2(\mathbb{R})$ such that the system $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $W_0 \equiv V_0^\perp \cap V_1$ (see chapter 2 of [HW]). This function is easily seen to be an orthonormal wavelet, while the constructive method from which it is derived has enormous advantages in applications.

It was an open question for some time to determine which wavelets could be constructed in this way. That is, for what wavelets ψ, the sequence of spaces

$$V_J = \overline{\text{span}}\,\{\psi_{j,k} \;:\; k \in \mathbb{Z},\; j < J\}, \quad J \in \mathbb{Z},$$

forms a multiresolution analysis in $L^2(\mathbb{R})$. P. Auscher and P.G. Lemarié gave, in [AUS] and [LEM], very mild sufficient conditions for this to happen. However, it was not until some time later that a characterization theorem was obtained. It was proved independently by G. Gripenberg and X. Wang ([GRI], [WAN]) that a wavelet ψ arises from an MRA if and only if

$$(A5) \qquad \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^j(\xi+2k\pi))|^2 = 1, \quad a.e.\, \xi \in \mathbb{R}.$$

This new equation complements $(A1)-(A4)$ in the construction of examples. For instance, the Journé wavelet, whose Fourier transform is given by: $\widehat{\psi}(\xi) = \chi_I(|\xi|)$, where $I = [\frac{4\pi}{7}, \pi) \cup [4\pi, \frac{32\pi}{7})$, does not satisfy $(A5)$, and therefore, cannot arise from a multiresolution analysis. On the other hand, a wavelet ψ having its Fourier transform supported in $[-\frac{8\pi}{3}, \frac{8\pi}{3}]$ (these have been characterized in [HWW1]) does always satisfy $(A5)$, and therefore must arise from an MRA.

Equation $(A5)$ has also raised the interest of other researchers. Very recently it was proved in [PAP], using equation $(A5)$, that every orthonormal wavelet in $L^2(\mathbb{R})$ arises from a *generalized frame-MRA*. Related questions in an abstract harmonic analysis setting (with the role of $(A5)$ played by a suitable "multiplicity function") have been studied by L. Baggett *et al.* in [BMM] and [BAG].

The sufficient conditions of Auscher and Lemarié were given in great generality, including wavelet families in $L^2(\mathbb{R}^n)$, formed after lattice translations and matrix dilations. They also studied whether these could arise from a multiresolution analysis with *more than one* scaling function, and considered in addition the possibility of having biorthogonal wavelet families, at that time just introduced by Cohen *et al.* ([CDF], [COH],...).

The theorem of Gripenberg-Wang mentioned above, on the contrary, was only stated for dyadic orthonormal single wavelets in $L^2(\mathbb{R})$. The extension to more general cases is not so straightforward as one might think since, for example, just with the biorthogonal assumption, new necessary conditions appear. This was already announced in [WAN], where the following theorem was proved (precise definitions are given below):

THEOREM 1.2 *Let $\psi, \tilde{\psi}$ be a pair of biorthogonal dyadic wavelets in $L^2(\mathbb{R})$. Then, the couple $\{\psi, \tilde{\psi}\}$ arises from a pair of biorthogonal MRA's if and only if there exists a constant $C = C(\psi, \tilde{\psi}) > 0$ such that:*

$$(1.3) \qquad \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \widehat{\tilde{\psi}}(2^j(\xi+2k\pi))\overline{\widehat{\psi}(2^j(\xi+2k\pi))} = 1, \quad a.e.\ \xi \in \mathbb{R};$$

$$(1.4) \qquad \textit{for} \quad a.e.\ \xi \in \mathbb{R}, \quad \exists\ j = j(\xi) \in \mathbb{Z}^+ \ \textit{such that}$$

$$\frac{\left(\sum_{k\in\mathbb{Z}} |\widehat{\tilde{\psi}}(2^j(\xi+2k\pi))|^2\right)^{1/2} \left(\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^j(\xi+2k\pi))|^2\right)^{1/2}}{\left|\sum_{k\in\mathbb{Z}} \widehat{\tilde{\psi}}(2^j(\xi+2k\pi))\overline{\widehat{\psi}(2^j(\xi+2k\pi))}\right|} \le C\,.$$

Note that, when $\psi = \tilde{\psi}$, condition (1.4) in the theorem is void, while (1.3) becomes precisely $(A5)$, giving us the statement of the original Gripenberg-Wang result.

The goal of this paper is to find a characterization theorem for MRA wavelets covering all the possible cases mentioned above, but still in terms of simple equations of the type (A1)-(A5), or (1.3)-(1.4). In higher dimensions, such equations do exist and can be written in a clean matrix form, but their proofs require non-trivial constructions that were avoided in the scalar (1-dimensional) situation. We point out that, compared to the sufficient conditions already existing in the literature, we do not assume smoothness or decay in our functions, obtaining a result which is most general possible, and that allows us to test with it even the most "irregular" wavelet families.

The paper will be structured as follows. In §2 we introduce some preliminary material and state our theorem. The main ideas of the periodization argument we shall use in the proof are presented here. §3 is devoted to the proof of the theorem, after a careful study of the properties of biorthogonal multiresolution analyses in $L^2(\mathbb{R}^n)$ that has interest in its own. Finally, §§4 and 5 contain a series of corollaries and applications of our result, which include mild sufficient conditions for the theorem to hold (milder even than those of Lemarié and Auscher), and a variety of new examples of wavelets whose MRA nature does not fall in the scope of previous works. Among these examples, there are constructions of non-MRA wavelets in $\mathbb{R}^2$, and *multi-wavelet sets* in $\mathbb{R}^n$.

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2 Biorthogonal MRA's and wavelets in $L^2(\mathbb{R}^n)$

During the rest of the paper, Γ will be a fixed lattice in $\mathbb{R}^n$; that is, $\Gamma = A\mathbb{Z}^n$, for some $n \times n$ real matrix with $a =: \det(A) > 0$. We also fix M, a dilation matrix preserving the lattice; that is, $M\Gamma \subset \Gamma$, and all the eigenvalues of M lie outside the closed unit disk of $\mathbb{C}$. Note that, in this case, $m := |\det(M)|$ is an integer greater than or equal to 2. The definitions of wavelet and MRA in this context can be found in several places ([AUS] and [LEM], for example). We shall give them here in order to establish the notation to be used later on. Before doing so, we recall some definitions from the abstract Hilbert space setting (see also [YOU]).

A system of vectors $\{e_j\}$, $j = 1, 2, \ldots$, in a Hilbert space $\mathcal{H}$ is said to be a *Bessel* *system* if there exists a constant $B > 0$ such that

$$\sum_{j=1}^{\infty} |\langle f, e_j \rangle|^2 \leq B \, \|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

A system $\{e_j\}_{j=1}^{\infty}$ is said to form a *Riesz basis* for $\mathcal{H}$ if its linear expansion is dense, and there are two constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \sum_{j=1}^{\infty} |\alpha_j|^2 \leq \| \sum_{j=1}^{\infty} \alpha_j e_j \|^2 \leq c_2 \sum_{j=1}^{\infty} |\alpha_j|^2, \quad \text{for all sequences } (\alpha_j)_{j=1}^{\infty} \in \ell^2(\mathbb{Z}^+).$$

Finally, two systems of vectors $\{e_j\}$, $\{\tilde{e}_j\}$, $j = 1, 2, \ldots$, are said to be *biorthogonal* (with respect to the inner product $\langle \cdot, \cdot \rangle$ in $\mathcal{H}$) if $\langle e_j, \tilde{e}_{j'} \rangle = \delta_{j,j'}$, $\forall\, j, j'$, where δ denotes the usual Krönecker symbol. We recall that if $\{e_j\}$ is a Riesz basis for $\mathcal{H}$, then there is another Riesz basis $\{\tilde{e}_j\}$, such that these two systems are biorthogonal and

$$f = \sum_{j=1}^{\infty} \langle f, \tilde{e}_j \rangle e_j = \sum_{j=1}^{\infty} \langle f, e_j \rangle \tilde{e}_j, \quad \text{for all } f \in \mathcal{H},$$

where the convergence of the series is unconditional in $\mathcal{H}$.

DEFINITION 2.1 *A system $\Psi = \{\psi^1, \ldots, \psi^L\}$ of vectors in $L^2(\mathbb{R}^n)$ is said to be a* wavelet family (for the lattice Γ and dilation matrix M) *if*

$$\psi^{\ell}_{j,\gamma}(x) = m^{j/2} \psi^{\ell}(M^j x - \gamma), \quad j \in \mathbb{Z}, \ \gamma \in \Gamma, \ \ell = 1, \ldots, L,$$

is a Riesz basis of $L^2(\mathbb{R}^n)$. Two sets $\Psi = \{\psi^1, \ldots, \psi^L\}, \widetilde{\Psi} = \{\tilde{\psi}^1, \ldots, \tilde{\psi}^L\} \subset L^2(\mathbb{R}^n)$ are said to form a pair of biorthogonal wavelet families*, whenever they are wavelet families, and the systems $\{\psi^{\ell}_{j,\gamma}\}, \{\tilde{\psi}^{\ell}_{j,\gamma}\}$, are biorthogonal in $L^2(\mathbb{R}^n)$; that is,*

$$\int_{\mathbb{R}^n} \psi^{\ell}_{j,\gamma}(x) \, \overline{\tilde{\psi}^{\ell'}_{j',\gamma'}(x)} \, dx \; = \; \delta_{\ell,\ell'} \, \delta_{j,j'} \, \delta_{\gamma,\gamma'} \,, \quad j, j' \in \mathbb{Z}, \ \gamma, \gamma' \in \Gamma, \ \ell, \ell' = 1, \ldots, L.$$

When $\Psi = \widetilde{\Psi}$, we obtain an orthonormal basis of $L^2(\mathbb{R}^n)$, and Ψ is called, then, an orthonormal wavelet family.

A matter that has raised the interest of several researchers is the characterization of this general type of wavelets in terms of equations like $(A1) - (A4)$ ([FGWW], [CAL], [RS], [HAN1]). If we denote by $N = M'$, the transpose matrix of M, by $\Gamma' = 2\pi (A')^{-1} \mathbb{Z}^n$, the *dual lattice* of Γ, and $\mathcal{O} = \Gamma' \setminus N\Gamma'$, we have the following result:

THEOREM 2.2 *Let $\Psi = \{\psi^1, \ldots, \psi^L\}, \widetilde{\Psi} = \{\tilde{\psi}^1, \ldots, \tilde{\psi}^L\} \subset L^2(\mathbb{R}^n)$. Then, they form a pair of biorthogonal wavelet families (for $\Gamma = A\mathbb{Z}^n$ and M) if and only if* *the systems* $\{\psi^\ell_{j,\gamma}\}, \{\tilde\psi^\ell_{j,\gamma}\}$*,* $j \in \mathbb{Z}, \gamma \in \Gamma, \ell = 1, \ldots, L$*, are Bessel, and the following equations hold for almost every* $\xi \in \mathbb{R}^n$*:*

$$(B1) \qquad \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \widehat{\tilde\psi^\ell}(N^j \xi)\, \overline{\widehat{\psi^\ell}(N^j\xi)} = \det(A)$$

$$(B2) \qquad t_\gamma(\xi) \equiv \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \widehat{\tilde\psi^\ell}(N^j \xi)\, \overline{\widehat{\psi^\ell}(N^j(\xi+\gamma))} = 0, \quad \forall\, \gamma \in \mathcal{O}$$

$$(B3) \qquad \sum_{\gamma \in \Gamma'} \widehat{\tilde\psi^{\ell_1}}(\xi+\gamma)\, \overline{\widehat{\psi^{\ell_2}}(\xi+\gamma)} = \det(A)\, \delta_{\ell_1,\ell_2}, \quad \ell_1, \ell_2 = 1, \ldots, L$$

$$(B4) \qquad \begin{cases} \sum_{\gamma\in\Gamma'} \widehat{\tilde\psi^{\ell_1}}(N^j(\xi+\gamma))\, \overline{\widehat{\psi^{\ell_2}}(\xi+\gamma)} = 0 \\ \sum_{\gamma\in\Gamma'} \widehat{\tilde\psi^{\ell_1}}(\xi+\gamma)\, \overline{\widehat{\psi^{\ell_2}}(N^j(\xi+\gamma))} = 0, \quad \forall\, j \in \mathbb{Z}^+, \ell_1, \ell_2 = 1, \ldots, L. \end{cases}$$

Equations $(B3)$ and $(B4)$ characterize the biorthogonality of the systems $\{\psi^\ell_{j,\gamma}\}$ and $\{\tilde\psi^\ell_{j,\gamma}\}$, while $(B1)$, $(B2)$ and the Bessel condition will give the completeness and unconditionality of the bases. While in the orthonormal situation the Bessel assumption is redundant (as we mentioned in §1; see [CAL] for more details), in the biorthogonal case there are counterexamples showing that it cannot be dropped (see [FGWW]). This has to do with the role played by (1.4) in Theorem 1.2, as we shall see below.

Remark 2.3 We point out that the version of Theorem 2.2 presented above is not stated in this generality in any of the papers mentioned. The authors of [FGWW], for instance, only treat the case $\Gamma = \mathbb{Z}^n$ and $M = 2I_n$ (I_n is the identity matrix in $\mathbb{R}^n$), while [CAL], that uses general lattices and matrix dilations, only considers the orthonormal case ($\Psi = \widetilde\Psi$). The papers [RS], [HAN1], allowing the same general context as ours, need to assume a (mild) smoothness condition on ψ^ℓ and $\tilde\psi^\ell$ for their arguments to work. However, it is not difficult to adapt the proofs of the first two papers to obtain Theorem 2.2 (this was also communicated to us by M. Bownik [BOW]).

Examples of wavelets constructed via the equations above have been considered by several authors: [SW], [DLS], [CAL], ... More useful for applications, however, are the wavelets constructed from multiresolution analyses: [CDF], [CD2], [CT], [HER], [DS], [CS], [GM], [STR], [RIS], ... We recall that, in this context, a *multiresolution analysis of multiplicity* D (for Γ and M) is a sequence, $\{V_j\}_{j\in\mathbb{Z}}$, of closed subspaces of $L^2(\mathbb{R}^n)$ satisfying:

(i) $\quad V_j \subset V_{j+1}, \quad \overline{\cup V_j} = L^2(\mathbb{R}^n)$

(ii) $\quad f(x) \in V_j$ if and only if $f(Mx) \in V_{j+1}$

(iii) $\quad \exists\, \varphi^1, \ldots, \varphi^D \in V_0 : \{\varphi^d(\cdot - \gamma) : \gamma \in \Gamma, d = 1, \ldots, D\}$ is a Riesz basis of V_0.

A set $\Phi = \{\varphi^1, \ldots, \varphi^D\}$ as in (iii) above is called a *scaling family* for the MRA. Two such MRA's, $\{V_j\}$, $\{\widetilde{V}_j\}$, are said to be *biorthogonal* if there are scaling families, $\Phi = \{\varphi^1, \ldots, \varphi^D\}$, $\tilde{\Phi} = \{\tilde{\varphi}^1, \ldots, \tilde{\varphi}^D\}$, of $\{V_j\}$ and $\{\widetilde{V}_j\}$, respectively, such that

$$\int_{\mathbb{R}^n} \varphi^d(x-\gamma)\, \overline{\tilde{\varphi}^{d'}(x-\gamma')}\, dx = \delta_{d,d'}\, \delta_{\gamma,\gamma'}, \quad \gamma,\, \gamma' \in \Gamma,\ d,\, d' = 1, \ldots, D.$$

The construction of wavelets from an MRA in the general context described above, can be very tricky when $(m-1)D \geq 2$, especially if one wishes to obtain "regular" wavelets (that is, smooth and with sufficiently rapid decay). Different analytic techniques have been used to overcome the topological impediments that appear in the process, obtaining all sorts of examples: [GRO], [JM], [LEM], [HER], [RIS], ...

The situation becomes simpler if one worries only about *measurable* wavelets. When $V_j = \widetilde{V}_j$, for instance, it was shown in [GLT] and [JS], using elementary methods, the existence of orthonormal wavelet families from the multiresolution analysis. We do not know whether this is possible in the biorthogonal setting without regularity assumptions in the scaling families (see [COH], or [CD1] for some necessary and sufficient conditions).

In this paper we will not be interested in this question, but rather in the converse. That is, we will characterize the pairs of biorthogonal wavelet families, $\Psi = \{\psi^1, \ldots, \psi^L\}, \tilde{\Psi} = \{\tilde{\psi}^1, \ldots, \tilde{\psi}^L\} \subset L^2(\mathbb{R}^n)$, for which the scales of spaces:

$$\left.\begin{aligned} V_J &= \overline{\operatorname{span}}\,\{\psi^\ell_{j,\gamma} \,:\, \ell = 1, \ldots, L,\ \gamma \in \Gamma,\ j < J\}, \quad \text{and} \\ \widetilde{V}_J &= \overline{\operatorname{span}}\,\{\tilde{\psi}^\ell{}_{j,\gamma} \,:\, \ell = 1, \ldots, L,\ \gamma \in \Gamma,\ j < J\}, \quad J \in \mathbb{Z}, \end{aligned}\right\} \tag{2.4}$$

form a pair of biorthogonal MRA's. P. Auscher gave a sufficient condition for this to happen in [AUS]. We will use some of his results to prove our characterization theorem. We begin with an algebraic lemma taken directly from [AUS] (see Proposition 5.2 there).

Lemma 2.5 *Let $\{e_j\}, \{\tilde{e}_j\}$, $j \in \mathbb{Z}^+$, be two families of vectors in a Hilbert space $\mathcal{H}$, and suppose that the sums $\sum_{j=1}^\infty \|e_j\|^2 = c$, $\sum_{j=1}^\infty \|\tilde{e}_j\|^2 = \tilde{c}$, are finite and:*

$$e_i = \sum_{j=1}^\infty \langle e_i, \tilde{e}_j\rangle e_j, \quad \tilde{e}_i = \sum_{j=1}^\infty \langle \tilde{e}_i, e_j\rangle \tilde{e}_j, \quad \forall\, i \in \mathbb{Z}^+.$$

Let $\mathcal{F} = \overline{\operatorname{span}}\,\{e_j\}$, $\widetilde{\mathcal{F}} = \overline{\operatorname{span}}\,\{\tilde{e}_j\}$, and define

$$Tv = \sum_{j=1}^\infty \langle v, \tilde{e}_j\rangle e_j, \quad \widetilde{T}v = \sum_{j=1}^\infty \langle v, e_j\rangle \tilde{e}_j, \quad v \in \mathcal{H}.$$

Then, $T, \widetilde{T}$ are bounded operators with $\|T\|$, $\|\widetilde{T}\| \leq \sqrt{c\,\tilde{c}}$, and the spaces $\mathcal{F}$, $\widetilde{\mathcal{F}}$ are finite-dimensional. Moreover, T is the projector onto $\mathcal{F}$ "parallel" to $\widetilde{\mathcal{F}}$ (that is,

$\mathcal{H} = \mathcal{R}(T) \oplus Ker(T) = \mathcal{F} \oplus \tilde{\mathcal{F}}^{\perp}$) *and*

$$(j) \qquad \dim \mathcal{F} = \dim \tilde{\mathcal{F}} = \sum_{j=1}^{\infty} \langle e_j, \tilde{e}_j \rangle,$$

$$(jj) \qquad \langle Tv, w \rangle = \langle v, \tilde{T} w \rangle, \quad \forall\, v, w \in \mathcal{H},$$

$$(jjj) \qquad S \equiv T|_{\tilde{\mathcal{F}}} : \tilde{\mathcal{F}} \longrightarrow \mathcal{F} \quad \textit{is an isomorphism with}$$
$$\|v\| \le \|Sv\| \le \sqrt{c\,\tilde{c}}\,\|v\|, \quad \forall\, v \in \tilde{\mathcal{F}}.$$

Towards the statement of our main result, we will translate the wavelet and MRA definitions into a *pointwise periodic setting* that will fit into Lemma 2.5.

Suppose now that $\Psi = \{\psi^1, \dots, \psi^L\}, \tilde{\Psi} = \{\tilde{\psi}^1, \dots, \tilde{\psi}^L\}$ are two biorthogonal wavelet families in $L^2(\mathbb{R}^n)$ associated with $\Gamma = A\mathbb{Z}^n$ and M. If $\mathbb{T}^n = [-\pi, \pi)^n$, and $\Gamma' = 2\pi (A')^{-1}\mathbb{Z}^n$, then $\{(A')^{-1}\mathbb{T}^n + \gamma\}_{\gamma \in \Gamma'}$ forms a partition of $\mathbb{R}^n$. Moreover, when $\eta \in L^2(\mathbb{R}^n)$, we have

$$\int_{(A')^{-1}\mathbb{T}^n} \sum_{j=1}^{\infty} \sum_{\gamma \in \Gamma'} |\hat{\eta}(N^j(\xi + \gamma))|^2 \, d\xi = \sum_{j=1}^{\infty} m^{-j} \int_{\mathbb{R}^n} |\hat{\eta}(\xi)|^2 \, d\xi < \infty.$$

Therefore,

$$\sum_{j=1}^{\infty} \sum_{\gamma \in \Gamma'} |\hat{\eta}(N^j(\xi + \gamma))|^2 < \infty, \quad \text{for} \quad a.e.\, \xi \in \mathbb{R}^n, \tag{2.6}$$

and the series is Γ'-periodic. Thus, for every $j \in \mathbb{Z}^+$, $\ell = 1, \dots, L$, and $a.e.\, \xi \in \mathbb{R}^n$, we can define the vector:

$$\omega_j^{\ell}(\xi) = \{\, a^{-\frac{1}{2}} \widehat{\psi}^{\ell}(N^j(\xi + \gamma)) \,\}_{\gamma \in \Gamma'} \in \mathcal{H} \equiv \ell^2(\Gamma')$$

and similarly $\tilde{\omega}_j^{\ell}(\xi)$, with ψ replaced by $\tilde{\psi}$. If we denote by $[\![\,\cdot\,]\!]$ and $[\![\cdot, \cdot]\!]$, respectively, the norm and inner product in $\mathcal{H}$, we have, by (2.6),

$$\sum_{\ell=1}^{L} \sum_{j=1}^{\infty} [\![\omega_j^{\ell}(\xi)]\!]^2, \quad \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} [\![\tilde{\omega}_j^{\ell}(\xi)]\!]^2 < \infty, \quad a.e.\, \xi \in \mathbb{R}^n. \tag{2.7}$$

To be in the setting of Lemma 2.5 we just need to show:

Lemma 2.8 *With the above assumptions, the following equalities hold, for every* $j \in \mathbb{Z}^+$, $\ell = 1, \dots, L$, *and* $a.e.\, \xi \in \mathbb{R}^n$*:*

$$\omega_j^{\ell}(\xi) = \sum_{h=1}^{L} \sum_{i=1}^{\infty} [\![\omega_j^{\ell}(\xi), \tilde{\omega}_i^{h}(\xi)]\!] \, \omega_i^{h}(\xi) \quad \textit{and} \quad \tilde{\omega}_j^{\ell}(\xi) = \sum_{h=1}^{L} \sum_{i=1}^{\infty} [\![\tilde{\omega}_j^{\ell}(\xi), \omega_i^{h}(\xi)]\!] \, \tilde{\omega}_i^{h}(\xi).$$

Proof: First of all, note that, because of (2.7), all the series appearing in the statement of the lemma converge absolutely for $a.e.\,\xi \in \mathbb{R}^n$. Now, let us denote the series on the right (multiplied by $a^{\frac{3}{2}}$) by:

$$G_j^\ell(\xi) = \sum_{h=1}^{L}\sum_{i=1}^{\infty}\sum_{\gamma\in\Gamma'} \widehat{\tilde{\psi}^\ell}(N^j(\xi+\gamma))\,\overline{\widehat{\psi^h}(N^i(\xi+\gamma))}\,\widehat{\tilde{\psi}^h}(N^i\xi).$$

It suffices to show that $G_j^\ell(\xi) = a\widehat{\tilde{\psi}^\ell}(N^j\xi)$, $a.e.\,\xi \in \mathbb{R}^n$. Rearranging the summations, and using equations $(B2), (B4)$ (since $j \geq 1$), we obtain:

$$\begin{aligned}
G_j^\ell(\xi) &= \sum_{\gamma\in\Gamma'} \widehat{\tilde{\psi}^\ell}(N^j(\xi+\gamma)) \sum_{h=1}^{L}\sum_{i=1}^{\infty} \widehat{\tilde{\psi}^h}(N^i\xi)\,\overline{\widehat{\psi^h}(N^i(\xi+\gamma))} \\
&= \sum_{\gamma\in\Gamma'} \widehat{\tilde{\psi}^\ell}(N^j(\xi+\gamma))\,[\,t_\gamma(\xi) - \sum_{h=1}^{L} \widehat{\tilde{\psi}^h}(\xi)\,\overline{\widehat{\psi^h}(\xi+\gamma)}\,] \\
&= \sum_{\gamma\in N\Gamma'} \widehat{\tilde{\psi}^\ell}(N^j(\xi+\gamma))\,t_\gamma(\xi) \\
&= \sum_{\gamma\in\Gamma'} \widehat{\tilde{\psi}^\ell}(N^{j+1}(N^{-1}\xi+\gamma)) \sum_{h=1}^{L}\sum_{i=0}^{\infty} \widehat{\tilde{\psi}^h}(N^{i+1}(N^{-1}\xi))\,\overline{\widehat{\psi^h}(N^{i+1}(N^{-1}\xi+\gamma))} \\
&= \sum_{h=1}^{L}\sum_{i=1}^{\infty}\sum_{\gamma\in\Gamma'} \widehat{\tilde{\psi}^\ell}(N^{j+1}(N^{-1}\xi+\gamma))\,\overline{\widehat{\psi^h}(N^i(N^{-1}\xi+\gamma))}\,\widehat{\tilde{\psi}^h}(N^i(N^{-1}\xi)) \\
&= G_{j+1}^\ell(N^{-1}\xi).
\end{aligned}$$

Iterating, we obtain $G_j^\ell(\xi) = G_1^\ell(N^{j-1}\xi)$. Now, we calculate $G_1^\ell(\xi)$ using similar arguments as before:

$$\begin{aligned}
G_1^\ell(\xi) &= \sum_{\gamma\in\Gamma'} \widehat{\tilde{\psi}^\ell}(N(\xi+\gamma)) \sum_{h=1}^{L}\sum_{i=1}^{\infty} \widehat{\tilde{\psi}^h}(N^i\xi)\,\overline{\widehat{\psi^h}(N^i(\xi+\gamma))} + \sum_{\gamma\in\mathcal{O}} \widehat{\tilde{\psi}^\ell}(N\xi+\gamma)\,t_\gamma(N\xi) \\
&= \sum_{\gamma\in\Gamma'} \widehat{\tilde{\psi}^\ell}(N\xi+\gamma) \sum_{h=1}^{L}\sum_{i=0}^{\infty} \widehat{\tilde{\psi}^h}(N^i(N\xi))\,\overline{\widehat{\psi^h}(N^i(N\xi+\gamma))} \\
&= \sum_{h=1}^{L}\sum_{i=0}^{\infty} \widehat{\tilde{\psi}^h}(N^i(N\xi))\,\delta_{i,0}\,\delta_{\ell,h}\,\det(A) \;=\; a\,\widehat{\tilde{\psi}^\ell}(N\xi).
\end{aligned}$$

Thus, $G_j^\ell(\xi) = a\widehat{\tilde{\psi}^\ell}(N^j\xi)$, and the lemma is proved.

□

Therefore, we can use Lemma 2.5 and define, for $a.e.\,\xi \in (A')^{-1}\mathbb{T}^n$, the spaces:

$$\mathcal{F}(\xi) = \overline{\operatorname{span}}\,\{\omega_j^\ell(\xi)\}_{\ell,j} \quad \text{and} \quad \widetilde{\mathcal{F}}(\xi) = \overline{\operatorname{span}}\,\{\tilde{\omega}_j^\ell(\xi)\}_{\ell,j}, \tag{2.9}$$

having both finite dimension equal to:

$$(B5)\qquad D_{\Psi,\tilde\Psi}(\xi)\equiv\sum_{\ell=1}^{L}\sum_{j=1}^{\infty}[\![\omega_j^\ell(\xi),\tilde\omega_j^\ell(\xi)]\!]=a^{-1}\sum_{\ell=1}^{L}\sum_{j=1}^{\infty}\sum_{\gamma\in\Gamma'}\widehat{\tilde\psi^\ell}(N^j(\xi+\gamma))\,\overline{\widehat{\psi}^\ell(N^j(\xi+\gamma))}.$$

Moreover, the map

$$v\longmapsto S(\xi)\,v=\sum_{\ell=1}^{L}\sum_{j=1}^{\infty}[\![v,\tilde\omega_j^\ell(\xi)]\!]\,\omega_j^\ell(\xi),\quad v\in\widetilde{\mathcal{F}}(\xi),\tag{2.10}$$

is an isomorphism from $\widetilde{\mathcal{F}}(\xi)$ onto $\mathcal{F}(\xi)$. The main theorem in this paper can then be stated as follows:

Theorem 2.11 *Let $\Psi=\{\psi^1,\dots,\psi^L\}$, $\tilde\Psi=\{\tilde\psi^1,\dots,\tilde\psi^L\}$ be a pair of biorthogonal wavelet families in $L^2(\mathbb{R}^n)$ for the lattice $\Gamma=A\mathbb{Z}^n$ and dilation matrix M. Then, the systems $\{V_j\}$, $\{\widetilde V_j\}$ in (2.4) form a pair of biorthogonal multiresolution analyses if and only if there is a positive integer D such that*

$$\begin{aligned}&(I)\qquad D_{\Psi,\tilde\Psi}(\xi)=D,\quad a.e.\,\xi\in(A')^{-1}\mathbb{T}^n,\\&(II)\qquad \text{there exists a constant } C=C(\Psi,\tilde\Psi)>0 \text{ such that}\\&\qquad\qquad \|S(\xi)\|\le C,\quad a.e.\,\xi\in(A')^{-1}\mathbb{T}^n.\end{aligned}$$

In this case, $L=(m-1)D$, where $m=|\det(M)|$ and the integer D coincides with the multiplicity of the MRA's.

For convenience, we have used in the statement of the theorem the notation introduced in (2.10). Below, we shall give an explicit expression for $\|S(\xi)\|$ only in terms of Ψ and $\widetilde\Psi$. Note that, as happened in the 1-dimensional situation, condition (II) is redundant when $\Psi=\widetilde\Psi$. Indeed, if this is the case, we have $\mathcal{F}(\xi)=\widetilde{\mathcal{F}}(\xi)$, and the projection $T(\xi)\colon\mathcal{H}\longrightarrow\mathcal{F}(\xi)$ is orthogonal (since $\mathcal{H}=\mathcal{F}(\xi)\oplus\mathcal{F}(\xi)^\perp$). In particular, $\|T(\xi)\|=\|S(\xi)\|=1$. We obtain, then, the following corollary, already announced in [CAL2] (see also [SW] and [BAG]).

Corollary 2.12 *Let $\Psi=\{\psi^1,\dots,\psi^L\}$ be an orthonormal wavelet family in $L^2(\mathbb{R}^n)$ with $L=(m-1)D$. Then, Ψ arises from a multiresolution analysis of multiplicity D if and only if*

$$\sum_{\ell=1}^{(m-1)D}\sum_{j=1}^{\infty}\sum_{\gamma\in\Gamma'}|\widehat{\psi}^\ell(N^j(\xi+\gamma))|^2=\det(A)D,\qquad a.e.\,\xi\in(A')^{-1}\mathbb{T}^n.$$

Before starting with the proof of the theorem in the next section, notice the following simplification:

REMARK 2.13 Without loss of generality, we will assume during the rest of the paper that the lattice $\Gamma = \mathbb{Z}^n$. Note that a function ψ is a wavelet, associated with the lattice $\Gamma = A\mathbb{Z}^n$ and matrix M, if and only if the function $\psi^\sharp = a^{\frac{1}{2}}\psi(A\cdot)$ is a wavelet associated with $\Gamma^\sharp = \mathbb{Z}^n$ and $M^\sharp = A^{-1}MA$. One does similarly with the MRA's V_j and $V_j^\sharp = \{f(A\cdot) \ : \ f \in V_j\}$.

3 The proof of the theorem

We start by proving the necessity of conditions (I) and (II) in Theorem 2.11. Assume then, that the sequences of spaces $\{V_j\}, \{\widetilde{V}_j\}$ in (2.4) form a pair of biorthogonal multiresolution analyses of, say, multiplicity D. We shall show first, using elementary properties of MRA's, that $L = (m-1)D$.

3.1 Necessity: the proof of $L = (m-1)D$

By definition, associated with the MRA's we have a pair of biorthogonal scaling families: $\Phi = \{\varphi^1, \dots, \varphi^D\}$, $\tilde{\Phi} = \{\tilde{\varphi}^1, \dots, \tilde{\varphi}^D\}$. For convenience, we will denote by $\widehat{\Phi}(\xi)$ and $\widehat{\tilde{\Phi}}(\xi)$ the column vectors in $\mathbb{C}^D$ with entries $\widehat{\varphi}^d(\xi)$ and $\widehat{\tilde{\varphi}}^d(\xi)$, $d = 1, \dots, D$, respectively. The biorthogonality of the systems $\{\varphi^d(\cdot - \gamma)\}_{\gamma,d}, \{\tilde{\varphi}^d(\cdot - \gamma)\}_{\gamma,d}$, is equivalent to equation $(B3)$ (with ψ replaced by φ), which in matrix notation can be written as:

$$\sum_{\gamma \in \Gamma'} \widehat{\Phi}(\xi + \gamma) \, \widehat{\tilde{\Phi}}(\xi + \gamma)^* = I_D, \quad a.e.\, \xi \in \mathbb{T}^n. \tag{3.1}$$

Here, A^* denotes the conjugate transpose of a matrix A, and I_D the identity matrix in $\mathbb{C}^D$. We have then, the following lemma:

LEMMA 3.2 *In the above conditions, the space V_1 consists precisely of the functions $f \in L^2(\mathbb{R}^n)$ whose Fourier transform can be written as*

$$\hat{f}(N\xi) = \sum_{d=1}^{D} m^{f,d}(\xi)\widehat{\varphi}^d(\xi), \quad a.e.\, \xi \in \mathbb{R}^n, \tag{3.3}$$

for (unique) Γ'-periodic functions $m^{f,1}, \dots, m^{f,D} \in L^2(\mathbb{T}^n)$. A similar statement holds for $\widetilde{V}_1$.

PROOF: The proof is straightforward. Since $\{\varphi^d(\cdot - \gamma)\}_{\gamma,d}$ is a Riesz basis for V_0, and a function $f \in V_1$ if and only if $f(M^{-1}\cdot) \in V_0$, associated with each such function there must exist coefficients $(c_\gamma^{f,d})_{d,\gamma} \in \ell^2(\{1, \dots, D\} \times \Gamma)$ such that:

$$m^{-1} f(M^{-1}x) = \sum_{d=1}^{D} \sum_{\gamma \in \Gamma} c_\gamma^{f,d} \, \varphi^d(x - \gamma), \quad \text{in } \ L^2(\mathbb{R}^n).$$

Taking Fourier transforms of both sides, we obtain precisely (3.3) with $m^{f,d}(\xi) = \sum_{\gamma\in\Gamma} c_\gamma^{f,d}\, e^{-i\,\gamma\cdot\xi}$ in $L^2(\mathbb{T}^n)$. The uniqueness of the functions $m^{f,d}$ follows from the uniqueness of the Riesz basis coefficients.

□

In particular, since $\varphi^d \in V_0 \subset V_1$, there are Γ'-periodic functions $m_0^{d,d'} \in L^2(\mathbb{T}^n)$, $d, d' = 1, \ldots, D$ such that $\widehat{\varphi}^d(N\xi) = \sum_{d'=1}^D m_0^{d,d'}(\xi)\widehat{\varphi}^{d'}(\xi)$. In matrix notation this can be written as:

$$\widehat{\Phi}(N\xi) = M_0(\xi)\,\widehat{\Phi}(\xi), \quad a.e.\,\xi \in \mathbb{R}^n, \tag{3.4}$$

where $M_0 = (m_0^{d,d'})_{d,d'=1\ldots,D}$ is usually called *low-pass filter* (or scaling matrix filter) associated with the scaling family Φ. Similarly, one has $\widehat{\widetilde{\Phi}}(N\xi) = \widetilde{M}_0(\xi)\,\widehat{\widetilde{\Phi}}(\xi)$ for the MRA $\{\widetilde{V}_j\}$.

Now, we choose a full set of representatives of the quotient space $\Delta \equiv \Gamma'/(N\Gamma')$. Since the number of distinct cosets is precisely $|\det(N)| = m$ (see, *e.g.*, [GM]), we can consider $\Delta = \{\delta_1, \ldots, \delta_m\}$, for appropriate lattice points $\delta_i \in \Gamma' = 2\pi\mathbb{Z}^n$ (from which we may assume $\delta_1 = 0$). Note that

$$\begin{aligned}\mathbb{R}^n &= \bigcup_{\gamma\in\Gamma'} [\,\mathbb{T}^n + \gamma\,] = \bigcup_{\delta\in\Delta}\bigcup_{\gamma\in\Gamma'} [\,(\mathbb{T}^n+\delta) + N\gamma\,] \\ &= \bigcup_{\gamma\in\Gamma'} \{\,[\,\bigcup_{\delta\in\Delta} N^{-1}(\mathbb{T}^n+\delta)\,] + \gamma\,\},\end{aligned} \tag{3.5}$$

where all the unions are disjoint.

Given $f \in V_1$, and $m^f = (m^{f,1} \cdots m^{f,D})$ as in Lemma 3.2, we can form, for $a.e.\,\xi \in \mathbb{R}^n$, the row vector in $\mathbb{C}^{mD}$:

$$u^f(\xi) = \Big(m^f(\xi + N^{-1}\delta_1)\ \ldots\ m^f(\xi + N^{-1}\delta_m)\Big).$$

When we let $f = \varphi^1, \ldots, \varphi^D$, we obtain a $D \times mD$ matrix of low-pass filters which will be denoted by:

$$U_0(\xi) \equiv \begin{pmatrix} u^{\varphi^1}(\xi) \\ \vdots \\ u^{\varphi^D}(\xi) \end{pmatrix} = \Big(M_0(\xi + N^{-1}\delta_1)\ \ldots\ M_0(\xi + N^{-1}\delta_m)\Big),$$

and similarly for $\widetilde{U}_0(\xi)$. Now, we have the following characterization for the elements of $W_0 \equiv \widetilde{V}_0^{\perp} \cap V_1$.

LEMMA 3.6 *In the same conditions as above, the space W_0 consists precisely of the functions $f \in V_1$ whose associated vector $m^f = (m^{f,1} \cdots m^{f,D})$, as in Lemma 3.2, satisfies the matrix equation in $\mathbb{C}^D$:*

$$\widetilde{U}_0(\xi)\,u^f(\xi)^* = \mathbf{0}, \quad a.e.\,\xi \in N^{-1}\mathbb{T}^n. \tag{3.7}$$

When this is the case, the equation above holds for $a.e.\,\xi \in \mathbb{R}^n$.

PROOF: The proof of this lemma follows from the biorthogonality properties of the MRA. Indeed, by definition we have that $f \in V_1$ belongs to W_0 if and only if $\int_{\mathbb{R}^n} \overline{f(x)}\tilde{\varphi}^d(x-\gamma)\,dx = 0,\ \forall\,\gamma \in \Gamma,\ d = 1,\dots,D$. Applying the Plancherel theorem to these integrals and periodizing, we arrive at the expression

$$0 = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\,\widehat{\tilde{\varphi}^d}(\xi)\,e^{-i\,\gamma\cdot\xi}\,d\xi = \int_{\mathbb{T}^n} \left(\sum_{\gamma'\in\Gamma'} \overline{\hat{f}(\xi+\gamma')}\,\widehat{\tilde{\varphi}^d}(\xi+\gamma') \right) e^{-i\,\gamma\cdot\xi}\,d\xi, \quad \forall\,\gamma,\, d.$$

Clearly, this is equivalent to have, for every d, and $a.e.\,\xi \in \mathbb{T}^n$, that the series in the inner parenthesis is 0. Using this, and equations (3.1), (3.3) and (3.4), we obtain that $f \in W_0$ if and only if for every $d = 1,\dots,D$ and $a.e.\,\xi \in N^{-1}\mathbb{T}^n$,

$$\begin{aligned}
0 &= \sum_{\gamma\in\Gamma'} \overline{\hat{f}(N\xi+\gamma)}\,\widehat{\tilde{\varphi}^d}(N\xi+\gamma) \\
&= \sum_{\gamma\in\Gamma'} \Big[\sum_{h=1}^{D} \overline{(m^{f,h}\,\widehat{\varphi}^h)(\xi+N^{-1}\gamma)} \Big] \Big[\sum_{h'=1}^{D} (\tilde{m}_0^{d,h'}\,\widehat{\tilde{\varphi}^{h'}})(\xi+N^{-1}\gamma) \Big] \\
&= \sum_{\delta\in\Delta} \sum_{h,h'=1}^{D} (\overline{m^{f,h}}\,\tilde{m}_0^{d,h'})(\xi+N^{-1}\delta) \Big[\sum_{\gamma\in\Gamma'} (\overline{\widehat{\varphi}^h}\,\widehat{\tilde{\varphi}^{h'}})(\xi+N^{-1}\delta+\gamma) \Big] \\
&= \sum_{\delta\in\Delta} \sum_{h=1}^{D} \overline{m^{f,h}(\xi+N^{-1}\delta)}\,\tilde{m}_0^{d,h}(\xi+N^{-1}\delta),
\end{aligned}$$

which is precisely the formula in (3.7). For the last statement note that the series above are $N^{-1}\Gamma'$-periodic, and therefore, the equalities hold for $a.e.\,\xi \in \mathbb{R}^n$. □

From our definitions and the biorthogonality of the systems $\{\psi^\ell_{j,\gamma}\}$, $\{\tilde{\psi}^\ell{}_{j,\gamma}\}$, it follows easily that $W_0 = \tilde{V}_0^\perp \cap V_1 = \overline{\text{span}}\,\{\psi^\ell(\cdot-\gamma)\}_{\ell,\gamma}$. Therefore, we can apply the previous lemma to each ψ^ℓ and find Γ'-periodic functions $m_1^{\ell,d} \equiv m^{\psi^\ell,d} \in L^2(\mathbb{T}^n)$, $\ell = 1,\dots,L$, such that the $L\times D$ matrix $M_1(\xi) = \left(m_1^{\ell,d}(\xi)\right)$ satisfies the equation:

$$\widehat{\Psi}(N\xi) = M_1(\xi)\widehat{\Phi}(\xi), \qquad a.e.\,\xi \in \mathbb{R}^n, \tag{3.8}$$

and similarly with $\widetilde{M}_1$. The matrices $M_1, \widetilde{M}_1$ are sometimes called a *pair of biorthogonal high-pass filters associated with* $(\Psi,\tilde{\Psi})$ *and* $(\Phi,\tilde{\Phi})$. Moreover, if we form the $L \times mD$ matrices:

$$U_1(\xi) = \begin{pmatrix} u^{\psi^1}(\xi) \\ \vdots \\ u^{\psi^L}(\xi) \end{pmatrix} = \Big(M_1(\xi+N^{-1}\delta_1)\ \dots\ M_1(\xi+N^{-1}\delta_m) \Big),$$

$$\tilde{U}_1(\xi) = \begin{pmatrix} u^{\tilde{\psi}^1}(\xi) \\ \vdots \\ u^{\tilde{\psi}^L}(\xi) \end{pmatrix} = \Big(\widetilde{M}_1(\xi+N^{-1}\delta_1)\ \dots\ \widetilde{M}_1(\xi+N^{-1}\delta_m) \Big),$$

then, Lemma 3.6 implies that

$$\widetilde{U}_0(\xi)U_1(\xi)^* = U_0(\xi)\widetilde{U}_1(\xi)^* = \mathbf{0}, \quad a.e.\, \xi \in \mathbb{R}^n. \tag{3.9}$$

We claim, further, that, for $a.e.\, \xi \in \mathbb{R}^n$,

$$U_0(\xi)\widetilde{U}_0(\xi)^* = I_D \quad \text{and} \quad U_1(\xi)\widetilde{U}_1(\xi)^* = I_L. \tag{3.10}$$

Indeed, it follows by exactly the same arguments used in the last part of the proof of Lemma 3.6 that

$$\delta_{d,d'} = \sum_{\gamma\in\Gamma'} \overline{\widehat{\varphi}^d(N\xi+\gamma)}\, \widehat{\tilde{\varphi}^{d'}}(N\xi+\gamma) = \sum_{\delta\in\Delta}\sum_{h=1}^{D} \overline{m_0^{d,h}(\xi+N^{-1}\delta)}\, \tilde{m}_0^{d',h}(\xi+N^{-1}\delta).$$

Since the last sum is precisely the conjugate of the (d,d')-entry of $U_0(\xi)\widetilde{U}_0(\xi)^*$, the first equality in (3.10) must hold. The same reasoning can be applied for the second equality, after replacing φ's with ψ's and use (3.8) instead of (3.4).

Note that (3.10) implies, in particular, that for $a.e.\, \xi \in \mathbb{R}^n$, the systems of vectors in $\mathbb{C}^{mD}$: $\{u^{\tilde{\varphi}^1}(\xi),\dots,u^{\tilde{\varphi}^D}(\xi)\}$ and $\{u^{\psi^1}(\xi),\dots,u^{\psi^L}(\xi)\}$, are each linearly independent, and that they generate subspaces of dimension D and L, respectively. From (3.9) these two subspaces must be orthogonal and, therefore, $L \le (m-1)D$. Suppose that $L < (m-1)D$. Then, for $a.e.\, \xi \in \mathbb{R}^n$, we can find a unit vector $v(\xi) = (v_1(\xi),\dots,v_{mD}(\xi)) \in \mathbb{C}^{mD}$ such that $v(\xi) \perp \operatorname{span} u^{\tilde{\varphi}^d}(\xi), u^{\psi^\ell}(\xi)_{d,\ell}$. Moreover, we can choose $v(\xi)$ such that each of its components $v_i(\xi)$ is a Γ'-periodic measurable function*. We will show that $v(\xi)$ can then be written as a linear combination of the $u^{\psi^\ell}(\xi)$'s, for $a.e.\, \xi \in N^{-1}\mathbb{T}^n$, which contradicts our hypothesis, and implies $L = (m-1)D$. To see this, we first define Γ'-periodic functions m^d, $d = 1,\dots,D$, given by:

$$m^d(\xi) = v_{(i-1)D+d}(\xi - N^{-1}\delta_i), \quad \text{if } \xi \in N^{-1}(\mathbb{T}^n + \delta_i), \quad i = 1,\dots,m.$$

That these functions extend Γ'-periodically to the whole $\mathbb{R}^n$ follows from (3.5). Note, further, that $m^d \in L^\infty(\mathbb{R}^n)$, $d = 1,\dots,D$. Then, Lemma 3.2 implies that the function f, whose Fourier transform is given by:

$$\hat{f}(N\xi) = \sum_{d=1}^{D} m^d(\xi)\widehat{\varphi}^d(\xi), \quad \xi \in \mathbb{R}^n,$$

belongs to V_1. Since $u^f(\xi) = v(\xi)$, $\xi \in N^{-1}\mathbb{T}^n$, and by assumption $\widetilde{U}_0(\xi)v^*(\xi) = \mathbf{0}$, Lemma 3.6 implies that $f \in W_0$. Now, $\{\psi^\ell(\cdot-\gamma)\}_{\gamma,\ell}$ is a Riesz basis for this space and, therefore, there must also exist (unique) Γ'-periodic functions $a^1,\dots,a^L \in L^2(\mathbb{T}^n)$ such that

$$\hat{f}(\xi) = \sum_{\ell=1}^{L} a^\ell(\xi)\widehat{\psi}^\ell(\xi), \quad a.e.\, \xi \in \mathbb{R}^n.$$

*That this can be done is a consequence, for example, of Lemma 3.17 below.

Using (3.8), together with the two equations above defining f, we obtain:

$$\sum_{d=1}^{D} m^d(\xi)\widehat{\varphi}^d(\xi) = \hat{f}(N\xi) = \sum_{d=1}^{D}\,[\,\sum_{\ell=1}^{L} a^\ell(N\xi)m_1^{\ell,d}(\xi)\,]\,\widehat{\varphi}^d(\xi), \quad a.e.\,\xi\in\mathbb{R}^n.$$

By the uniqueness part of Lemma 3.2, we must have

$$m^d(\xi) = \sum_{\ell=1}^{L} a^\ell(N\xi)m_1^{\ell,d}(\xi), \quad d=1,\dots,D, \qquad a.e.\,\xi\in\mathbb{R}^n.$$

In particular, if we let $\xi = \zeta + N^{-1}\delta_i$, for $\zeta \in N^{-1}\mathbb{T}^n$ and each $i = 1,\dots,m$, we obtain the vector equality:

$$v(\zeta) = \sum_{\ell=1}^{L} a^\ell(N\zeta)u^{\psi^\ell}(\zeta), \qquad a.e.\,\zeta\in N^{-1}\mathbb{T}^n,$$

which implies the desired contradiction. Therefore, we obtain $L = (m-1)D$ and, in particular, the following corollary:

COROLLARY 3.11
For a.e. $\xi\in\mathbb{R}^n$, the matrices $U(\xi)\equiv\begin{pmatrix}U_0(\xi)\\ U_1(\xi)\end{pmatrix}$ and $\widetilde{U}(\xi)\equiv\begin{pmatrix}\widetilde{U}_0(\xi)\\ \widetilde{U}_1(\xi)\end{pmatrix}$ are square matrices in $\mathbb{C}^{mD}$, whose inverses are, respectively, $\widetilde{U}^*(\xi)$ and $U^*(\xi)$.

PROOF: Immediate consequence of $L=(m-1)D$ and (3.9), (3.10). □

3.2 Necessity: the proof of (I), (II)

We now turn to the proof of the necessity of condition (I) in Theorem 2.11. From (3.4) and (3.8), we have, for $a.e.\,\xi\in\mathbb{R}^n$, the following *scalar* equation:

$$\widehat{\Phi}(N\xi)^*\widehat{\widetilde{\Phi}}(N\xi) + \widehat{\Psi}(N\xi)^*\widehat{\widetilde{\Psi}}(N\xi) = \widehat{\Phi}(\xi)^*\,[M_0^*(\xi)\widetilde{M}_0(\xi) + M_1^*(\xi)\widetilde{M}_1(\xi)]\,\widehat{\widetilde{\Phi}}(\xi).$$

Now, by Corollary 3.11 one also has $U(\xi)^*\widetilde{U}(\xi) = I_{mD}$, and hence, the matrix in the middle bracket above is the identity in $\mathbb{C}^D$. Iterating the obtained equality, we arrive at the expression, for every $J\geq 1$,

$$\widehat{\Phi}(\xi)^*\widehat{\widetilde{\Phi}}(\xi) = \sum_{j=1}^{J}\widehat{\Psi}(N^j\xi)^*\widehat{\widetilde{\Psi}}(N^j\xi) + \widehat{\Phi}(N^J\xi)^*\widehat{\widetilde{\Phi}}(N^J\xi), \qquad a.e.\,\xi\in\mathbb{R}^n. \tag{3.12}$$

The following lemma allows us to obtain a simple equation relating $\widehat{\Phi}^*\widehat{\widetilde{\Phi}}$ and $\widehat{\Psi}^*\widehat{\widetilde{\Psi}}$.

LEMMA 3.13 *If $f\in L^1(\mathbb{R}^n)$, then, for $a.e.\,\xi\in\mathbb{R}^n$, $\lim_{J\to+\infty} f(N^J\xi) = 0$.*

Proof: The proof is very simple. If there exists $\varepsilon > 0$ and a set of positive measure $E \subset \mathbb{R}^n$ such that $\underline{\lim}_{J\to\infty} |f(N^J\xi)| \geq \varepsilon$, then, by Fatou's lemma, we obtain:

$$\varepsilon|E| \leq \int_E \underline{\lim}_{J\to\infty} |f(N^J\xi)|\, d\xi \leq \underline{\lim}_{J\to\infty} \int_E |f(N^J\xi)|\, d\xi \leq \lim_{J\to\infty} m^{-J}\|f\|_1 = 0,$$

which is a contradiction.

□

From (3.12) and the previous lemma, we obtain the following scalar equality, for $a.e.\, \xi \in \mathbb{R}^n$:

$$\widehat{\Phi}(\xi)^* \widehat{\tilde{\Phi}}(\xi) = \sum_{j=1}^{\infty} \widehat{\Psi}(N^j\xi)^* \widehat{\tilde{\Psi}}(N^j\xi). \tag{3.14}$$

Periodizing this expression, and using the biorthogonality relations in (3.1), we obtain:

$$D = \sum_{\gamma\in\Gamma'} \widehat{\Phi}(\xi+\gamma)^* \widehat{\tilde{\Phi}}(\xi+\gamma) = \sum_{\gamma\in\Gamma'} \sum_{j=1}^{\infty} \widehat{\Psi}(N^j(\xi+\gamma))^* \widehat{\tilde{\Psi}}(N^j(\xi+\gamma)) = D_{\Psi,\tilde{\Psi}}(\xi),$$

for $a.e.\, \xi \in \mathbb{R}^n$, which is precisely the statement of (I) in Theorem 2.11.

Finally, we turn to the proof of condition (II). We claim that, for $a.e.\, \xi \in \mathbb{R}^n$, the spaces $\mathcal{F}(\xi), \widetilde{\mathcal{F}}(\xi)$, defined from the families Ψ and $\widetilde{\Psi}$ in (2.9), can also be written as:

$$\mathcal{F}(\xi) = \overline{\operatorname{span}}\, \{v^d(\xi)\}_{d=1}^D, \quad \widetilde{\mathcal{F}}(\xi) = \overline{\operatorname{span}}\, \{\tilde{v}^d(\xi)\}_{d=1}^D, \tag{3.15}$$

where $v^d(\xi) = \{\widehat{\varphi}^d(\xi+\gamma)\}_{\gamma\in\Gamma'}$ and $\tilde{v}^d(\xi) = \{\widehat{\tilde{\varphi}^d}(\xi+\gamma)\}_{\gamma\in\Gamma'}$, $d = 1, \ldots, D$. Indeed, since $\psi^\ell_{j,0} \in V_0$, for all $j < 0$, $\ell = 1, \ldots, L$, an argument as in Lemma 3.2 implies that:

$$\widehat{\psi}^\ell(N^j\xi) = \sum_{d=1}^{D} a_d^{\ell,j}(\xi)\widehat{\varphi}^d(\xi), \quad \text{for some} \quad a_d^{\ell,j} \in L^2(\mathbb{T}^n),$$

or equivalently, in the notation of the space $\mathcal{F}(\xi)$, $\omega^\ell_j(\xi) = \sum_{d=1}^D a_d^{\ell,j}(\xi) v^d(\xi)$. Now, from (3.1) it follows that $\{v^d(\xi)\}_d$ and $\{\tilde{v}^d(\xi)\}_d$ are biorthogonal (hence, linearly independent) systems in $\mathcal{H} = \ell^2(\Gamma')$. Since $\dim \mathcal{F}(\xi) = \dim \widetilde{\mathcal{F}}(\xi) = D$, the claim in (3.15) follows. In fact, the systems $\{v^d(\xi)\}_d$, $\{\tilde{v}^d(\xi)\}_d$ are (biorthogonal) bases for $\mathcal{F}(\xi)$ and $\widetilde{\mathcal{F}}(\xi)$. To study the norm of the operator $S(\xi)\colon \widetilde{\mathcal{F}}(\xi) \to \mathcal{F}(\xi)$, defined in (2.10), it suffices to look at the associated matrix in terms of these bases. The (d, d')-entry of such matrix will be given by:

$$[\![S(\xi)\tilde{v}^d(\xi), \tilde{v}^{d'}(\xi)]\!] = [\![\tilde{v}^d(\xi), \tilde{v}^{d'}(\xi)]\!] = \sum_{\gamma\in\Gamma'} \widehat{\tilde{\varphi}^d}(\xi+\gamma)\overline{\widehat{\tilde{\varphi}^{d'}}(\xi+\gamma)},$$

where in the first equality we have used (jj) of Lemma 2.5 and the fact that $\widetilde{S}(\xi)$ is a projection onto $\widetilde{\mathcal{F}}(\xi)$. Thus, the matrix associated with $S(\xi)$ is precisely

$$\widetilde{C}(\xi) = \sum_{\gamma\in\Gamma'} \widehat{\tilde{\Phi}}(\xi+\gamma)\, \widehat{\tilde{\Phi}}(\xi+\gamma)^*, \quad a.e.\, \xi \in \mathbb{R}^n. \tag{3.16}$$

It is a well-known property of shift-invariant systems like $\{\tilde{\varphi}^d(\cdot - \gamma)\}_{d,\gamma}$ that they generate a Riesz basis for their closed span if and only if, for *a.e.* $\xi \in \mathbb{R}^n$, the matrix $\widetilde{C}(\xi)$ in (3.16) is positive definite, and there are two constants $0 < c_1 \le c_2 < \infty$, such that

$$c_1\|\alpha\|^2 \le \langle \widetilde{C}(\xi)\alpha, \alpha\rangle \le c_2\|\alpha\|^2, \quad \forall\, \alpha \in \mathbb{C}^D,$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and inner product in $\mathbb{C}^D$ (see, *e.g.*, Theorem 1 in [DS]). Thus, the operator norm of $S(\xi)$ is bounded above by c_2, and the necessity of (II) in Theorem 2.11 is also established.

3.3 Sufficiency

We now turn to the sufficiency part of the theorem. Suppose then, that $\Psi = \{\psi^1, \ldots, \psi^L\}, \tilde{\Psi} = \{\tilde{\psi}^1, \ldots, \tilde{\psi}^L\}$ are two biorthogonal wavelet families in $L^2(\mathbb{R}^n)$ such that conditions (I) and (II) from Theorem 2.11 hold. Then, (I) and Lemma 2.5 imply that the spaces $\mathcal{F}(\xi)$, $\widetilde{\mathcal{F}}(\xi)$ in (2.9) are, each, D-dimensional. We show in the next lemma how to construct in a measurable way an orthonormal basis for the second space.

Lemma 3.17 *Let $\mathbb{X}$ be a measure space and $\boldsymbol{\alpha}_j : \mathbb{X} \to \ell^2 = \ell^2(\mathbb{Z}^+)$, $j = 1, 2, \ldots$, be measurable vector-valued functions (that is, each component a_j^i, $i \in \mathbb{Z}^+$, of $\boldsymbol{\alpha}_j$ is measurable). Suppose that the vector spaces*

$$\mathbb{F}(x) = \overline{\operatorname{span}}\,\{\boldsymbol{\alpha}_j(x) \ : \ j = 1, 2, \ldots\}, \quad x \in \mathbb{X},$$

have constant finite dimension $d \ge 1$. Then, there exist measurable ℓ^2-valued functions $\mathbf{b}_1, \ldots, \mathbf{b}_d$ such that $\{\mathbf{b}_1(x), \ldots, \mathbf{b}_d(x)\}$ is an orthonormal basis of $\mathbb{F}(x)$, for every $x \in \mathbb{X}$.

Remark 3.18 This lemma can also be found in [JS] (see Lemma 3.8 there). For completeness, we include an elementary proof here.

Proof: Let us define, when $j \ge 1$,

$$X_j = \{x \in \mathbb{X} \ : \ \boldsymbol{\alpha}_j(x) \ne \mathbf{0} \text{ and } \boldsymbol{\alpha}_k(x) = \mathbf{0},\ 1 \le k < j\}.$$

Then, $\{X_j\}_{j=1}^\infty$ forms a measurable partition of $\mathbb{X}$. Without loss of generality, we may assume that $\mathbb{X} = X_1$ (otherwise, one proceeds separately with X_1, X_2, etc...). If $d = 1$, the vector-valued function $\mathbf{b}_1(x) = \frac{\boldsymbol{\alpha}_1(x)}{\|\boldsymbol{\alpha}_1(x)\|}$ will do for the lemma. If on the contrary $d \ge 2$, we consider the orthogonal projections

$$\begin{aligned} P_1(x) \ : \ \ell^2 &\longrightarrow \mathbb{F}_1(x) \equiv \operatorname{span} \mathbf{b}_1(x) \\ \mathbf{v} &\mapsto P_1(x)\mathbf{v} = \langle \mathbf{v}, \mathbf{b}_1(x)\rangle\, \mathbf{b}_1(x), \end{aligned}$$

and the measurable partition of $\mathbb{X}$ consisting of the sets, for $j \geq 2$,

$$X_j = \{x \in \mathbb{X} \ : \ \|P_1(x)\boldsymbol{\alpha}_j(x)\| \neq \|\boldsymbol{\alpha}_j(x)\| \text{ and } \|P_1(x)\boldsymbol{\alpha}_k(x)\| = \|\boldsymbol{\alpha}_k(x)\|, \ 2 \leq k < j\,\}.$$

When $x \in X_j$, the system $\{\mathbf{b}_1(x), \boldsymbol{\alpha}_j(x)\}$ is linearly independent and generates a 2-dimensional space: $\mathbb{F}_2(x)$. We can then apply a Gram-Schmidt process to find a measurable $\mathbf{b}_2(x)$ such that $\{\mathbf{b}_1(x), \mathbf{b}_2(x)\}$ is an orthonormal basis for $\mathbb{F}_2(x)$ (letting, for example, $\mathbf{b}_2(x) = [\boldsymbol{\alpha}_2 - \langle \mathbf{b}_1, \boldsymbol{\alpha}_2\rangle \mathbf{b}_1](x)/\|\cdot\|$). If $d = 2$, then $\mathbb{F}_2(x) = \mathbb{F}(x)$ and we are done. If $d \geq 3$, we assume (again, without loss of generality) that $\mathbb{X} = X_2$ and continue the process with the orthogonal projection:

$$\begin{aligned} P_2(x) \ &: \ \ell^2 \longrightarrow \mathbb{F}_2(x) \\ & \qquad \mathbf{v} \mapsto P_2(x)\mathbf{v} = \sum_{j=1}^{2} \langle \mathbf{v}, \mathbf{b}_j(x)\rangle\, \mathbf{b}_j(x), \end{aligned}$$

and the measurable partition of $\mathbb{X}$ consisting of the sets, for $j \geq 3$,

$$X_j = \{x \in \mathbb{X} \ : \ \|P_2(x)\boldsymbol{\alpha}_j(x)\| \neq \|\boldsymbol{\alpha}_j(x)\| \text{ and } \|P_2(x)\boldsymbol{\alpha}_k(x)\| = \|\boldsymbol{\alpha}_k(x)\|, \ 3 \leq k < j\,\}.$$

Since d is finite, this process will end, and we will obtain the desired basis for $\mathbb{F}(x)$. $\square$

In view of this lemma, we can find measurable vectors $\tilde{v}^d : \mathbb{T}^n \to \mathcal{H}$ such that $\{\tilde{v}^1(\xi), \ldots, \tilde{v}^D(\xi)\}$ is an orthonormal basis of $\widetilde{\mathcal{F}}(\xi)$, for every $\xi \in \mathbb{T}^n$. Applying the isomorphism $S(\xi)$, we define

$$v^d(\xi) := S(\xi)\tilde{v}^d(\xi), \quad d = 1\ldots, D, \ \ \xi \in \mathbb{T}^n,$$

which will be a basis for $\mathcal{F}(\xi)$. Moreover, these two bases are biorthogonal since, from Lemma 2.5, we have

$$[\![\tilde{v}^d(\xi), v^{d'}(\xi)]\!] = [\![\tilde{v}^d(\xi), S(\xi)\tilde{v}^{d'}(\xi)]\!] = [\![\tilde{v}^d(\xi), \tilde{v}^{d'}(\xi)]\!] = \delta_{d,d'}.$$

Now, we let

$$\left.\begin{aligned} \widehat{\varphi}^d(\xi) \ &:= \ v^d(\xi - \gamma) \\ \widehat{\tilde{\varphi}^d}(\xi) \ &:= \ \tilde{v}^d(\xi - \gamma) \end{aligned}\right\} \quad \text{if } \xi \in \mathbb{T}^n + \gamma, \quad \gamma \in \Gamma', \quad d = 1, \ldots, D.$$

We claim that, with this definition, each $\widehat{\varphi}^d$, $\widehat{\tilde{\varphi}^d}$ belongs to $L^2(\mathbb{R}^n)$ and, moreover, the systems $\{\varphi^d(\cdot - \gamma)\}$, $\{\tilde{\varphi}^d(\cdot - \gamma)\}$, $\gamma \in \Gamma$, $d = 1, \ldots, D$, are biorthogonal and form Riesz bases of their closed linear expansions. Indeed, for the first assertion note that

$$\int_{\mathbb{R}^n} |\widehat{\tilde{\varphi}^d}(\xi)|^2\, d\xi = \int_{\mathbb{T}^n} \sum_{\gamma \in \Gamma'} |\widehat{\tilde{\varphi}^d}(\xi + \gamma)|^2\, d\xi = \int_{\mathbb{T}^n} [\![\tilde{v}^d(\xi)]\!]^2\, d\xi = (2\pi)^n,$$

while

$$\int_{\mathbb{R}^n} |\widehat{\varphi}^d(\xi)|^2 \, d\xi = \int_{\mathbb{T}^n} [\![v^d(\xi)]\!]^2 \, d\xi = \int_{\mathbb{T}^n} [\![S(\xi)\tilde{v}^d(\xi)]\!]^2 \, d\xi \le C^2 \, (2\pi)^n,$$

where in the last inequality we have used (II) of Theorem 2.11.
Now, to see that $\{\tilde{\varphi}^d(\cdot - \gamma)\}_{\gamma,d}$ forms an orthonormal system is equivalent to check that

$$\sum_{\gamma \in \Gamma'} \overline{\widehat{\tilde{\varphi}}^d(\xi+\gamma)} \widehat{\tilde{\varphi}}^{d'}(\xi+\gamma) = \delta_{d,d'}, \quad a.e.\, \xi \in \mathbb{T}^n.$$

But this is immediate since the series above equals $[\![\tilde{v}^{d'}(\xi), \tilde{v}^d(\xi)]\!]$, when $\xi \in \mathbb{T}^n$, and $\{\tilde{v}^d(\xi)\}_d$ is orthonormal in $\widetilde{\mathcal{F}}(\xi)$. On the other hand, to show that $\{\varphi^d(\cdot - \gamma)\}_{\gamma,d}$ is a Riesz system, it suffices to see that the matrix $C(\xi)$, with entries:

$$C_{d,d'}(\xi) = \sum_{\gamma \in \Gamma'} \overline{\widehat{\varphi}^d(\xi+\gamma)} \widehat{\varphi}^{d'}(\xi+\gamma), \quad \xi \in \mathbb{T}^n,$$

satisfies, for some constants $0 < c_1 \le c_2 < \infty$, the inequalities:

$$c_1 \|\alpha\|^2 \le \langle C(\xi)\alpha, \alpha \rangle \le c_2 \|\alpha\|^2, \quad \forall\, \alpha \in \mathbb{C}^D, \, a.e.\, \xi \in \mathbb{T}^n, \tag{3.19}$$

(see the comment right after (3.16)). But this follows from the fact that, if $\alpha = (\alpha_1, \ldots, \alpha_D) \in \mathbb{C}^D$, then

$$\begin{aligned} \langle C(\xi)\alpha, \alpha \rangle &= \sum_{d,d'=1}^{D} [\![v^{d'}(\xi), v^d(\xi)]\!] \, \alpha_{d'} \overline{\alpha_d} = [\![S(\xi) (\textstyle\sum_{d=1}^{D} \alpha_d \tilde{v}^d(\xi))]\!]^2 \\ &\le C^2 \, [\![\textstyle\sum_{d=1}^{D} \alpha_d \tilde{v}^d(\xi)]\!]^2 = C^2 \displaystyle\sum_{d=1}^{D} |\alpha_d|^2, \end{aligned}$$

after using (II) in Theorem 2.11 and, again, the orthonormality of $\{\tilde{v}^d(\xi)\}_d$. The estimation from below in (3.19) follows similarly, now using the left inequality in (jjj) of Lemma 2.5. Finally, the biorthogonality of the two systems generated by φ^d and $\tilde{\varphi}^d$, is a simple consequence of

$$\sum_{\gamma \in \Gamma'} \overline{\widehat{\varphi}^d(\xi+\gamma)} \widehat{\tilde{\varphi}}^{d'}(\xi+\gamma) = [\![v^d(\xi), \tilde{v}^{d'}(\xi)]\!] = \delta_{d,d'}, \quad a.e.\, \xi \in \mathbb{T}^n.$$

To conclude the proof of the theorem we will show that the sequences $\{V_j\}$ and $\{\widetilde{V}_j\}$ in (2.4) form a pair of biorthogonal MRA's. If we define the spaces

$$V_0^\sharp = \overline{\mathrm{span}} \, \{\varphi^d(\cdot - \gamma)\}_{\gamma,d} \quad \text{and} \quad \widetilde{V}_0^\sharp = \overline{\mathrm{span}} \, \{\tilde{\varphi}^d(\cdot - \gamma)\}_{\gamma,d},$$

it will suffice to verify the statement of the following lemma.

Lemma 3.20 *In the above conditions,* $V_0 = V_0^\sharp$, *and* $\widetilde{V}_0 = \widetilde{V}_0^\sharp$.

PROOF:

We begin by showing $V_0 \subseteq V_0^\sharp$. For this, it clearly suffices to check that each $\psi^\ell_{j,\gamma} \in V_0^\sharp$, $\gamma \in \Gamma$, $j < 0$, $\ell = 1, \ldots, L$. Moreover, since $V_0^\sharp$ is Γ-translation invariant, we may assume $\gamma = 0$. Note that $(\psi^\ell_{j,0})\hat{}(\xi) = m^{-j/2}\widehat{\psi}^\ell(N^{-j}\xi)$.

Now, for every $j < 0$ and $\xi \in \mathbb{T}^n$, we have $\omega^\ell_{-j}(\xi) \in \mathcal{F}(\xi)$. Therefore, we can find functions $a^{\ell,j}_d : \mathbb{T}^n \to \mathbb{C}$ such that:

$$\omega^\ell_{-j}(\xi) = \sum_{d=1}^{D} a^{\ell,j}_d(\xi)\, v^d(\xi), \quad \xi \in \mathbb{T}^n. \tag{3.21}$$

In fact, the coefficient functions are given by $a^{\ell,j}_d(\xi) = [\![\omega^\ell_{-j}(\xi), \tilde{v}^d(\xi)]\!]$, which are measurable and belong to $L^2(\mathbb{T}^n)$, since

$$\int_{\mathbb{T}^n} |a^{\ell,j}_d(\xi)|^2\, d\xi \le \int_{\mathbb{T}^n} [\![\omega^\ell_{-j}(\xi)]\!]^2\, d\xi = \int_{\mathbb{R}^n} |\widehat{\psi}^\ell(N^{-j}\xi)|^2\, d\xi.$$

Thus, we may extend Γ'-periodically the functions $a^{\ell,j}_d$ to $\mathbb{R}^n$ and write (3.21) in the form:

$$\psi^\ell(N^{-j}\xi) = \sum_{d=1}^{D} a^{\ell,j}_d(\xi)\, \widehat{\varphi}^d(\xi), \quad \xi \in \mathbb{R}^n.$$

Taking inverse Fourier transforms of both sides, we are led to $\psi^\ell_{j,0} \in V_0^\sharp$, establishing the proof of $V_0 \subseteq V_0^\sharp$.

In order to establish the other inclusion, it suffices to show that $V_0^\sharp \perp \widetilde{W}_j \equiv \overline{\mathrm{span}}\, \{\tilde{\psi}^\ell_{j,\gamma}\}_{\ell,\gamma}$, for all $j \ge 0$. Indeed, when this is the case, the properties of the Riesz bases $\{\psi^\ell_{j,\gamma}\}$, $\{\tilde{\psi}^\ell_{j,\gamma}\}$, imply that:

$$V_0^\sharp \subset \bigcap_{j=0}^{\infty} \left(\widetilde{W}_j^\perp\right) = \left(\bigoplus_{j=0}^{\infty} \widetilde{W}_j\right)^\perp = V_0.$$

Again, by the translation invariance of the spaces $\widetilde{W}_j$, $j \ge 0$, it suffices to show that $(\varphi^d, \tilde{\psi}^\ell_{j,\gamma}) = 0$, for all γ, ℓ, d and $j \ge 0$. Writting the inner product as an integral and using the Plancherel theorem, we obtain:

$$\begin{aligned}
(\widehat{\tilde{\psi}^\ell}_{j,\gamma}, \widehat{\varphi}^d) &= m^{-j/2} \int_{\mathbb{R}^n} \overline{\widehat{\varphi}^d(\xi)}\, \widehat{\tilde{\psi}}^\ell(N^{-j}\xi)\, e^{-iN^{-j}\xi\cdot\gamma}\, d\xi \\
&= m^{j/2} \int_{\mathbb{R}^n} \overline{\widehat{\varphi}^d(N^j\xi)}\, \widehat{\tilde{\psi}}^\ell(\xi)\, e^{-i\xi\cdot\gamma}\, d\xi \\
&= m^{j/2} \int_{\mathbb{T}^n} \Big[\sum_{\delta\in\Gamma'} \overline{\widehat{\varphi}^d(N^j(\xi+\delta))}\, \widehat{\tilde{\psi}}^\ell(\xi+\delta)\Big]\, e^{-i\xi\cdot\gamma}\, d\xi.
\end{aligned}$$

We claim that :

$$\tau^{d,\ell}_j(\xi) \equiv \sum_{\delta\in\Gamma'} \overline{\widehat{\varphi}^d(N^j(\xi+\delta))}\; \widehat{\tilde{\psi}}^\ell(\xi+\delta) = 0, \quad a.e.\, \xi \in \mathbb{T}^n, \tag{3.22}$$

for all values of ℓ, d and $j \geq 0$. Indeed, using the notation in §2, we can write, for every $\xi \in \mathbb{T}^n$,

$$v^d(\xi) = T(\xi)v^d(\xi) = \sum_{\ell'=1}^{L} \sum_{i=1}^{\infty} [\![v^d(\xi), \tilde{\omega}_i^{\ell'}(\xi)]\!] \, \omega_i^{\ell'}(\xi).$$

If we denote the inner product: $[\![v^d(\xi), \tilde{\omega}_i^{\ell'}(\xi)]\!] \equiv b_{\ell',i}^d(\xi)$, extended Γ'-periodically to $\mathbb{R}^n$, and write the above expression coordenatewise, we obtain:

$$\widehat{\varphi}^d(\xi) = \sum_{\ell'=1}^{L} \sum_{i=1}^{\infty} b_{\ell',i}^d(\xi) \, \widehat{\psi}^{\ell'}(N^i\xi), \quad \xi \in \mathbb{R}^n.$$

Thus, substituting in the definition of $\tau_j^{d,\ell}$ given in (3.22), and using equation $(B4)$, we arrive at:

$$\tau_j^{d,\ell}(\xi) = \sum_{\ell'=1}^{L} \sum_{i=1}^{\infty} \overline{b_{\ell',i}^d(N^j\xi)} \sum_{\delta \in \Gamma'} \overline{\widehat{\psi}^{\ell'}(N^{i+j}(\xi+\delta))} \, \widehat{\tilde{\psi}}^{\ell}(\xi+\delta) = 0, \quad a.e.\, \xi \in \mathbb{T}^n.$$

This concludes the proof of $V_0 = V_0^\sharp$. One obtains $\widetilde{V}_0 = \widetilde{V}_0^\sharp$ in a similar fashion, establishing the lemma and, with it, completing the proof of Theorem 2.11.

□

4 Corollaries

In this section, we present some corollaries of the theorem just proved, including an explicit computation of the operator norm $\|S(\xi)\|$ in terms of Ψ and $\widetilde{\Psi}$, and very mild sufficient conditions for the theorem to hold.

During the course of the proof we have seen the crucial role played by the *dimension function* $D_{\Psi,\tilde{\Psi}}(\xi)$ in order to construct scaling functions from the "periodized versions" of V_0 and $\widetilde{V}_0$ (namely, $\mathcal{F}(\xi)$ and $\widetilde{\mathcal{F}}(\xi)$). When doing so, an upper bound for the expression $\|S(\xi)\|$ was needed to guarantee the Bessel conditions in the constructed biorthogonal systems. When $\Psi = \widetilde{\Psi}$, we always have $\|S(\xi)\| = 1$, and the Bessel condition comes for free (as happened in Theorem 2.2), giving us the simple statement in Corollary 2.12. In the biorthogonal situation, however, $\|S(\xi)\|$ can take larger values, and a good understanding of its boundedness would be desireable. When the multiplicity of the MRA is $D = 1$ we have a simple formula for $\|S(\xi)\|$ involving only ψ^ℓ and $\tilde{\psi}^\ell$. We state it in the following corollary, which includes as a particular case Theorem 1.2.

Corollary 4.1 *Let $\Psi = \{\psi^1, \ldots, \psi^{m-1}\}$, $\tilde{\Psi} = \{\tilde{\psi}^1, \ldots, \tilde{\psi}^{m-1}\}$ be two biorthogonal wavelet families in $L^2(\mathbb{R}^n)$ (for $\Gamma = \mathbb{Z}^n$ and M). Then, they arise from a pair* *of biorthogonal multiresolution analyses (necessarily of multiplicity* 1*) if and only if there is a constant* $C = C(\Psi, \widetilde{\Psi}) > 0$ *such that*

$$(I) \qquad D_{\Psi,\tilde{\Psi}}(\xi) \neq 0, \quad a.e.\, \xi \in \mathbb{T}^n;$$

$$(II) \qquad \text{for} \quad a.e.\, \xi \in \mathbb{T}^n,\ \exists\, j = j(\xi) \in \mathbb{Z}^+,\ \ell = \ell(\xi) \in \{1, \ldots, L\},$$

$$\text{such that} \qquad \frac{[\![\omega_j^\ell(\xi)]\!][\![\tilde{\omega}_j^\ell(\xi)]\!]}{|\,[\![\omega_j^\ell(\xi), \tilde{\omega}_j^\ell(\xi)]\!]\,|} \leq C.$$

Proof: We show first the direct implication. Note that, in view of Theorem 2.11, if $\Psi, \widetilde{\Psi}$ do arise from a pair of biorthogonal MRA's, they have to be of multiplicity 1 (since the number of wavelets in Ψ is $L = m - 1$). In addition,

$$D_{\Psi,\tilde{\Psi}}(\xi) = \dim \mathcal{F}(\xi) = \dim \widetilde{\mathcal{F}}(\xi) = 1, \quad a.e.\, \xi \in \mathbb{T}^n.$$

As a consequence, not only we obtain (I) in the corollary, but also find a simple expression for $\|S(\xi)\|$ that leads to (II). Indeed, if $\xi \in \mathbb{R}^n$ is fixed and v, $\tilde{v}$ is a biorthogonal couple generating $\mathcal{F}(\xi)$ and $\widetilde{\mathcal{F}}(\xi)$, respectively (as in (3.15)), we have

$$\|S(\xi)\| = \frac{[\![S(\xi)\tilde{v}]\!]}{[\![\tilde{v}]\!]} = \frac{[\![\,[\![S(\xi)\tilde{v}, \tilde{v}]\!]\, v\,]\!]}{[\![\tilde{v}]\!]} = [\![\tilde{v}]\!][\![v]\!]. \tag{4.2}$$

Now, from (I) in the corollary, there must exist $j = j(\xi) \in \mathbb{Z}^+$ and $\ell = \ell(\xi) \in \{1, \ldots, L\}$, such that

$$\sum_{\gamma \in \Gamma'} \overline{\widehat{\psi}^\ell(N^j(\xi + \gamma))}\, \widehat{\tilde{\psi}^\ell}(N^j(\xi + \gamma)) = [\![\tilde{\omega}_j^\ell(\xi), \omega_j^\ell(\xi)]\!] \neq 0.$$

In particular, there are non-zero scalars $\lambda = \lambda(\xi),\ \mu = \mu(\xi) \in \mathbb{C}$ such that:

$$\omega_j^\ell(\xi) = \lambda\, v \quad \text{and} \quad \tilde{\omega}_j^\ell(\xi) = \mu\, \tilde{v}.$$

Substituting this in (4.2), we obtain:

$$\|S(\xi)\| = \frac{[\![\omega_j^\ell(\xi)]\!]\, [\![\tilde{\omega}_j^\ell(\xi)]\!]}{|\lambda\, \mu|} = \frac{[\![\omega_j^\ell(\xi)]\!][\![\tilde{\omega}_j^\ell(\xi)]\!]}{|\,[\![\omega_j^\ell(\xi), \tilde{\omega}_j^\ell(\xi)]\!]\,|},$$

establishing (II) in the corollary.

For the converse, it suffices to show that $D_{\Psi,\tilde{\Psi}}(\xi) = 1,\ a.e.\, \xi \in \mathbb{T}^n$, since from here, and the above calculations of $\|S(\xi)\|$, Theorem 2.11 will do. But from Lemma 2.5, $D_{\Psi,\tilde{\Psi}}(\xi)$ is always integer-valued and, therefore, it is always real and greater than or equal to 1. Now,

$$\begin{aligned}
\int_{\mathbb{T}^n} D_{\Psi,\tilde{\Psi}}(\xi)\, \frac{d\xi}{(2\pi)^n} &= \sum_{\ell=1}^{m-1} \sum_{j=1}^{\infty} \sum_{\gamma \in \Gamma'} \int_{\mathbb{T}^n} \overline{\widehat{\psi}^\ell(N^j(\xi + \gamma))}\, \widehat{\tilde{\psi}^\ell}(N^j(\xi + \gamma))\, \frac{d\xi}{(2\pi)^n} \\
&= \sum_{\ell=1}^{m-1} \sum_{j=1}^{\infty} m^{-j} \int_{\mathbb{R}^n} \overline{\widehat{\psi}^\ell(\xi)}\, \widehat{\tilde{\psi}^\ell}(\xi)\, \frac{d\xi}{(2\pi)^n} = \frac{m-1}{m-1} = 1.
\end{aligned}$$

Therefore, we must have $D_{\Psi,\tilde{\Psi}}(\xi) = 1$, $a.e.\, \xi \in \mathbb{T}^n$, and the corollary is proved. □

Note that condition (II) in the corollary is equivalent to saying that the *cosine* of the angle formed by the vectors $\omega_j(\xi)$ and $\tilde{\omega}_j(\xi)$ is bounded away from 0. In other words, the spaces $\mathcal{F}(\xi)$ and $\widetilde{\mathcal{F}}(\xi)^\perp$ must be "separated" by a uniform positive angle. The calculation of $\|S(\xi)\|$, when $D \geq 2$, is done in a similar fashion to the case just treated. However, the expression we obtain is somewhat more complicated this time, because we work with matrices and vectors in D-dimensional spaces ($\mathcal{F}(\xi)$ and $\widetilde{\mathcal{F}}(\xi)$) rather than scalars in $\mathbb{C}$. We begin with the definition of correlation matrix (a tool which was already used in (3.16) above).

Let $\mathbf{v} = \{v_1, \ldots, v_D\}$, $\boldsymbol{\omega} = \{\omega_1, \ldots, \omega_D\}$ be two systems of vectors in a Hilbert space $\mathcal{H}$. Then, the *correlation matrix of* $\mathbf{v}$ *and* $\boldsymbol{\omega}$ is the complex $D \times D$ matrix $C = C_{\mathbf{v},\boldsymbol{\omega}}$, whose entries are $(\,\langle v_d, \omega_{d'}\rangle\,)_{d,d'=1,\ldots D}$, where $\langle\cdot\,,\cdot\rangle$ denotes the inner product in $\mathcal{H}$.

LEMMA 4.3 *If* $\mathbf{v} = \{v_1, \ldots, v_D\}$ *is a system of vectors in a Hilbert space* $\mathcal{H}$*, then* $C_{\mathbf{v},\mathbf{v}}$ *is a hermitian, non-negative definite matrix (all eigenvalues are greater than or equal to* 0*). Moreover, if* $\boldsymbol{\omega} = \{\omega_1, \ldots, \omega_D\}$ *is another system in* $\mathcal{H}$ *then,* $C_{\mathbf{v},\boldsymbol{\omega}}$ *is invertible if and only if the systems* $\mathbf{v}$ *and* $\boldsymbol{\omega}$ *consist of linearly independent vectors and* $\operatorname{span}\mathbf{v}^\perp \cap \operatorname{span}\boldsymbol{\omega} = \{0\}$.

PROOF: That $C_{\mathbf{v},\mathbf{v}}$ is hermitian is obviuos. In fact, $C^*_{\mathbf{v},\boldsymbol{\omega}} = C_{\boldsymbol{\omega},\mathbf{v}}$. For the other assertions, note that, for every $\alpha = (\alpha^1, \ldots, \alpha^D)$, $\beta = (\beta^1, \ldots, \beta^D) \in \mathbb{C}^D$, then

$$\alpha C_{\mathbf{v},\boldsymbol{\omega}} \beta^* = \sum_{d,d'} \alpha^d \,\langle v_d, \omega_{d'}\rangle\, \overline{\beta^{d'}} = \langle \sum_d \alpha^d v_d, \sum_{d'} \beta^{d'} \omega_{d'}\rangle. \tag{4.4}$$

Clearly, then, $C_{\mathbf{v},\mathbf{v}}$ is non-negative definite. Moreover, if for some $\beta \in \mathbb{C}^D$ we have $\sum_d \beta^d \omega_d \in \operatorname{span}\mathbf{v}^\perp$, by (4.4) we must have $C_{\mathbf{v},\boldsymbol{\omega}}\beta^* = 0$. Thus, if $C_{\mathbf{v},\boldsymbol{\omega}}$ is invertible, $\beta = 0$, and it follows that $\operatorname{span}\mathbf{v}^\perp \cap \operatorname{span}\boldsymbol{\omega} = \{0\}$, and the vectors in $\boldsymbol{\omega}$ are linearly independent. Since $C^*_{\mathbf{v},\boldsymbol{\omega}} = C_{\boldsymbol{\omega},\mathbf{v}}$, also the vectors in $\mathbf{v}$ are linearly independent. Conversely, if $C_{\mathbf{v},\boldsymbol{\omega}}$ is not invertible, there is a vector $\beta \in \mathbb{C}^D$, $\beta \neq 0$, such that $C_{\mathbf{v},\boldsymbol{\omega}}\beta^* = 0$. From (4.4), we also have $\sum_d \beta^d \omega_d \in \operatorname{span}\mathbf{v}^\perp$, which in turn implies that, either $\operatorname{span}\mathbf{v}^\perp \cap \operatorname{span}\boldsymbol{\omega} \neq \{0\}$, or $\sum_d \beta^d \omega_d = 0$ and $\boldsymbol{\omega}$ is not linearly independent. □

In our situation, the space $\mathcal{H}$ will be $\ell^2(\Gamma')$, with the inner product $[\![\cdot\,,\cdot]\!]$ defined in §2. We obtain, then, the following expression for $\|S(\xi)\|$, which resembles very much the one given in Corollary 4.1 for the case $D = 1$ (compare also with (1.4)):

PROPOSITION 4.5 *Let* Ψ *and* $\widetilde{\Psi}$ *be two biorthogonal wavelet families and* $\xi \in \mathbb{T}^n$ *fixed. If* $\boldsymbol{\omega} \equiv \{\omega_1, \ldots, \omega_D\} \subset \mathcal{F}(\xi)$ *and* $\tilde{\boldsymbol{\omega}} \equiv \{\tilde{\omega}_1, \ldots, \tilde{\omega}_D\} \subset \widetilde{\mathcal{F}}(\xi)$ *are two linearly independent systems, then the operator* $S = S(\xi)$ *defined in (2.10) has norm:*

$$\|S\| = \|C^{\frac{1}{2}}_{\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}}\, C^{-1}_{\boldsymbol{\omega},\tilde{\boldsymbol{\omega}}}\, C^{\frac{1}{2}}_{\boldsymbol{\omega},\boldsymbol{\omega}}\|. \tag{4.6}$$

PROOF: We know that

$$\|S\|^2 = \sup_{\substack{p\in\mathbb{C}^d \\ p\neq 0}} \frac{[\![\, S[\sum_d p^d \tilde{\omega}_d] \,]\!]^2}{[\![\, \sum_d p^d \tilde{\omega}_d \,]\!]^2}. \tag{4.7}$$

Using (4.4), and considering p as a row matrix, the denominator in the expression above can be written as: $pC_{\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}}p^*$. To find a simpler expression for the numerator, let $\mathbf{v} = \{v_1, \ldots, v_D\}$ and $\tilde{\mathbf{v}} = \{\tilde{v}_1, \ldots, \tilde{v}_D\}$ be a pair of biorthogonal bases for $\mathcal{F} = \mathcal{F}(\xi)$ and $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}(\xi)$, and let $\Lambda = (\Lambda_{d,d'})$, $\tilde{\Lambda} = (\tilde{\Lambda}_{d,d'})$ be two invertible matrices of scalars so that

$$v_d = \sum_{d'=1}^{D} \Lambda_{d,d'}\omega_{d'} = \Lambda_d \cdot \boldsymbol{\omega} \quad \text{and} \quad \tilde{v}_d = \sum_{d'=1}^{D} \tilde{\Lambda}_{d,d'}\tilde{\omega}_{d'} = \tilde{\Lambda}_d \cdot \tilde{\boldsymbol{\omega}}, \tag{4.8}$$

where Λ_d and $\tilde{\Lambda}_d$ denote the d^{th}-rows of Λ and $\tilde{\Lambda}$, and $\boldsymbol{\omega}$ and $\tilde{\boldsymbol{\omega}}$ are column matrices of vectors in $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, respectively. Then, for $d = 1, ..., D$, we have

$$S\tilde{\omega}_d = \sum_{d'} [\![S\tilde{\omega}_d, \tilde{v}_{d'}]\!]\, v_{d'} = \sum_{d'} [\![\tilde{\omega}_d, \tilde{v}_{d'}]\!]\, v_{d'}.$$

Using (4.8) and (4.4), the vector in the numerator of (4.7) equals

$$\begin{aligned} S(p\cdot\tilde{\boldsymbol{\omega}}) &= \sum_{d'} [\![p\cdot\tilde{\boldsymbol{\omega}}, \tilde{v}_{d'}]\!]\, v_{d'} = \sum_{d'} [\![p\cdot\tilde{\boldsymbol{\omega}}, \tilde{\Lambda}_{d'}\cdot\tilde{\boldsymbol{\omega}}]\!]\, \Lambda_{d'}\cdot\boldsymbol{\omega} \\ &= \sum_{d'=1}^{D} (pC_{\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}}\tilde{\Lambda}^*_{d'})\Lambda_{d'}\cdot\boldsymbol{\omega} = pC_{\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}}\,(\sum_{d'} \tilde{\Lambda}^*_{d'}\Lambda_{d'})\cdot\boldsymbol{\omega} \;=\; pC_{\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}}\tilde{\Lambda}^*\Lambda\cdot\boldsymbol{\omega}. \end{aligned}$$

Now, from (4.4), (4.8), and the biorthogonality of $\mathbf{v}$ and $\tilde{\mathbf{v}}$ one easily verifies that $\Lambda C_{\boldsymbol{\omega},\tilde{\boldsymbol{\omega}}}\tilde{\Lambda}^* = C_{\mathbf{v},\tilde{\mathbf{v}}} = I_D$. Thus, one more use of (4.4) gives us:

$$\|S\|^2 = \sup_{p\neq 0} \frac{(pC_{\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}}C^{-1}_{\boldsymbol{\omega},\tilde{\boldsymbol{\omega}}})C_{\boldsymbol{\omega},\boldsymbol{\omega}}(pC_{\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}}C^{-1}_{\boldsymbol{\omega},\tilde{\boldsymbol{\omega}}})^*}{pC_{\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}}p^*}.$$

Changing variables $q = pC^{\frac{1}{2}}_{\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}}$ (the square root matrix is well-defined by Lemma 4.3), we arrive at the desired expression in (4.6).

□

We turn now to a sufficient condition for wavelet families to arise from a pair of biorthogonal MRA's. As we shall see later, this condition is implied by minimal smoothness and decay assumptions on the wavelets, as was proved by P. Auscher and P.G. Lemarié in [AUS], and [LEM]. Their construction of the scaling families, however, requires a finer analysis than ours, since they can obtain them with the same smoothness and decay they assume for the wavelets.

COROLLARY 4.9 *Let $\Psi = \{\psi^1, \ldots, \psi^L\}$, $\widetilde{\Psi} = \{\tilde{\psi}^1, \ldots, \tilde{\psi}^L\}$ be a pair of biorthogonal wavelet families in $L^2(\mathbb{R}^n)$, and suppose that each $\eta \in \Psi \cup \widetilde{\Psi}$ has continuous Fourier transform and the series*

$$D_\eta(\xi) \equiv \sum_{j=1}^{\infty} \sum_{\gamma \in \Gamma'} |\hat{\eta}(N^j(\xi+\gamma))|^2, \quad \xi \in \mathbb{R}^n, \tag{4.10}$$

converges uniformly to a bounded function in compact sets of $(0, 2\pi)^n$. Then, there exists an integer $D \geq 1$ such that $L = (m-1)D$, and the couple $\{\Psi, \widetilde{\Psi}\}$ arises from a pair of biorthogonal multiresolution analyses (of multiplicity D).

PROOF: We only need to verify that conditions (I) and (II) in Theorem 2.11 hold. Now, the second one follows immediately from Lemma 2.5, part (jjj), since we have the bound:

$$\|S(\xi)\|^2 \leq \Big(\sum_{\ell=1}^{L} D_{\psi^\ell}(\xi)\Big)\Big(\sum_{\ell=1}^{L} D_{\tilde{\psi}^\ell}(\xi)\Big), \quad \xi \in \mathbb{T}^n, \tag{4.11}$$

and the last quantity belongs to $L^\infty(\mathbb{T}^n)$ by assumption. To see the first condition, note that the series defining $D_{\Psi,\tilde{\Psi}}(\xi)$ also converges uniformly in compact sets of $(0, 2\pi)^n$ to a continuous function. Since, by Lemma 2.5, this function can only take integer values, there must exist $D \in \mathbb{Z}^+$ such that $D_{\Psi,\tilde{\Psi}}(\xi) = D,\ \xi \in (0, 2\pi)^n$. $\square$

The sufficient conditions in terms of smoothness and decay of the wavelets given by Auscher and Lemarié are the following:

DEFINITION 4.12 *We say that a function $f \in L^2(\mathbb{R}^n)$ is* regular* *if $\hat{f} \in C(\mathbb{R}^n)$ and there exist ε, $C > 0$ such that $|\hat{f}(\xi)| \leq C\,(1+|\xi|)^{-(n/2+\varepsilon)}$, $\xi \in \mathbb{R}^n$.*

DEFINITION 4.13 *We say that a function $f \in L^2(\mathbb{R}^n)$ is* localized *if for some $\varepsilon > 0$ $|\xi|^\varepsilon \hat{f}(\xi) \in L^2(\mathbb{R}^n)$ and $|x|^{n/2+\varepsilon} f(x) \in L^2(\mathbb{R}^n)$. Using Sobolev space terminology, this is equivalent to $f \in H^\varepsilon(\mathbb{R}^n)$ and $\hat{f} \in H^{n/2+\varepsilon}(\mathbb{R}^n)$.*

The following lemma shows how $D_f(\xi) = \sum_{j=1}^{\infty} \sum_{\gamma \in \Gamma'} |\hat{f}(N^j(\xi+\gamma))|^2$ behaves in each of these cases. We will fix a real number μ so that $1 < \mu < \min\{\,|\lambda| : \lambda$ eigenvalue of $N\,\}$. Note that, in this case, there is a constant $c > 0$ such that

$$|N^j\xi| \geq c\,\mu^j |\xi|, \quad \forall\, \xi \in \mathbb{R}^n, \ \ j \geq 0$$

(see, *e.g.*, Remark 2 in [CAL]).

---

*We point out that Auscher's regularity condition is a bit stronger than the one given in Definition 4.12. A decay of the form $|\hat{f}(\xi)| \leq C\,|\xi|^\varepsilon$, when $|\xi| \sim 0$, is also assumed in [AUS]. However, in what concerns to our results, the definition we give suffices.

Lemma 4.14 *If $f \in L^2(\mathbb{R}^n)$ is regular or localized, then the series defining $D_f(\xi)$ converges uniformly in compact sets of $\mathbb{T}^n \setminus \{0\}$, and there is a constant $C > 0$ such that*

$$D_f(\xi) \leq C + \sum_{j=1}^{\infty} |\hat{f}(N^j\xi)|^2, \quad \xi \in (-\pi,\pi)^n \setminus \{0\}. \tag{4.15}$$

Proof:

Assume first that f is regular. If $\xi \in \mathbb{R}^n \setminus \{0\}$, we have:

$$s(\xi) \equiv \sum_{j=1}^{\infty} |\hat{f}(N^j\xi)|^2 \leq C \sum_{j=1}^{\infty} \frac{1}{(1+|N^j\xi|)^{n+2\varepsilon}} \leq C\,(c|\xi|)^{-(n+2\varepsilon)} \sum_{j=1}^{\infty} \mu^{-j(n+2\varepsilon)}.$$

Thus, the series defining $s(\xi)$ converges uniformly in compact sets of $\mathbb{R}^n \setminus \{0\}$ to a continuous function. Moreover, the decay of $s(\xi)$ when ξ is away from 0 implies that the series:

$$D_f(\xi) = \sum_{\gamma\in\Gamma'} s(\xi+\gamma) = s(\xi) + \sum_{\gamma\neq 0} s(\xi+\gamma), \quad \xi \in [-\pi,\pi]^n \setminus \{0\},$$

converges uniformly in compact sets of $\mathbb{T}^n \setminus \{0\}$, and (4.15) holds.

In the localized situation, one may use complex interpolation to find $\varepsilon' > 0$ and $\alpha > n/2$ such that $|\xi|^{\varepsilon'}\,\hat{f}(\xi) \in H^{\alpha}(\mathbb{R}^n)$ (see Lemma 2 in Chapter 2 of [KL]). Sobolev space properties imply then, that the periodized series

$$t(\xi) = \sum_{\gamma\in\Gamma'} |\xi+\gamma|^{2\varepsilon'}\,|\hat{f}(\xi+\gamma)|^2, \quad \xi \in [-\pi,\pi]^n,$$

converges uniformly to a continuous periodic function in $\mathbb{R}^n$ (see, *e.g.*, Chapter 3 of [KL]). Then, if $\xi \in [-\pi,\pi]^n$ and $|\xi| > \delta$, for some $0 < \delta < 1$, we have:

$$D_f(\xi) \leq \sum_{j=1}^{\infty}\sum_{\gamma\in\Gamma'} \frac{|N^j(\xi+\gamma)|^{2\varepsilon'}}{(\mu^j\delta c)^{2\varepsilon'}}\,|\hat{f}(N^j(\xi+\gamma))|^2 \leq (\delta c)^{-2\varepsilon'} \sum_{j=1}^{\infty} \mu^{-2j\varepsilon'} t(N^j\xi) \leq C_\delta,$$

which implies the convergence on compact sets announced in the statement of the lemma. Moreover, if $\xi \in (-\pi,\pi)^n$, we obtain:

$$\begin{aligned} D_f(\xi) &\leq s(\xi) + \sum_{j=1}^{\infty}\sum_{\gamma\neq 0} \frac{|N^j(\xi+\gamma)|^{2\varepsilon'}}{(\mu^j c)^{2\varepsilon'}}\,|\hat{f}(N^j(\xi+\gamma))|^2 \\ &\leq s(\xi) + c^{-2\varepsilon'} \sum_{j=1}^{\infty} \mu^{-2j\varepsilon'} t(N^j\xi) \leq s(\xi) + C. \end{aligned}$$

□

Corollary 4.16 *If $\Psi = \{\psi^1,\ldots,\psi^L\}, \widetilde{\Psi} = \{\tilde{\psi}^1,\ldots,\tilde{\psi}^L\}$ are biorthogonal wavelet families in $L^2(\mathbb{R}^n)$, and the functions in $\Psi \cup \widetilde{\Psi}$ are either regular or localized, then $(\Psi,\widetilde{\Psi})$ arises from a pair of biorthogonal multiresolution analyses.*

PROOF: This is an immediate consequence of Lemma 4.14, Corollary 4.9, and the fact that $\sum_{j\in\mathbb{Z}} |\hat{\eta}(N^j\xi)|^2 \in L^\infty(\mathbb{R}^n)$ whenever $\{\eta_{j,\gamma}\}_{j,\gamma}$ is a Bessel system (see, *e.g.* Remark 7 in [FGWW]).

□

The reasoning we just gave has another interesting consequence which shows how mild condition (II) in Theorem 2.11 can be (see also the sufficient condition given by Bin Han in [HAN2]):

COROLLARY 4.17 *If* $\Psi = \{\psi^1, \dots, \psi^L\}, \tilde{\Psi} = \{\tilde{\psi}^1, \dots, \tilde{\psi}^L\}$ *are biorthogonal wavelet families in* $L^2(\mathbb{R}^n)$*, and every function* $\eta \in \Psi \cup \tilde{\Psi}$ *has* $|\hat{\eta}(\xi)| \le C(1+|\xi|)^{-(n/2+\varepsilon)}$*, for constants* $C, \varepsilon > 0$*, then* $\operatorname{ess-sup} \|S(\xi)\| < \infty$.

PROOF: The decay condition on η implies that the series $\sum_{j=1}^\infty \sum_{\gamma\ne 0} |\hat{\eta}(N^j(\xi+\gamma))|^2$ converges uniformly, when $\xi \in \mathbb{T}^n$, to a bounded function (see the first lines of the proof of Lemma 4.14). Since we have $\sum_{j\in\mathbb{Z}} |\hat{\eta}(N^j\xi)|^2 \in L^\infty(\mathbb{R}^n)$, it follows that $D_\eta(\xi)$ is (essentially) bounded, and (4.11) establishes the corollary.

□

5 Examples

The rest of the paper will be devoted to present a series of examples that illustrate how to use Theorem 2.11. The first example will be of an orthonormal wavelet in $\mathbb{R}^2$, associated with an expansive matrix of determinant 2, and not arising from an MRA. The second is a general construction of orthonormal wavelet families in $\mathbb{R}^n$ arising from MRA's of multiplicity 2^J. Finally, the third one is a collection of non-trivial biorthogonal wavelets in $L^2(\mathbb{R})$, also arising from multiresolution analyses.

One of the simplest ways of constructing orthonormal wavelets is by finding measurable sets $T \subset \mathbb{R}^n$ satisfying:

$$\begin{array}{lll} (C1) & \{N^jT\}_{j\in\mathbb{Z}} & \text{is a partition } a.e. \text{ of } \mathbb{R}^n \\ (C3) & \{T+\gamma\}_{\gamma\in\Gamma'} & \text{is a partition } a.e. \text{ of } \mathbb{R}^n. \end{array}$$

Any such T is called a *wavelet set* since the function $\psi = (\chi_T)^\vee$ is an orthonormal wavelet in $L^2(\mathbb{R}^n)$ (see [FW], [HW] for $n=1$, and [DLS], [SW], [CAL] for $n \ge 2$). Although ψ will not even belong to $L^1(\mathbb{R}^n)$, an appropriate smoothing of the characteristic function χ_T produces, in many cases, single wavelets in the Schwartz class which can be very useful in applications (see [HWW1] for $n=1$, and [CAL2], [SPE] for $n=2$). Note that, when dealing with wavelet sets, $\widehat{\psi}$ is not continuous, and therefore, one cannot use Corollary 4.9 to determine whether ψ arises from a multiresolution analysis. However, the statement we gave of Theorem 2.11 simplifies very much in this situation.

COROLLARY 5.1 *Let ψ be an orthonormal wavelet in $L^2(\mathbb{R}^n)$ so that $\widehat{\psi} = \chi_T$, for some measurable set $T \subset \mathbb{R}^n$. Then, ψ arises from an MRA (necessarily of multiplicity 1) if and only if $m = |\det(M)| = 2$ and*

$$(C5) \qquad \{ [\cup_{j=1}^{\infty} N^{-j}T] + \gamma \}_{\gamma \in \Gamma'} \quad \textit{is a partition a.e. of } \mathbb{R}^n.$$

PROOF: Immediate consequence of $(C1)$ and Corollary 2.12. □

This result (which in the 1-dimensional setting is due to [FW]) can be effectively used to test wavelet sets already existing in the literature. For instance, when $n \geq 2$, most of the examples in [CAL2], [DLS] and [SW], are either related to matrices with $m = |\det(M)| > 2$ (and hence, cannot arise from an MRA), or have been obtained directly from an MRA after finding appropriate scaling functions. Below, we give an example of a set $T \subset \mathbb{R}^2$, associated with $\Gamma = \mathbb{Z}^2$ and $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, such that $\psi = (\chi_T)\check{}$ is a wavelet not arising from an MRA. Note that $m = 2$ and, therefore, we need to use (I) in Theorem 2.11 to show this (or equivalently, see that condition $(C5)$ above does not hold).

EXAMPLE 5.2 As usually happens with the constructions of wavelet sets in $\mathbb{R}^2$, T will be of "fractal" form (see Figure 5.1 below). The iterative process producing it is as follows. Let us define the sequence of complex numbers:

$$z_0 = 0 \quad \text{and} \quad z_n = z_{n-1} + \pi 2^{-\frac{3n-2}{2}} e^{-i(n-2)\frac{\pi}{4}}, \quad n \in \mathbb{Z}^+. \tag{5.3}$$

Let us consider the sequence of (convex) rectangular isosceles triangles $\{T_n\}_{n \geq 0}$ such that, for every n, T_n has the point $z_n = (\text{Re } z_n, \text{Im } z_n)$ as vertex of the right angle, while the point $z_{n+1} = (\text{Re } z_{n+1}, \text{Im } z_{n+1})$ is the middle point of the hypotenusa (see Figure 5.1).

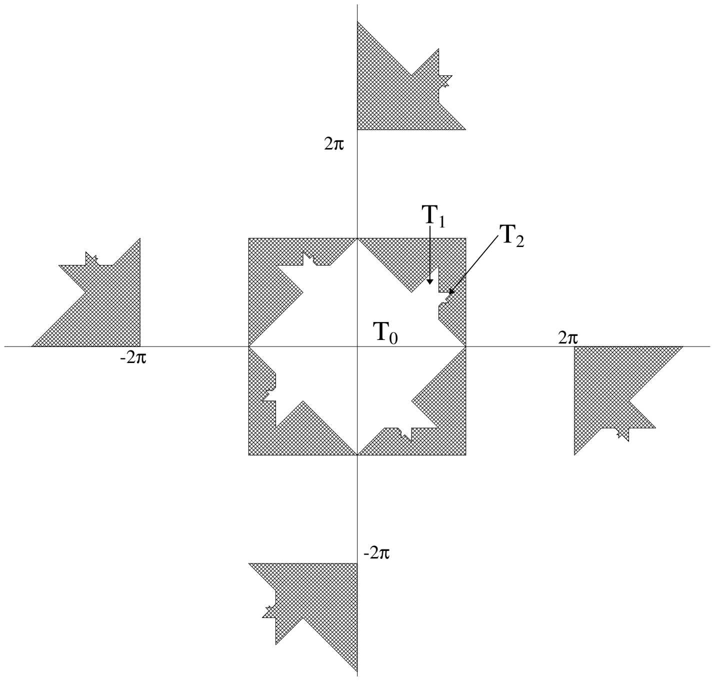


Figure 5.1: The wavelet set T, and the sequence of triangles that defines it.

In other words, we start with the triangle T_0, with vertices $\{(\pi, 0),\ (0, \pi), (0, 0)\}$, and construct T_1 by first applying $\frac{1}{2}N^{-1}$ (that is, rotate T_0 around $z_0 = 0$ by $-\pi/4$ radians, and dilate by $2^{-3/2}$), and then translating by $z_1 = (\frac{\pi}{2}, \frac{\pi}{2})$ (that is, place the obtained triangle in the middle point of the hypotenusa of T_0). One proceeds similarly to construct T_n from T_{n-1}, which in mathematical language can be written as:

$$T_n = \frac{1}{2}N^{-1}(T_{n-1} - z_{n-1}) + z_n, \quad n \in \mathbb{Z}^+.$$

Note that this is possible since $\frac{1}{2}N^{-1}(z_n - z_{n-1}) + z_n = z_{n+1}$, and the triangle T_n is determined precisely by the points z_n and z_{n+1}. Now, let

$$T^I = \left([0,\pi)^2 \setminus \bigcup_{n\geq 0} T_n \right) \cup \left(\bigcup_{n\geq 0} T_n + (0, 2\pi) \right),$$

and $T = T^I \cup T^{II} \cup T^{III} \cup T^{IV}$, where the last three sets correspond to the rotations of T^I into the second, third and fourth quadrants, respectively. From the definition of T^I, one immediately verifies that the set T satisfies condition $(C3)$. To show $(C1)$, we claim that the component of T on the left of the second quadrant (see Figure 5.1) "dilates" (by N^{-3}) precisely into the union of the triangles $T_1 \cup T_2 \cup \ldots$, in the first quadrant. Indeed, from (5.3) it is easy to see that

$$N^{-3}(\, e^{i\frac{\pi}{2}} z_{n-1} + (-2\pi, 0)\,) = z_n, \quad n \in \mathbb{Z}^+,$$

and, hence,

$$N^{-3}(\, e^{i\frac{\pi}{2}} T_{n-1} + (-2\pi, 0)\,) = T_n, \quad n \in \mathbb{Z}^+. \tag{5.4}$$

Now, by definition

$$T^{II} \setminus [-\pi, 0)^2 = e^{i\frac{\pi}{2}} \bigcup_{n=0}^{\infty} [\, T_n + (0, 2\pi)\,] = \bigcup_{n=1}^{\infty} [\, e^{i\frac{\pi}{2}} T_{n-1} + (-2\pi, 0)\,]$$

and, by (5.4), the last quantity dilates (with N^{-3}) precisely onto $\cup_{n=1}^{\infty} T_n$, establishing our claim. Proceeding similarly with each of the quadrants, one obtains the *a.e.* disjoint union:

$$[\, T \cap [-\pi, \pi)^2\,] \cup [\, N^{-3}(T \setminus [-\pi, \pi)^2)\,] = [-\pi, \pi)^2 \setminus N^{-1}[-\pi, \pi)^2,$$

which clearly implies $(C1)$, and proves that T is a wavelet set.

To see that $(C5)$ does not hold, note that the triangle $\widetilde{T} = [(\frac{\pi}{2}, 0),\, (\frac{3\pi}{2}, \frac{\pi}{4}),\, (\pi, 0)]$ is contained in the torus $\mathbb{T}^2 = [-\pi, \pi)^2$, but the dilations

$$\begin{cases} N\widetilde{T} & = \; [\,(\frac{\pi}{2}, \frac{\pi}{2}),\, (\frac{\pi}{2}, \pi),\, (\pi, \pi)\,] \\ \\ N^2(\,\widetilde{T} - (2\pi, 0)\,) & = \; [\,(0, -3\pi),\, (-\frac{\pi}{2}, -\frac{5\pi}{2}),\, (0, -2\pi)\,], \end{cases}$$

are each contained in T. Hence, $N^{-1}T \cap [N^{-2}T + (2\pi, 0)\,] \supset \widetilde{T}$, which has positive measure, and establishes our claim. $\square$

The example we just gave, like all the constructions with wavelet sets, is associated with a "single" wavelet $\psi = (\chi_T)\check{}$. One can also derive (orthonormal) wavelet families $\Psi = \{\psi^1, \ldots, \psi^L\}$ such that $\widehat{\psi^\ell} = \chi_{T_\ell}$, for appropriate choices of T_ℓ; the sets $\{T_1, \ldots, T_L\}$ will be called a family of *multi-wavelet sets*. An example is given in Figure 5.2 below (see also [CAL2], [BL]).

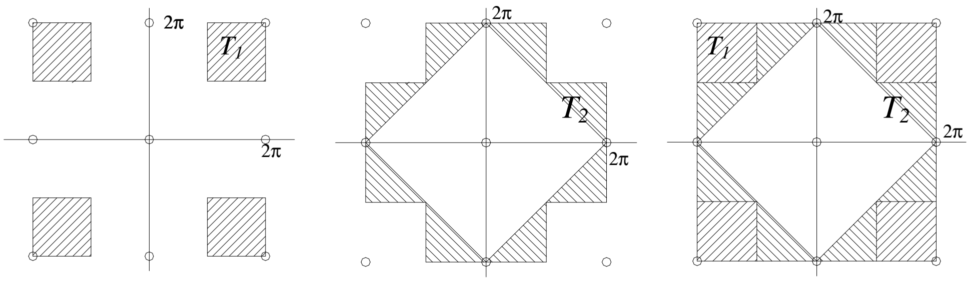


Figure 5.2: Sets T_1, T_2, leading to a wavelet family $\Psi = \{(\chi_{T_1})\check{}, (\chi_{T_2})\check{}\}$, when $\Gamma = \mathbb{Z}^2$ and $M = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$.

Note that, in this example, T_1 and T_2 form an appropriate partition of NT, where $T = N\mathbb{T}^2 \setminus \mathbb{T}^2$ is a wavelet set in $\mathbb{R}^2$. Using Corollary 2.12 one can easily show that Ψ arises from an MRA of multiplicity 2. This is also a particular instance of the following general construction.

EXAMPLE 5.5 Let $T \subset \mathbb{R}^n$ be a wavelet set, and $\psi = (\chi_T)\check{}$ the associated wavelet. Let us fix $J \in \mathbb{Z}^+$, and recall (from the arguments leading to (3.5)) that the quotient $\Delta = \Delta_J \equiv \Gamma'/N^J\Gamma'$ has exactly m^J distinct representatives: $\{\delta_1 = 0, \delta_2, \ldots, \delta_{m^J}\}$, while

$$\{\, C(\ell) \equiv \cup_{\gamma\in\Gamma'}[\mathbb{T}^n + N^J\gamma] \,+\, \delta_\ell \,\}_{\ell=1}^{m^J},$$

forms a partition of $\mathbb{R}^n$. Then, we can split the set N^JT into m^J disjoint parts as follows:

$$T_\ell \equiv N^JT \cap C(\ell), \quad \ell = 1, \ldots, m^J. \tag{5.6}$$

(Compare with Figure 5.2.) We claim that $\Psi = \{(\chi_{T_1})\check{}, \ldots, (\chi_{T_{m^J}})\check{}\}$ forms an (orthonormal) wavelet family in $L^2(\mathbb{R}^n)$. This will be an immediate consequence of Theorem 2.2 and the following lemma:

LEMMA 5.7 *In the above conditions, if for some $\gamma \in \Gamma'$ and $\ell = 1, \ldots, m^J$, we have $|\,(T_\ell + \gamma) \cap T_\ell\,| > 0$, then $\gamma = 0$.*

Similarly, if for some $j \in \mathbb{Z}$ and $\ell, \ell' = 1, \dots, m^J$, we have $|\,(N^j T_\ell) \cap T_{\ell'}\,| > 0$, then $j = 0$ and $\ell = \ell'$.

PROOF: For the first statement, write $\gamma = N^J \gamma' + \delta$, for $\gamma' \in \Gamma'$, $\delta \in \Delta$. Then, $(T_\ell + \gamma) \cap T_\ell \subset (C(\ell) + \delta) \cap C(\ell)$, which by (3.5) implies that $\delta = 0$. Now $\gamma = N^J \gamma'$, and therefore, $(T_\ell + \gamma) \cap T_\ell \subset N^J[\,(T + \gamma') \cap T\,]$. Property $(C3)$ implies then that $\gamma' = \gamma = 0$.

For the second statement, note that $N^j T_\ell \cap T_{\ell'} \subset N^{j+J} T \cap N^J T$, which by $(C1)$ implies $j = 0$. Then, $|\,T_\ell \cap T_{\ell'}\,| > 0$, which can only happen if $\ell = \ell'$.

□

From the lemma it is immediate that equations $(B2)$ and $(B4)$ in Theorem 2.2 hold (each of the terms in the series involved will be 0). Moreover, to check $(B1)$ and $(B3)$ it suffices to see that $\cup_\ell \cup_j N^j T_\ell = \mathbb{R}^n$ and $\cup_\gamma (T_\ell + \gamma) = \mathbb{R}^n$, $\ell = 1, \dots, m^J$. But this follows clearly from (5.6) since

$$\bigcup_{\ell=1}^{m^J} \bigcup_{j \in \mathbb{Z}} N^j T_\ell = \bigcup_{j \in \mathbb{Z}} N^{j+J} T = \mathbb{R}^n$$

and

$$\bigcup_{\gamma \in \Gamma'} (T_\ell + \gamma) = \bigcup_{\delta \in \Delta} [\; \bigcup_{\gamma' \in \Gamma'} N^J T \cap C(\ell) + N^J \gamma'\,] + \delta \; = \cup_{\delta \in \Delta} C(\ell) + \delta = \mathbb{R}^n .$$

Thus, Ψ is a wavelet family*. Now, we make use of Corollary 2.12 to determine whether Ψ arises from an MRA. Note that, for $\xi \in \mathbb{R}^n$,

$$\begin{aligned} D_\Psi(\xi) &= \sum_{\ell=1}^{m^J} \sum_{j=1}^{\infty} \sum_{\gamma \in \Gamma'} \chi_{T_\ell}(N^j(\xi + \gamma)) \\ &= \sum_{j=1}^{\infty} \sum_{\gamma \in \Gamma'} \chi_{N^J T}(N^j(\xi + \gamma)) = \sum_{j=J+1}^{\infty} \{\cdots\} + \sum_{j=1}^{J} \{\cdots\} \\ &= D_\psi(\xi) + \sum_{j=0}^{J-1} \{ \sum_{\gamma \in \Gamma'} \chi_{N^j T}(\xi + \gamma) \}. \end{aligned}$$

Using $(C3)$, we see that the series inside the brackets equals

$$\sum_{\delta \in \Delta_j} \sum_{\gamma' \in \Gamma'} \chi_T(N^{-j}\xi + \gamma' + N^{-j}\delta) = \mathrm{Card}(\Delta_j) = m^j, \quad j = 0, \dots J-1, \quad a.e.\, \xi \in \mathbb{T}^n.$$

Thus,

$$D_\Psi(\xi) = D_\psi(\xi) + \sum_{j=0}^{J-1} m^j, \quad a.e.\, \xi \in \mathbb{T}^n,$$

---

*In this case, since $\Psi = \widetilde{\Psi}$, the Bessel condition in the statement of Theorem 2.2 needs not be checked (see the comment immediately before Remark 2.3, or [BL]).

and $D_\Psi(\xi)$ is constant $(a.e.)$ in $\mathbb{T}^n$ if and only if $D_\psi(\xi)$ is. But since $\int_{\mathbb{T}^n} D_\psi(\xi)\,\frac{d\xi}{(2\pi)^n} = \frac{1}{m-1}$ (see the computation right before the end of the proof of Corollary 4.1) and D_ψ is integer-valued, this can happen only if $m = 2$ and $D_\psi(\xi) \equiv 1$. We can summarize all this in the following proposition:

PROPOSITION 5.8 *Let $J \in \mathbb{Z}^+$ be fixed, and $T \subset \mathbb{R}^n$ be a wavelet set (that is, T satisfies $(C1)$ and $(C3)$). Define $T_\ell = N^J T \cap C(\ell)$, $\ell = 1, \ldots, m^J$, and let $\Psi = \{(\chi_{T_1})\check{}, \ldots, (\chi_{T_{m^J}})\check{}\}$. Then, Ψ is an orthonormal wavelet family. Moreover, Ψ arises from an MRA of multiplicity d if and only if T satisfies $(C5)$, in which case $m = 2$ and $d = 2^J$.*

One more example of multiwavelet sets, still with the matrix $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, is given in Figure 5.3 below.

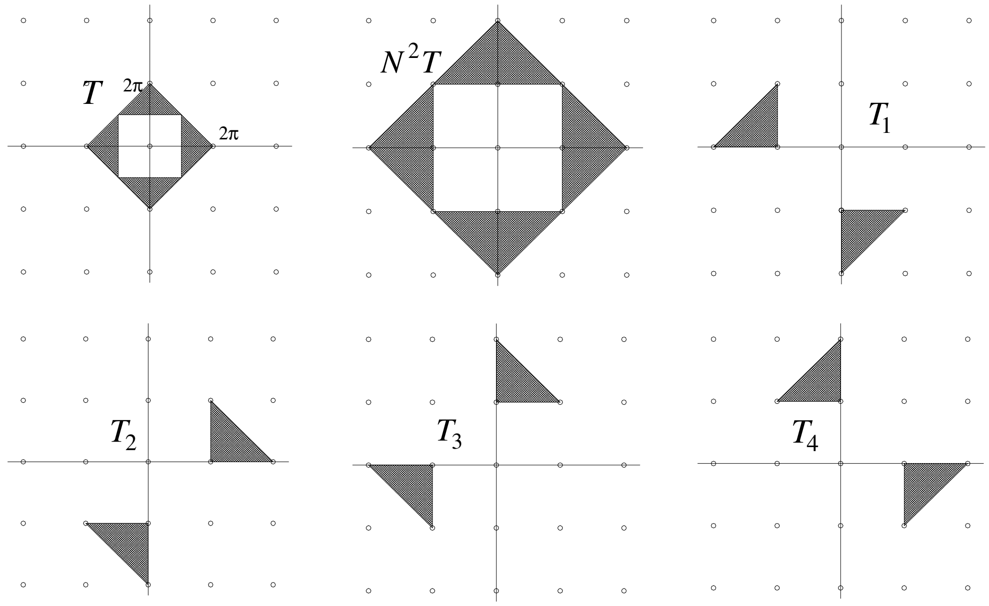


Figure 5.3: Sets producing a wavelet family arising from an MRA of multiplicity 4.

□

Finally, we present an example of a non-trivial pair of biorthogonal wavelets in $L^2(\mathbb{R})$, and use Theorem 1.2 to determine whether it arises from an MRA. Note that, in the biorthogonal situation, we cannot have a *non-trivial* analogue of wavelet sets. Indeed, it is an easy exercise to show (from equations $(B1)-(B4)$) that two functions ψ, $\tilde{\psi} \in L^2(\mathbb{R}^n)$, of the form $\widehat{\psi} = a\chi_T$ and $\widehat{\tilde{\psi}} = \tilde{a}\chi_{\tilde{T}}$, ($a$, $\tilde{a}$ complex constants) can only form a pair of biorthogonal wavelets if $\tilde{a} = \overline{a}^{-1}$ and $T = \widetilde{T}$ *a.e.* Therefore, modulo a constant, ψ and $\tilde{\psi}$ will be equal to an orthonormal wavelet $\eta = (\chi_T)\check{}$. Although these constructions may have a theoretical interest when a is a non constant function†, we seek less trivial examples where $\widehat{\psi}$ and $\widehat{\tilde{\psi}}$ have disjoint supports. The choice $a(\xi) = \tilde{a}(\xi) = e^{i\xi/2}$, and T, $\widetilde{T}$ as in Figure 5.4 below is an instance of what we are saying.

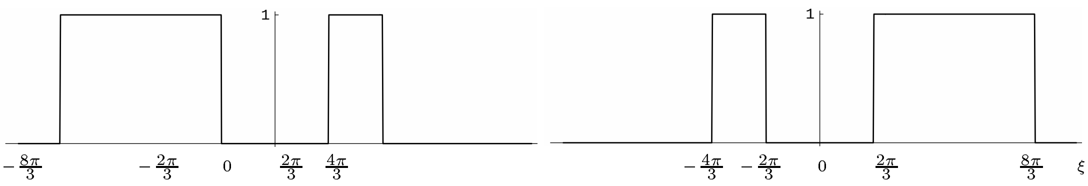


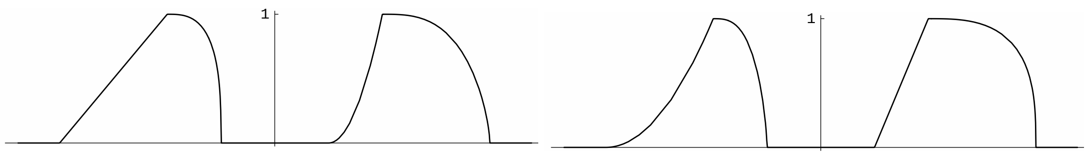
Figure 5.4: Sets T and $\widetilde{T}$ producing a pair of biorthogonal wavelets in $L^2(\mathbb{R})$.

More generally, we can construct the following family of biorthogonal wavelets in $L^2(\mathbb{R})$ (with dilation factor 2^j, as in §1).

EXAMPLE 5.9 We start with an arbitrary measurable function $b : [\frac{2\pi}{3}, \frac{4\pi}{3}) \to \mathbb{R}$ such that $0 \le b \le 1$, and two fixed real numbers $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$. Then, define:

$$|\widehat{\psi}(\xi)| = \begin{cases} b(\xi)^\alpha, & \frac{2\pi}{3} \le \xi < \frac{4\pi}{3} \\ (1 - b(\xi/2))^\alpha, & \frac{4\pi}{3} \le \xi < \frac{8\pi}{3} \\ (1 - b(\xi + 2\pi))^\beta, & -\frac{4\pi}{3} \le \xi < -\frac{2\pi}{3} \\ b(\xi/2 + 2\pi)^\beta, & -\frac{8\pi}{3} \le \xi < -\frac{4\pi}{3} \\ 0 & \text{elsewhere} \end{cases} \tag{5.10}$$

and let $\widehat{\psi}(\xi) := e^{i\frac{\xi}{2}}|\widehat{\psi}(\xi)|$, $\xi \in \mathbb{R}$. One defines $\widehat{\tilde{\psi}}$ in exactly the same way, but interchanging the roles of α and β in (5.10) (see Figure 5.5 below).



†The function a could be thought as a generalization of the notion of *wavelet multiplier*; see [WUTAM].

$-\frac{8\pi}{3}$ $-\frac{4\pi}{3}$ $-\frac{2\pi}{3}$ 0 $\frac{2\pi}{3}$ $\frac{4\pi}{3}$ $\frac{8\pi}{3}$ $-\frac{8\pi}{3}$ $-\frac{4\pi}{3}$ $-\frac{2\pi}{3}$ 0 $\frac{2\pi}{3}$ $\frac{4\pi}{3}$ $\frac{8\pi}{3}$ ξ

Figure 5.5: Graphs of $|\widehat{\psi}|$ and $|\widehat{\tilde{\psi}}|$, with $b(\xi) = c(\xi - \frac{2\pi}{3})^3$ and $\alpha = \frac{2}{3}$, $\beta = \frac{1}{3}$.

It is easy to check that the pair $\widehat{\psi}, \widehat{\tilde{\psi}}$ just constructed satisfies the equations $(B1)$ through $(B4)$, as well as equation (1.3) in Theorem 1.2 (the reader can compare with the proof in the orthonormal case, given in [HWW1] or Chapter 3 of [HW]). Moreover, the compact support (in $\mathbb{R} \setminus \{0\}$) and boundedness of $|\widehat{\psi}|$ and $|\widehat{\tilde{\psi}}|$ imply that each $\{\psi_{j,k}\}_{j,k}$ and $\{\tilde{\psi}_{j,k}\}_{j,k}$ is a Bessel system in $L^2(\mathbb{R})$ (see, *e.g.*, Remark 6 in [FGWW]), and hence $\{\psi, \tilde{\psi}\}$ is a biorthogonal wavelet pair. We just need to verify that $\|S(\xi)\| \in L^\infty(\mathbb{T})$ to show that it arises from a pair of biorthogonal MRA's. But this is an immediate consequence of Corollary 4.17. Note that one may choose the function b highly discontinuous, so that $\{\psi, \tilde{\psi}\}$ cannot be tested with Corollary 4.16. $\square$

6 References

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