A SUFFICIENT CONDITION FOR HAAR MULTIPLIERS IN TRIEBEL-LIZORKIN SPACES

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In memory of Guido Weiss

ABSTRACT. We consider Haar multiplier operators T_m acting on Sobolev spaces, and more generally Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R})$, for indices in which the Haar system is not unconditional. When m depends only on the Haar frequency, we give a sufficient condition for the boundedness of T_m in $F^s_{p,q}$, in terms of the variation norms $\|m\|_{V_u}$, which is optimal in u (up to endpoints) when $p, q > 1$.

1. Introduction

Consider the classical Haar system in $\mathbb{R}$,

$$\mathscr{H} = \big\{h_{j,\mu} \;:\; j \geq -1,\, \mu \in \mathbb{Z}\big\}, \tag{1.1}$$

where, if $h = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$, we let

$$h_{j,\mu}(x) = h(2^j x - \mu)\,, \quad \text{for} \quad j = 0, 1, 2, \dots, \;\; \mu \in \mathbb{Z},$$

while for $j = -1$ we let

$$h_{-1,\mu} = \mathbb{1}_{[\mu,\mu+1)}, \quad \mu \in \mathbb{Z}.$$

We shall refer to the elements of the family $\mathscr{H}_j = \{h_{j,\mu} : \mu \in \mathbb{Z}\}$ as *Haar functions of frequency* 2^j.

Let $F^s_{p,q}$ denote the usual Triebel-Lizorkin space in $\mathbb{R}$; see [16]. It is known from the work of Triebel [17, Theorem 2.9.ii] that $\mathscr{H}$ is an unconditional basis of $F^s_{p,q}(\mathbb{R})$ when s belongs to the range

$$\max\Big\{1/p - 1, 1/q - 1\Big\} < s < \min\Big\{1/p, 1/q, 1\Big\}. \tag{1.2}$$

That this range is actually optimal was shown by the last two authors in [12, 13]. More recently, we proved in [3] that $\mathscr{H}$ is a Schauder basis of $F^s_{p,q}(\mathbb{R})$ (with respect to natural enumerations) in the larger range

$$1/p - 1 < s < \min\Big\{1/p, 1\Big\}, \quad (\text{for all } 0 < q < \infty), \tag{1.3}$$

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while at the endpoints (see [5]) the property holds if and only if

$$s = 1/p - 1 \quad \text{and} \quad 1/2 < p \leq 1, \tag{1.4}$$

also for all $0 < q < \infty$. These regions are depicted in Figure 1 below.

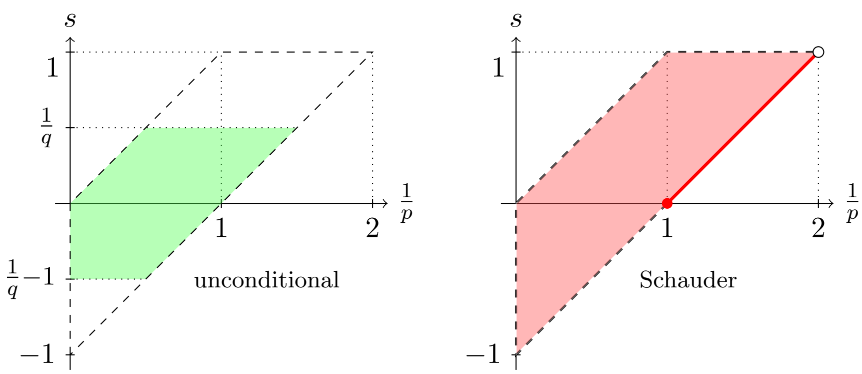

FIGURE 1. Parameter domain for $\mathscr{H}$ to be an unconditional basis (left figure) or a Schauder basis (right figure) in $F^s_{p,q}(\mathbb{R})$.

We shall mainly be interested in values of the parameters outside the region of unconditionality. In that range, it becomes a natural question to find sufficient conditions on a sequence $\{m_{j,\mu}\}$ so that the mapping

$$f \longmapsto \sum_{j\geq 0} \sum_{\mu\in\mathbb{Z}} m_{j,\mu}\, 2^j \langle f, h_{j,\mu}\rangle h_{j,\mu},$$

defined say for $f \in \operatorname{span} \mathscr{H}$, extends as a bounded linear operator in the space $F^s_{p,q}$.

In this paper we regard this problem in the special case when the sequence is constant in each frequency level, namely, if $m = \{m(j)\}_{j\geq 0}$, we consider the operators

$$T_m f = \sum_{j\geq 0} m(j)\, \mathbb{D}_j f,$$

where $\mathbb{D}_j$ denotes the orthogonal projection onto the space generated by $\mathscr{H}_j$, that is

$$\mathbb{D}_j f = \sum_{\mu\in\mathbb{Z}} 2^j \langle f, h_{j,\mu}\rangle\, h_{j,\mu}, \quad j \geq 0.$$

It is well known that one can write

$$\mathbb{D}_j = \mathbb{E}_{j+1} - \mathbb{E}_j,$$

where $\mathbb{E}_j$ is the conditional expectation operator defined by

$$\mathbb{E}_j f(x) = \sum_{\mu\in\mathbb{Z}} \mathbb{1}_{I_{j,\mu}}(x)\, 2^j \int_{I_{j,\mu}} f(y)dy\,, \tag{1.5}$$

associated with the dyadic intervals $I_{j,\mu} = [\mu 2^{-j}, (\mu+1)2^{-j})$, $\mu \in \mathbb{Z}$.

The uniform boundedness of the operators $\mathbb{E}_N$ in $F^s_{p,q}$ (and $B^s_{p,q}$) has been throughly studied in the papers [3, 4, 5]. In particular, it is shown in those papers that $\mathscr{H}$ is a Schauder basis of $F^s_{p,q}$ (with respect to natural enumerations) if and only if

$$\sup_{N\geq 0} \big\|\mathbb{E}_N\big\|_{F^s_{p,q}\to F^s_{p,q}} < \infty \quad \text{and} \quad \operatorname{span}\mathscr{H} \text{ is dense in } F^s_{p,q},$$

and this in turn is equivalent to (s,p,q) belonging to the ranges in (1.3) and (1.4). In those cases, an elementary summation by parts argument and the σ-triangle inequality, with $\sigma = \min\{1,p,q\}$, imply that

$$\|T_m f\|_{F^s_{p,q}} \lesssim \|m\|_{\ell^\infty} + \Big[\sum_{j=1}^{\infty} |m(j) - m(j-1)|^\sigma\Big]^{\frac{1}{\sigma}}, \tag{1.6}$$

for all $f \in \operatorname{span}\mathscr{H}$ with $\|f\|_{F^s_{p,q}} \leq 1$.

We shall next formulate a stronger multiplier result which involves the Wiener space notion of sequences of bounded u-variation. We recall how these are defined. If $u \geq 1$, we let $\mathcal{V}_u(m)$ be the u-variation of the sequence $\{m(j)\}_{j\geq 0}$, defined by

$$\mathcal{V}_u(m) = \sup\Big(\sum_{n=1}^{N} |m(j_n) - m(j_{n-1})|^u\Big)^{1/u}$$

with the supremum taken over all finite strings of numbers $\{j_0,\dots,j_N\}$ satisfying $j_{n-1} < j_n$ for $1 \leq n \leq N$, and $j_n \in \mathbb{N}\cup\{0\}$. Note that if $u = 1$ we simply have

$$\mathcal{V}_1(m) = \sum_{j=1}^{\infty} |m(j) - m(j-1)|.$$

We denote by V_u the space of all $m : \mathbb{N}\cup\{0\} \to \mathbb{C}$ for which

$$\|m\|_{V_u} := \|m\|_\infty + \mathcal{V}_u(m) < \infty.$$

In particular, if $1 \leq u_1 \leq u_2 < \infty$, it holds

$$V_1 \hookrightarrow V_{u_1} \hookrightarrow V_{u_2} \hookrightarrow \ell^\infty,$$

As an example, observe that $m_\alpha(n) = 1/(n+1)^\alpha$ belongs to V_1 for all $\alpha > 0$, while the alternate sequence $(-1)^n m_\alpha(n)$ belongs to V_u iff $\alpha > 1/u$.

We wish to find, in the region of exponents (s,p,q) where $\mathscr{H}$ is a conditional basis of $F^s_{p,q}$, the largest possible u for which $m \in V_u$ implies the boundedness of the operator T_m in $F^s_{p,q}$. The examples given in [12], based on multipliers taking the values 0 and 1 (suitable characteristic functions of finite sets of integers), show that for $1/u < s - 1/q$ there are $m \in V_u$ such that the corresponding operators T_m are unbounded on $F^s_{p,q}$; see also §3.3 below. Our main result in this note shows that, in the case $1 < p,q < \infty$, boundedness holds in the complementary range, except perhaps at the endpoint.

Theorem 1.1. *Let $1 < p < q < \infty$, $1/q \le s < 1/p$ and $u \ge 1$. Then*

$$\|T_m\|_{F^s_{p,q} \to F^s_{p,q}} + \|T_m\|_{F^{-s}_{p'q'} \to F^{-s}_{p'q'}} \le C\|m\|_{V_u}, \quad \textit{if} \quad \frac{1}{u} > s - \frac{1}{q}.$$

Remark 1.2. The appearance of the variation norms is inspired by a result of Coifman, Rubio de Francia and Semmes [2] on Fourier multipliers (which is based on the square function result of Rubio de Francia [8], see also [14, 7]). However the variation spaces come up in quite different ways in [2] where the variation norm is taken over dyadic intervals $[2^j, 2^{j+1})$, with a bound uniformly in j. This has no analogue in our situation as for each interval $[2^j, 2^{j+1})$ there is only one Haar frequency; instead our conditions involve the variation norms in the parameter j.

2. Subspaces of V_u

As in [2], in order to analyze functions in V_u it is convenient to consider certain subspaces R_u of V_u built on convex combinations of characteristic functions of unions of disjoint intervals. This is sketched in [2], but for the convenience of the reader we give a detailed exposition in the setting of variation spaces for functions on the integers.

For $1 \le u < \infty$, let r_u be the class of functions $g : \mathbb{N}_0 \to \mathbb{C}$ which are of the form

$$g = \sum_\nu a_\nu \chi_{I_\nu}, \quad \text{with} \quad (\sum_\nu |a_\nu|^u)^{1/u} \le 1,$$

where the I_ν are mutually disjoint intervals. Then R_u is the space of all sequences of the form

$$m = \sum_l c_l g_l, \quad \text{with} \quad g_l \in r_u \quad \text{and} \quad \sum |c_l| < \infty. \tag{2.1}$$

The norm $\|m\|_{R_u}$ is defined as the infimum of $\sum_l |c_l|$ over all representations as in (2.1). These definitions (for functions on the real line) can be found in [2]. The following result is a discrete analogue of [2, Lemme 2], whose proof is sketched for completeness.

Proposition 2.1. *For $\varepsilon > 0$, and $1 \le u < \infty$ we have*

$$R_u \subset V_u \subset R_{u+\varepsilon} \tag{2.2}$$

with continuous embedding.

Proof. Consider first $g \in r_u$, with $g = \sum_\nu a_\nu \chi_{I_\nu}$. It is straightforward to see that $\mathcal{V}_u(g) \le 2\|a\|_{\ell^u}$, thus $R_u \subset V_u$.

For the second inclusion assume that $f \in V_u$, with

$$\mathcal{V}_u(f) = 1,$$

for some $u < \infty$. This implies that $\lim_{n\to\infty} f(n)$ exists and is finite.

Let $w(0) = 0$, and for $n \geq 1$, let $w(n)$ be the u-th power of the u-variation of f over $[0, n]$, that is

$$w(n) = \sup_{0 \leq n_0 < n_1 < \cdots < n_N \leq n} \sum_{i=1}^{N} |f(n_i) - f(n_{i-1})|^u.$$

Clearly w is positive, increasing and $\mathcal{V}_u(f) = \lim_{n \to \infty} [w(n)]^{1/u}$ so that w takes values in $[0, 1]$. There are two situations (i) $w(n) < 1$ for all $n \in \mathbb{N}$, and (ii) $w(n) = 1$ for $n \geq N_0$ (and some N_0). In what follows we assume (i) and omit the minor modification for (ii) (in the second case one works with finite sequences instead of infinite sequences).

As stated in [1, Theorem 2] and used in [2] one can express

$$f = \rho \circ w, \quad \text{where } \rho \in C^{1/u}[0, 1].$$

To verify this we may choose a strictly increasing sequence of nonnegative integers $\{j_n\}_{n=0}^{\infty}$ so that $j_0 = 0$ and

$$w(j_n) < w(j_{n+1}), \quad w(j_n) = w(k) \quad \text{for } j_n \leq k < j_{n+1}.$$

This implies $f(k) = f(j_n)$ for $j_n \leq k < j_{n+1}$. We now define a piecewise linear function $\rho(t)$, $0 \leq t < 1$, as follows

$$\rho(t) = f(j_n) + \frac{f(j_{n+1}) - f(j_n)}{w(j_{n+1}) - w(j_n)} (t - w(j_n)), \quad w(j_n) \leq t < w(j_{n+1}).$$

So $\rho(0) = f(0)$, and if we let $\rho(1) = \lim_{n \to \infty} f(n)$, then ρ is continuous in $[0, 1]$. Observe also that $\rho \circ w = f$, since for $j_n \leq k < j_{n+1}$ we have

$$\rho(w(k)) = \rho(w(j_n)) = f(j_n) = f(k).$$

We now show the Hölder condition

$$|\rho(t) - \rho(t')| \leq 3|t - t'|^{1/u}, \quad 0 \leq t' < t \leq 1. \tag{2.3}$$

By the definitions of ρ and w we have for $n' < n$

$$|\rho(w(j_n)) - \rho(w(j_{n'}))| = |f(j_n) - f(j_{n'})| \leq (w(j_n) - w(j_{n'}))^{1/u}. \tag{2.4a}$$

For $w(j_n) \leq t < w(j_{n+1})$

$$\begin{aligned} |\rho(t) - \rho(w(j_n))| &= \Big| \frac{f(j_{n+1}) - f(j_n)}{w(j_{n+1}) - w(j_n)} \Big| |t - w(j_n)| \\ &\leq \big| w(j_{n+1}) - w(j_n) \big|^{-1+\frac{1}{u}} |t - w(j_n)| \leq |t - w(j_n)|^{1/u}, \end{aligned} \tag{2.4b}$$

since $u \geq 1$. Similarly

$$\begin{aligned} |\rho(w(j_{n+1})) - \rho(t)| &= \Big| \frac{f(j_{n+1}) - f(j_n)}{w(j_{n+1}) - w(j_n)} \Big| |w(j_{n+1}) - t| \\ &\leq \big| w(j_{n+1}) - w(j_n) \big|^{-1+\frac{1}{u}} |w(j_{n+1}) - t| \leq |w(j_{n+1}) - t|^{1/u}. \end{aligned} \tag{2.4c}$$

Combining the three cases we obtain (2.3) for all $t, t' \in [0, 1)$, and by continuity the result also holds true on $[0, 1]$.

We now use the expansion of ρ in terms of the Haar system in $[0,1]$, that is $\{\mathbb{1}_{[0,1)}, h_{j,\mu}\}$ with $j \ge 0$ and $0 \le \mu < 2^j$. Here

$$h_{j,\mu} = \mathbb{1}_{I^{\text{left}}_{j,\mu}} - \mathbb{1}_{I^{\text{right}}_{j,\mu}}$$

with $I^{\text{left}}_{j,\mu}$ and $I^{\text{right}}_{j,\mu}$ the left and right halves of $I_{j,\mu} = [2^{-j}\mu, 2^{-j}(\mu+1))$. Then

$$\rho(t) - \int_0^1 \rho(s)ds = \sum_{j=0}^{\infty} \rho_j(t)$$

where

$$\rho_j(t) = \sum_{\mu=0}^{2^j-1} 2^j \langle h_{j,\mu}, \rho\rangle h_{j,\mu}(t) = \rho_{j,1}(t) - \rho_{j,2}(t)$$

with

$$\rho_{j,1}(t) = \sum_{\mu=0}^{2^j-1} 2^j \langle h_{j,\mu}, \rho\rangle \mathbb{1}_{I^{\text{left}}_{j,\mu}}(t), \quad \text{and} \quad \rho_{j,2}(t) = \sum_{\mu=0}^{2^j-1} 2^j \langle h_{j,\mu}, \rho\rangle \mathbb{1}_{I^{\text{right}}_{j,\mu}}(t).$$

Now

$$|2^j \langle h_{j,\mu}, \rho\rangle| = 2^j \Big| \int h_{j,\mu}(t)\big[\rho(t) - \rho\big(2^{-j}(\mu+\tfrac{1}{2})\big)\big]dt\Big| \le 3\cdot 2^{-1/u}\, 2^{-j/u},$$

by (2.3). Thus, if $\varepsilon > 0$ we have

$$\Big(\sum_{\mu=0}^{2^j-1} |2^j \langle h_{j,\mu}, \rho\rangle|^{u+\varepsilon}\Big)^{\frac{1}{u+\varepsilon}} \le 3\cdot 2^{-\frac{1}{u}}\, 2^{-\frac{j}{u}}\, 2^{\frac{j}{u+\varepsilon}} =: c_{j,\varepsilon}.$$

Since w is increasing it is clear that the functions

$$n \mapsto \mathbb{1}_{I^{\text{left}}_{j,\mu}}\big(w(n)\big), \quad n \mapsto \mathbb{1}_{I^{\text{right}}_{j,\mu}}\big(w(n)\big)$$

are characteristic functions of intervals restricted to the integers. For fixed j these intervals are also mutually disjoint, so we see that

$$g_{j,\varepsilon} := \frac{1}{2c_{j,\varepsilon}}\,\big(\rho_j \circ w\big) \in r_{u+\varepsilon}.$$

Since $C_\varepsilon := 2\sum_{j\ge 0} |c_{j,\varepsilon}| < \infty$, it then follows that

$$f = \rho\circ w = \int_0^1 \rho + \sum_{j=0}^{\infty} 2c_{j,\varepsilon}\, g_{j,\varepsilon} \in R_{u+\varepsilon}, \quad \text{with } \|f\|_{R_{u+\varepsilon}} \le 1 + C_\varepsilon.$$

□

Remark 2.2. Observe that the previous proof actually shows that, if $m \in V_u$ and $\varepsilon > 0$, then one can write $m = \sum_{j=0}^{\infty} c_j m_j$ with $m_j \in r_{u+\varepsilon}$ and $\sum_{j=1}^{\infty} |c_j|^\sigma < \infty$, for all $\sigma > 0$.

3. *The proof of Theorem 1.1*

We shall actually prove a stronger result than Theorem 1.1, which provides optimal boundedness (up to endpoints) in a slightly larger region of indices. Given a fixed $q > 1$, we denote by $\mathbf{T}_q$ the open triangle in the plane $(1/p, s)$ with vertices $(1, 1), (1/q, 1/q)$ and $(1 + 1/q, 1/q)$; see Figure 2.

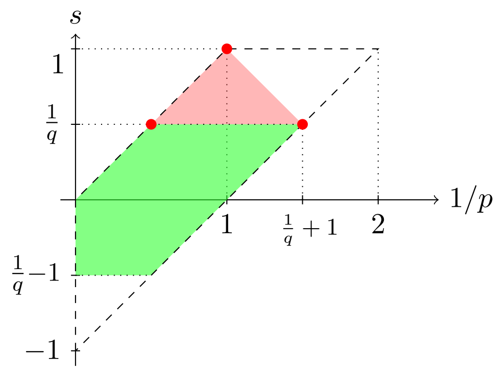

FIGURE 2. In red, the region $\mathbf{T}_q$, for $q > 1$.

For this region we give the following result, which includes Theorem 1.1 as a special case.

Theorem 3.1. *Let $1 < q < \infty$ and $(1/p, s) \in \mathbf{T}_q$. Then, for all $u \geq 1$ such that $1/u > s - 1/q$, it holds*

$$\|T_m f\|_{F^s_{p,q} \to F^s_{p,q}} \leq c\, \|m\|_{V_u}, \quad m \in V_u. \tag{3.1}$$

Moreover, a necessary condition for (3.1) to hold for all such m is that $1/u \geq s - 1/q$.

In view of Proposition 2.1, we shall first consider sequences from the class r_u, that is multipliers m of the form

$$m[a, \mathcal{I}] = \sum_\nu a_\nu \mathbb{1}_{I_\nu}, \tag{3.2}$$

where $\mathcal{I} = \{I_\nu\}$ is a family of disjoint intervals and $a = \{a_\nu\}$ a sequence in ℓ^u. For these multipliers we have the following result.

Proposition 3.2. *Let $1 < q < \infty$ and $(1/p, s) \in \mathbf{T}_q$. Then, for all $u \geq 1$ such that $1/u > s - 1/q$, there exists $c = c(p, q, s, u) > 0$ such that*

$$\|T_{m[a,\mathcal{I}]} f\|_{F^s_{p,q}} \leq c\, \|a\|_{\ell^u} \|f\|_{F^s_{p,q}}, \quad \forall\, f \in F^s_{p,q}(\mathbb{R}),\ a \in \ell^u, \tag{3.3}$$

for every multiplier $m[a, \mathcal{I}]$ defined as in (3.2). Moreover, a necessary condition for (3.3) to hold for all such $m[a, \mathcal{I}]$ is that $1/u \geq s - 1/q$.

Remark 3.3. We emphasize that the constant c in (3.3) does not depend on the family of disjoint intervals $\mathcal{I} = \{I_\nu\}$.

In the next subsections we shall prove Proposition 3.2. For the sufficiency part we shall use complex interpolation applied to the bilinear operator

$$(a, f) \longmapsto \mathcal{T}[a, f] := T_{m[a,\mathcal{I}]} f.$$

For simplicity we shall remove the dependence on $\mathcal{I}$ in the subsequent notation, as it will be clear from the proofs that the involved constants do not depend on it.

3.1. *Interpolation with varying q and $p = 1$.* We first prove an inequality for $p = 1$ and a fixed s near 1, namely

$$\|\mathcal{T}[a, f]\|_{F^s_{1,q}} \lesssim \|a\|_{\ell^u} \|f\|_{F^s_{1,q}}, \quad s < 1, \quad 1/u > 1 - 1/q. \tag{3.4}$$

Since the Haar system is an unconditional basis on $F^s_{1,1} = B^s_{1,1}$, $0 < s < 1$ (see Theorem 2.9 in [17]) we have

$$\|\mathcal{T}[a, f]\|_{F^s_{1,1}} \lesssim \|a\|_{\ell^\infty} \|f\|_{F^s_{1,1}}, \quad 0 < s < 1.$$

Next, the uniform boundedness of the operators $\mathbb{E}_N$ in F^s_{1,q_1} (see [3, Corollary 1.3]) and the trivial estimate in (1.6) (with $\sigma = 1$) imply that for any $q_1 \in (q, \infty)$

$$\|\mathcal{T}[a, f]\|_{F^s_{1,q_1}} \lesssim \|a\|_1 \|f\|_{F^s_{1,q_1}}, \quad 0 < s < 1.$$

By complex interpolation [6] we then obtain (3.4) for $1/u = (1 - 1/q)/(1 - 1/q_1)$, which after choosing q_1 large enough implies (3.4) whenever $1/u > 1 - 1/q$.

3.2. *Interpolation with fixed q.* Let $q \in (1, \infty)$ be fixed, and let $(1/p, s) \in \mathbf{T}_q$. We shall prove (3.3) by interpolating sufficiently close to the upper vertex $(1, 1)$ and the lower segment $(1/p_1, 1/q)$ of $\mathbf{T}_q$.

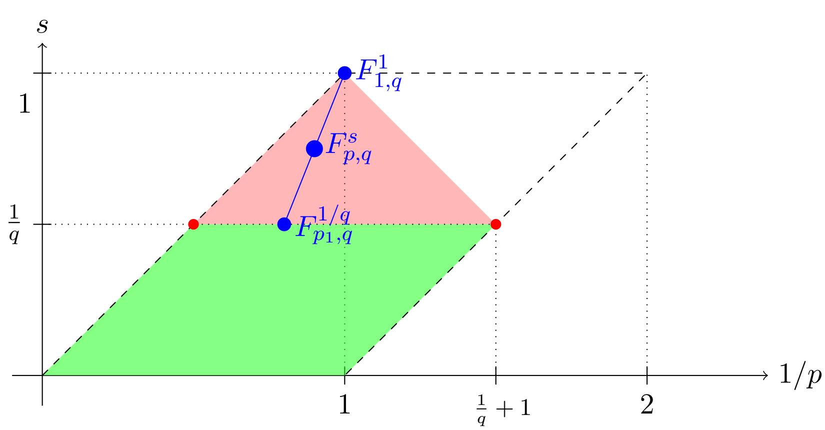

FIGURE 3. Interpolation strategy for points $(1/p, s) \in \mathbf{T}_q$.

Let $P = (1/p, s) \in \mathbf{T}_q$ and $\varepsilon_1 > \varepsilon_0 > 0$ be sufficiently small, to be chosen. Let $P_0 = (1, s_0)$ with $s_0 = 1 - \varepsilon_0$. Draw a line through P_0 and P, and let

$P_1 = (1/p_1, s_1)$ be the intersection with the horizontal line $s_1 = 1/q - \varepsilon_1$. That is,

$$P = (1-\theta)\cdot P_0 + \theta \cdot P_1, \quad \text{with} \quad \theta = \frac{s_0 - s}{s_0 - s_1}.$$

Choosing $\varepsilon_0, \varepsilon_1$ sufficiently small we can guarantee that $s_1 < s < s_0$ (and hence $\theta \in (0,1)$), and that P_1 lies in the green region. Next, take

$$\frac{1}{u_0} := 1 - \frac{1}{q} + \varepsilon_1 - \varepsilon_0 > 1 - \frac{1}{q}.$$

From the previous step and the unconditional basis property we have

$$\mathcal{T} : \ell^{u_0} \times F^{s_0}_{1,q} \to F^{s_0}_{1,q} \quad \text{and} \quad \mathcal{T} : \ell^{\infty} \times F^{s_1}_{p_1,q} \to F^{s_1}_{p_1,q}.$$

Using complex interpolation this yields

$$\|\mathcal{T}[a,f]\|_{F^s_{p,q}} \lesssim \|a\|_{\ell^u} \|f\|_{F^s_{1,q}}, \tag{3.5}$$

with

$$\frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{\infty} = \frac{s - s_1}{s_0 - s_1} \cdot \frac{1}{u_0}.$$

Since the above definitions give $s_0 - s_1 = 1/u_0$, then we have

$$\frac{1}{u} = s - s_1 = s - \frac{1}{q} + \varepsilon_1.$$

Letting $\varepsilon_1 \searrow 0$ we deduce the validity of (3.5) whenever $\frac{1}{u} > s - \frac{1}{q}$. This completes the proof of the sufficient condition in Proposition 3.2.

3.3. *Necessary condition.* Suppose first that $1 < p < q$ with $1/q < s < 1/p$. Then, the example constructed in [12, §5] gives a multiplier of the form $m = \mathbb{1}_E$, so that $\operatorname{card} E = 2^N$ (with the elements in E being N-separated), and with the property that

$$\|T_m\|_{F^s_{p,q} \to F^s_{p,q}} \gtrsim 2^{N(s - \frac{1}{q})}. \tag{3.6}$$

Since we can write m in the form (3.2) (with $I_\nu = \{\nu\}$ and $a_\nu = 1$, for $\nu \in E$), then the validity of (3.3) will imply that

$$2^{N(s-\frac{1}{q})} \lesssim \|T_m\|_{F^s_{p,q} \to F^s_{p,q}} \lesssim \|a\|_{\ell^u} = 2^{N/u}.$$

Thus, we must necessarily have $1/u \geq s - 1/q$.

Arguing by interpolation as in [12, §7] one can show that (3.6) (with an ε loss) continues to hold for all $(1/p, s)$ with

$$\max\{1/q, 1/p - 1\} < s < \min\{1/p, 1\} \tag{3.7}$$

which is a larger region than $\mathbf{T}_q$; see Figure 2.

To be more precise, let $P_1 = (1/p, s)$ belong to the open quadrilateral defined by (3.7), where we assume $p \leq 1$. We shall interpolate close to the points shown in Figure 4.

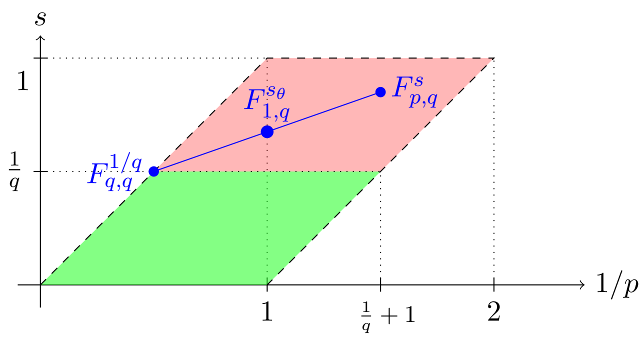

FIGURE 4

Namely, given $\varepsilon > 0$, let $P_0 = (\frac{1}{q}, \frac{1}{q} - \varepsilon)$. Draw a segment from P_0 to P_1, and consider the convex combination of P_0 and P_1 with first coordinate $(1+\varepsilon)^{-1}$; i.e let $\theta \in (0,1)$ and s_θ be such that

$$\big(\tfrac{1}{1+\varepsilon}, s_\theta\big) = (1-\theta)\,\big(\tfrac{1}{q}, \tfrac{1}{q} - \varepsilon\big) \,+\, \theta\,\big(\tfrac{1}{p}, s\big). \tag{3.8}$$

Then, by complex interpolation [6] we have

$$\|T_m\|_{F^{s_\theta}_{1+\varepsilon,q}} \lesssim \|T_m\|^{1-\theta}_{F^{\frac{1}{q}-\varepsilon}_{q,q}} \|T_m\|^{\theta}_{F^s_{p,q}}$$

By unconditionality, $\|T_m\|_{F^{\frac{1}{q}-\varepsilon}_{q,q}} \lesssim 1$, so we arrive at

$$\|T_m\|_{F^s_{p,q}} \gtrsim \|T_m\|^{1/\theta}_{F^{s_\theta}_{1+\varepsilon,q}} \gtrsim 2^{\frac{N}{\theta}\,(s_\theta - \frac{1}{q})},$$

the last bound due to (3.6). Now, solving for s_θ in (3.8) we see that

$$s_\theta - \frac{1}{q} = \big(s - \frac{1}{q}\big)\theta - (1-\theta)\varepsilon.$$

Thus,

$$\|T_m\|_{F^s_{p,q}} \gtrsim 2^{N\,[(s-\frac{1}{q}) - \frac{1-\theta}{\theta}\,\varepsilon]}.$$

So, if (3.3) was true, arguing as above we would arrive at

$$\frac{1}{u} \geq (s - \frac{1}{q}) - \frac{1-\theta}{\theta}\,\varepsilon,$$

which letting $\varepsilon \searrow 0$ leads to $\frac{1}{u} \geq s - \frac{1}{q}$.

3.4. *Conclusion of the proof of Theorem 3.1.* Let $q > 1$ and let $(1/p, s) \in \mathbf{T}_q$ be fixed. Let $1/u > s - 1/q$ and $m \in V_u$ with $u \geq 1$. Then, for some $u_1 > u$ we also have $1/u_1 > s - 1/q$. By Remark 2.2 we can write $m = \sum_{j=0}^\infty c_j m_j$ with $m_j \in r_{u_1}$ and $\sum_{j=0}^\infty |c_j|^\sigma \lesssim \|m\|^\sigma_{V_u}$, with $\sigma = \min\{1, p\}$. Then, using the σ-triangle inequality, we have

$$\|T_m f\|^\sigma_{F^s_{p,q}} \leq \sum_{j=0}^\infty |c_j|^\sigma \, \|T_{m_j} f\|^\sigma_{F^s_{p,q}}, \quad f \in F^s_{p,q}.$$

By Proposition 3.2, $\|T_{m_j} f\|_{F^s_{p,q}} \lesssim \|f\|_{F^s_{p,q}}$, for all $j \geq 0$, so we conclude that

$$\|T_m\|_{F^s_{p,q} \to F^s_{p,q}} \lesssim \|m\|_{V_u}.$$

□

Remark 3.4. When $p > 1$, the assertion

$$\|T_m\|_{F^{-s}_{p',q'} \to F^{-s}_{p',q'}} \lesssim \|m\|_{V_u}$$

stated in Theorem 1.1 follows from (3.1) by duality. So, when $q > 1$, the condition on V_u is optimal (up to endpoints) also in the lower triangle on the left of Figure 1.

Remark 3.5. When $1/2 < p \leq 1$, we did not state any result for the right upper triangle in Figure 2. It is also possible to obtain, by complex interpolation, a sufficient condition for multipliers of the form $m[a, \mathcal{I}]$ in terms of $\|a\|_{\ell^u}$, although in this range the value of u will no longer match the necessary condition from §3.3.

Remark 3.6. When $1/2 < q \leq 1$, one can also prove by interpolation, for multipliers of the form $m[a, \mathcal{I}]$, that

$$\|a\|_{\ell^u} < \infty, \quad \frac{1}{u} > \frac{1}{q} - 1 - s,$$

is a sufficient condition in the open triangle with vertices $(0, -1)$, $(1/q, 1/q - 1)$ and $(1/q - 1, 1/q - 1)$; see Figure 5 below. This matches the necessary condition from the examples in [12, §5] (except for the endpoint). In the remaining part of the figure, however, the sufficient condition obtained by interpolation will be weaker than this one.

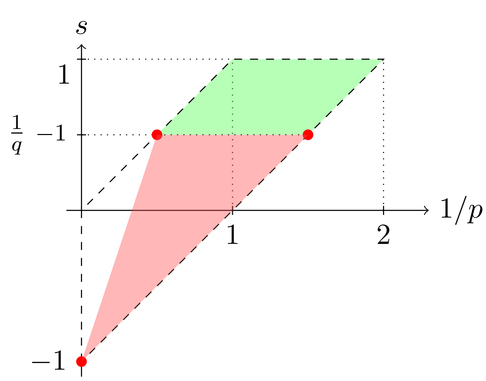

FIGURE 5. Parameter domain for the cases $1/2 < q < 1$.

Remark 3.7. It may be interesting to note that even for the special case of Sobolev spaces $H^s_p = F^s_{p,2}$, $1 < p < 2$, $1/2 \leq s < 1/p$ and $u < \frac{2}{2s-1}$ the use of Triebel-Lizorkin spaces $F^s_{p,q}$ with $q \neq 2$ is crucial. Such interpolation arguments were used for multiplier transformations in other contexts to establish endpoint results on Lorentz spaces $L_{p,2}$, see [9, 10] for basic versions, and [11, 15] for more advanced versions.

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