Basic properties of wavelets

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Dedicated to Eugene Fabes

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Abstract

A wavelet multiplier is a function whose product with the Fourier transform of a wavelet is the Fourier transform of a wavelet. We characterize the wavelet multipliers, as well as the scaling function multipliers and low pass filter multipliers. We then prove that, if the set of all wavelet multipliers acts on the set of all MRA wavelets, the orbits are the sets of all MRA wavelets whose Fourier transforms have equal absolute values, and these are also equal to the sets of all MRA wavelets with the corresponding scaling functions having same absolute values of their Fourier transforms. As an application of these techniques we prove that the set of MRA wavelets is arcwise connected in $L^2(\mathbf{R})$.

Foreword

Two groups of collaborators, one led by Xingde Dai and David Larson and the other led by Eugenio Hernández and Guido Weiss, became aware that they were involved in the study of wavelets and obtained results that had much in common, even though different methods were employed. Early in 1996 we decided to exchange our ideas by holding meetings attended by us, our collaborators and students and by correspondence. This paper is what we hope is the first of a series of reports describing our joint results. Fourteen researchers are involved in this work. A few shorter notes describe more technical aspects of the work performed by different subgroups of this collection of fourteen researchers. At the end of this article, we shall describe the group that we call the "WUTAM CONSORTIUM". This is not the name of a Chinese city; it is the acronym for **Washington University, Texas A & M**, the two institutions associated with each of us.

1 Introduction

We shall begin by examining the most basic facts about orthonormal wavelets on the real line $\mathbf{R}$ and show that there is a partition of this set that is particularly useful for understanding wavelets and their properties. In particular, we will show that the wavelets, as a subset of the unit sphere in $L^2(\mathbf{R})$ have some unexpected connectivity properties; moreover, we will obtain several other properties of wavelets. In order to describe precisely the results we obtain, we must present appropriate definitions, notation and the facts that we assume the reader knows.

By an *orthonormal wavelet* (or, simply, a *wavelet*) we mean a function $\psi \in L^2(\mathbf{R})$ such that the system $\{\psi_{j,k}(x)\} = \{2^{j/2}\psi(2^j x - k)\}$, where j and k range through the integers $\mathbf{Z}$, forms an orthonormal basis of $L^2(\mathbf{R})$. There are two equations, involving the Fourier transform, $\hat{\psi}$, that completely characterize all wavelets. Let us agree to choose the following definition of the Fourier transform:

$$\hat{\psi}(\xi) = \int_{\mathbf{R}} \psi(x) e^{-i\xi x} dx$$

considered as a Lebesgue integral when $\psi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ and as a limit in the mean in the general case (see [11]). The characterization of wavelets is given by the following:

Proposition 1.1 *$\psi \in L^2(\mathbf{R})$ is an orthonormal wavelet if and only if $\|\psi\|_2 \geq 1$ and the following are true:*

(i) $\quad \sum_{j\in\mathbf{Z}} |\hat{\psi}(2^j\xi)|^2 = 1$ a.e.;

(ii) $\quad t_q(\xi) \equiv \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + 2q\pi))} = 0$ a.e.

when q is an odd integer.

For a proof of this Proposition see chapter 7 of [6] where one can also find the appropriate references to G. Gripenberg and X. Wang who discovered this result independently.

These two equations can be used for constructing wavelets (and we shall come back to this later on); however, there is an elegant method that is often used to construct a large class of wavelets known as the *Multiresolution Analysis Method.* A *Multiresolution Analysis*, or, simply, an **MRA**, is a sequence $\mathcal{M} = \{V_j\}$, $j \in \mathbf{Z}$, of closed subspaces of $L^2(\mathbf{R})$ satisfying

(1.2) $\qquad V_j \subseteq V_{j+1} \quad \text{for all} \quad j \in \mathbf{Z}$

(1.3) $\qquad f \in V_j \ \text{ if and only if } \ f(2(\cdot)) \in V_{j+1} \quad \text{for all} \quad j \in \mathbf{Z}$

(1.4) $\qquad \overline{\cup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$

(1.5) There exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}$, $k \in \mathbf{Z}$, is an orthonormal basis for V_0.

It is not hard to show that properties (1.2), (1.3) and (1.5) imply

(1.6) $\qquad \cap_{j\in\mathbf{Z}} V_j = \{0\}$

It follows readily from these properties that the spaces $W_j := V_{j+1} \cap V_j^{\perp}$, $j \in \mathbf{Z}$, are mutually orthogonal and their (orthogonal) direct sum equals $L^2(\mathbf{R})$:

$$L^2(\mathbf{R}) = \bigoplus_{j\in\mathbf{Z}} W_j \ . \tag{1}$$

It is clear from the above that a function $\psi \in W_0$ such that $\{\psi(\cdot - k)\}$, $k \in \mathbf{Z}$, is an orthonormal basis for W_0 is an orthonormal wavelet. The collection of all such wavelets will be referred to as the *set of all wavelets associated with the* **MRA** $\mathcal{M} = \{V_j\}$. We shall also say that an orthonormal wavelet ψ *is an* MRA *wavelet* if it is so associated with an MRA $\mathcal{M} = \{V_j\}$.

It is necessary that we review the well known explicit construction of such an MRA wavelet; this will allow us to make some observations that do not appear in the literature that form the basis for many of the results we will present. Given an MRA $\mathcal{M} = \{V_j\}$, let $\varphi \in V_0$ be a function satisfying (1.5). Such a function is called a *scaling function* associated with $\mathcal{M}$. Since $\frac{1}{2}\varphi(\frac{\cdot}{2}) \in V_{-1} \subseteq V_0$ (from (1.2), (1.3) and (1.5)) we can express this function in terms of the orthonormal basis $\{\varphi(\cdot + k)\}$, $k \in \mathbf{Z}$:

$$\frac{1}{2}\varphi\left(\frac{x}{2}\right) = \sum_{k\in\mathbf{Z}} \alpha_k \varphi(x+k) \ , \tag{2}$$

where

$$\alpha_k = \frac{1}{2}\int_{\mathbf{R}} \varphi(\frac{x}{2})\overline{\varphi(x+k)}dx$$

with $\sum_{k\in\mathbf{Z}} |\alpha_k|^2 < \infty$ and the convergence of the series in (2) is in the norm of $L^2(\mathbf{R})$. The Plancherel theorem shows that we can take the Fourier transform of both sides of equality (2) and obtain

$$\hat{\varphi}(2\xi) = \hat{\varphi}(\xi)\sum_{k\in\mathbf{Z}} \alpha_k e^{ik\xi} \equiv \hat{\varphi}(\xi)m(\xi) \ \ , \tag{3}$$

where

$$m(\xi) = \sum_{k\in\mathbf{Z}} \alpha_k e^{ik\xi}$$

is a 2π periodic function in $L^2(\mathbf{T}) = L^2([-\pi,\pi))$. This function m is uniquely determined by the scaling function φ via equation (3) and is called the *low pass filter* associated with φ. One can then show (see chapter 2 of [6]) that $\psi \in L^2(\mathbf{R})$ satisfying

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{m\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right) \tag{4}$$

is an orthonormal wavelet associated with $\mathcal{M}$. More generally, if

$$\hat{\psi}(\xi) = s(\xi)e^{i\xi/2}\overline{m\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right) \tag{5}$$

with s a *unimodular* (that is, $|s(\xi)| = 1$ a.e.) 2π periodic measurable function, then ψ is an orthonormal wavelet associated with $\mathcal{M}$. In fact, Proposition 2.13 in chapter 2 of [6] asserts ψ is an orthonormal wavelet associated with $\mathcal{M}$ if and only if (5) is satisfied. It is often useful to know if we can find a scaling function $\varphi \in V_0$ such that the simpler equality (4) is true (that is if (5) with $s(\xi) \equiv 1$ is satisfied). This is the content of the first theorem we shall prove:

Theorem I. *Suppose ψ is an orthonormal wavelet associated with an MRA $\mathcal{M} = \{V_j\}$, $j \in \mathbf{Z}$, then there exists a scaling function $\varphi \in V_0$ such that φ and ψ satisfy (4) where m is the low pass filter determined by φ .*

We have seen that the class of all wavelets can be characterized (see Proposition 1.1). There is also a simple (but not elementary) characterization of the subclass of all MRA wavelets; moreover, one can characterize all $\varphi \in L^2(\mathbf{R})$ that are scaling functions for an MRA. These characterizations are stated in the following two propositions.

Proposition 1.2 *An orthonormal wavelet ψ is an MRA wavelet if and only if*

$$D_\psi(\xi) \equiv \sum_{j=1}^{\infty} \sum_{k \in \mathbf{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2 = 1 \qquad a.e.$$

Proposition 1.3 *A function $\varphi \in L^2(\mathbf{R})$ is a scaling function for an MRA if and only if*

(i) $\quad \sum_{k \in \mathbf{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1$ for a.e. $\xi \in \mathbf{R}$;

(ii) $\quad \lim_{j \to \infty} |\hat{\varphi}(2^{-j}\xi)| = 1$ for a.e.$\xi \in \mathbf{R}$;

(iii) there exists a 2π periodic function $m \in L^2([-\pi, \pi))$ such that $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$ for a.e. $\xi \in \mathbf{R}$.

The proofs for these results, as well as proper credits, can be found in chapter 7 of [6]. It is useful to point out that equality (i) in the last proposition is equivalent to the property that the sequence $\{\varphi(\cdot - k)\}$, $k \in \mathbf{Z}$, is an orthonormal system. The easy proof of this fact can also be found in [6].

An immediate consequence of Proposition 1.3 is that if φ is a scaling function, then the function $\tilde{\varphi}$ whose Fourier transform is $|\hat{\varphi}|$ is also a scaling function; however, $\tilde{\varphi}$ may be associated with a different MRA. An example of this phenomenon is provided by the Haar wavelet which is associated with the MRA determined by the scaling function φ that is the characteristic function of the interval $[-1, 0]$. Then

$$\hat{\varphi}(\xi) = e^{i\xi/2} \frac{\sin(\xi/2)}{(\xi/2)} .$$

Let $\tilde{\varphi} = (|\hat{\varphi}|)^\vee$. If $\tilde{\varphi}$ were a scaling function of this MRA, then $\widehat{\tilde{\varphi}}(\xi) = l(\xi)\hat{\varphi}(\xi)$ with l a 2π periodic unimodular measurable function. Indeed, if $\tilde{\varphi}$, $\varphi \in V_0$ are each a scaling function for the MRA $\{V_j\}$, $j \in \mathbf{Z}$, then

$$\tilde{\varphi}(x) = \sum_{k \in \mathbf{Z}} \alpha_k \varphi(x - k)$$

with $\sum_{k\in\mathbf{Z}} |\alpha_k|^2 < \infty$ and the convergence is in the norm of $L^2(\mathbf{R})$. Taking Fourier transforms of both sides, we then have

$$\hat{\tilde{\varphi}}(\xi) = \left\{ \sum_{k\in\mathbf{Z}} \alpha_k e^{-ik\xi} \right\} \hat{\varphi}(\xi) \equiv l(\xi)\hat{\varphi}(\xi).$$

Since l is the Fourier series of a function in $L^2(\mathbf{T})$ it is 2π periodic. This fact and equality (i) in Proposition 1.3 imply that l is unimodular:

$$1 = \sum_{k\in\mathbf{Z}} |\hat{\tilde{\varphi}}(\xi + 2k\pi)|^2 = \sum_{k\in\mathbf{Z}} |l(\xi + 2k\pi)\hat{\varphi}(\xi + 2k\pi)|^2 =$$

$$|l(\xi)|^2 \sum_{k\in\mathbf{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = |l(\xi)|^2 \cdot 1 = |l(\xi)|^2$$

almost everywhere. In the Haar wavelet situation we are considering, we would then have

$$l(\xi) = \{\mathrm{sgn}\xi\} \left\{ e^{-i\xi/2} \frac{|\sin(\xi/2)|}{\sin(\xi/2)} \right\} = e^{-i\xi/2}$$

for $\xi \in [-2\pi, 2\pi]\backslash\{0\}$, which shows that l cannot be 2π periodic.

It follows from our discussion that if φ_1 and φ_2 are scaling functions of the same MRA, then $|\hat{\varphi}_1(\xi)| = |\hat{\varphi}_2(\xi)|$ a.e. The same is true for two wavelets ψ_1 and ψ_2 associated with the same MRA: $|\hat{\psi}_1(\xi)| = |\hat{\psi}_2(\xi)|$. We have just seen that a scaling function whose Fourier transform equals $|\hat{\varphi}_1|$ (or $|\hat{\varphi}_2|$) may be associated with a different MRA and, as we shall presently see, the same is true for an MRA wavelet. We shall see that there are situations when it is desirable to reduce a problem involving wavelets to the case where the scaling function φ involved satisfies $\hat{\varphi}(\xi) \geq 0$. Toward this end, it is natural to study the properties of the class $\mathcal{S}_{\psi_0}$, ψ_0 an MRA wavelet, of all those MRA wavelets ψ such that $|\hat{\varphi}_0(\xi)| = |\hat{\varphi}(\xi)|$ a.e., where φ_0 is a scaling function associated with the same MRA as ψ_0 and φ is a scaling function related with ψ in the same way. It is not hard to see that $\mathcal{S}_{\psi_0}$ coincides with the class $\mathcal{W}_{\psi_0}$ of all those wavelets ψ such that $|\hat{\psi}_0(\xi)| = |\hat{\psi}(\xi)|$ a.e. (observe that it follows from Proposition 1.2 that any such ψ is also an MRA wavelet). Let us first observe that the inclusion $\mathcal{W}_{\psi_0} \subseteq \mathcal{S}_{\psi_0}$ is an immediate consequence of the equality

$$|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|^2 \tag{6}$$

(see (2.16) in chapter 2 of [6]). Now suppose that $\psi \in \mathcal{S}_{\psi_0}$ and φ, φ_0 are scaling functions related to ψ and ψ_0 by equality (5). It is clear from (3) that the corresponding low pass filters m and m_0 have the same absolute value almost everywhere (recall that the low pass filter is uniquely determined by the scaling function). It then follows from (5) that $\hat{\psi}$ and $\hat{\psi}_0$ have absolute values that are equal a.e. ; this shows that $\mathcal{S}_{\psi_0} \subseteq \mathcal{W}_{\psi_0}$. Thus, whenever ψ_0 is an MRA wavelet,

$$\mathcal{W}_{\psi_0} = \mathcal{S}_{\psi_0}. \tag{7}$$

There is another description of this class of wavelets that is most useful to us. This involves the notion of a *wavelet multiplier* : a measurable function ν such that $(\nu\hat{\psi})^\vee$ is an orthonormal wavelet whenever ψ is an orthonormal wavelet. There is an elegant and simple characterization of these functions. Before announcing this result let us make a few observations. It will be of interest to us to consider what might be a larger class of such multipliers: those measurable functions ν such that $(\nu\hat{\psi})^\vee$ is an MRA wavelet whenever ψ is an MRA wavelet. It is natural to consider this notion of multipliers in connection with scaling functions and low pass filters: we say that ν is a *scaling function multiplier* if $\nu\hat{\varphi}$ is the Fourier transform of a scaling function whenever φ is a scaling function; similarly we say that μ is a *filter multiplier* provided μm is a low pass filter whenever m is a low pass filter. The characterization of these multipliers is the content of the second theorem that we shall prove:

Theorem II. *The class of wavelet multipliers coincides with both the class of MRA wavelet multipliers and the set of scaling function multipliers. Moreover, a measurable function ν belongs to any one of these classes if and only if it is unimodular and $\nu(2\xi)/\nu(\xi)$ is almost everywhere equal to a 2π periodic function. A measurable function μ is a filter multiplier if and only if μ is unimodular and is almost everywhere equal to a 2π periodic function.*

Let us now return to another description of the class $\mathcal{W}_{\psi_0}$. If ψ_0 is a wavelet let $\mathcal{M}_{\psi_0}$ be the collection of all wavelets ψ whose Fourier transform equals $\nu\hat{\psi}_0$, where ν is a wavelet multiplier. Proposition 1.2 and the unimodularity of ν imply that any $\psi \in \mathcal{M}_{\psi_0}$ is an MRA wavelet when ψ_0 is an MRA wavelet. The third result we shall establish is

Theorem III. *If ψ_0 is an MRA wavelet, then $\mathcal{M}_{\psi_0} = \mathcal{W}_{\psi_0} = \mathcal{S}_{\psi_0}$.*

When we defined the class $\mathcal{M}_{\psi_0}$ we did not assume that ψ_0 is an MRA wavelet; moreover, the set $\mathcal{W}_{\psi_0}$ can also be defined whether or not ψ_0 is an MRA wavelet. Of course we must make this assumption in order to define the class $\mathcal{S}_{\psi_0}$; thus, we can only consider $\mathcal{M}_{\psi_0}$ and $\mathcal{W}_{\psi_0}$ when ψ_0 is not an MRA wavelet. These two sets are not always equal in this last case. This is one of the reasons we restrict our attention in this paper, for the most part, to MRA wavelets. Most of the rest of this work is devoted to applications of the equality obtained in Theorem III. Perhaps, the most interesting one is the fact that this last result can be used effectively for showing that the collection of MRA wavelets, as a subset of the unit sphere in $L^2(\mathbf{R})$ is arcwise connected. More precisely, we shall prove

Theorem IV. *If ψ_0 and ψ_1 are two MRA wavelets, then there exists a continuous map $A : [0,1] \longrightarrow L^2(\mathbf{R})$ such that $A(0) = \psi_0$, $A(1) = \psi_1$ and $A(t)$ is an MRA wavelet for all $t \in [0,1]$.*

If $f(\xi)$ is a complex valued function on $\mathbf{R}$, then $f(\xi) = e^{i\beta(\xi)}|f(\xi)|$ for some real valued function β (which is clearly not unique). We call such a function β a *phase* of the function f. Another application of Theorem III that we shall present is a characterization of the phases of the MRA wavelets. We will also make some observations about the phases of other wavelets. Finally, we will describe the "shorter notes" that accompany this paper.

2 The Proof of Theorem I

Given an MRA wavelet ψ satisfying equality (5) we must find a scaling function $\tilde{\varphi}$ with accompanying filter $\tilde{m}$ such that

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{\tilde{m}\left(\frac{\xi}{2} + \pi\right)}\hat{\tilde{\varphi}}\left(\frac{\xi}{2}\right) \tag{8}$$

where $\tilde{\varphi}$ and φ are scaling functions for the same MRA. But this means that $\hat{\tilde{\varphi}}(\xi) = t(\xi)\hat{\varphi}(\xi)$, with t a unimodular 2π periodic measurable function. A simple calculation shows that the filter m determined by φ is related to $\tilde{m}$

by the equality

$$\tilde{m}(\xi) = t(2\xi)\overline{t(\xi)}m(\xi). \tag{9}$$

Thus, equality (8) can be re-written in the form

$$\hat{\psi}(\xi) = e^{i\xi/2}\left\{\overline{t(\xi)}t\left(\frac{\xi}{2}\right)t\left(\frac{\xi}{2}+\pi\right)\right\}\overline{m\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right)$$

(we are using the 2π periodicity of t). Comparing this equality with (5) we see that we can reduce the proof of Theorem I to establishing the following Lemma:

Lemma 2.1 *Suppose s is a 2π periodic, unimodular and measurable function on $\mathbf{R}$, then there exists a 2π periodic, unimodular and measurable function t such that*

$$s(\xi) = \overline{t(\xi)}t\left(\frac{\xi}{2}\right)t\left(\frac{\xi}{2}+\pi\right) \tag{10}$$

Proof. We shall show that the solution $t(\xi)$ is completely determined by its restriction to the interval $[0,\pi)$ and this restriction can be an arbitrary unimodular measurable function. Toward this end let us rewrite equality (10) in the form

$$t(\xi) = \overline{t(\xi-\pi)}t(2(\xi-\pi))s(2\xi)\ , \tag{11}$$

where we have used the fact that the desired solution t is 2π periodic. Let us partition $[0,2\pi)$ into the subintervals $I_j = [2\pi(1-2^{-j}), 2\pi(1-2^{-j-1}))$, $j = 0,1,2,\dots$; thus,

$$\bigcup_{j=0}^{\infty} I_j = [0,\pi)\cup[\pi,3\pi/2)\cup[3\pi/2,7\pi/4)\dots = [0,2\pi)\ .$$

We begin by defining the restrictions, t_j, of t to the intervals I_j. Let us choose an arbitrary unimodular measurable function t_0 on $I_0 = [0,\pi)$. If $\xi \in I_1$ then $\xi - \pi$ and $2(\xi-\pi)$ belong to I_0. Hence, equality (11) defines the function t_1 on I_1 in terms of t_0 and s. In general, if $\xi \in I_{j+1}$ then $\xi - \pi \in I_0$ and $2(\xi-\pi)$ belongs to I_j . Thus, equality (11) defines t_{j+1} by letting

$$t_{j+1}(\xi) = \overline{t_0(\xi-\pi)}t_j(2(\xi-\pi))s(2\xi)$$

when $\xi \in I_{j+1}$. Having so defined t on $[0, 2\pi)$, we extend t to $\mathbf{R}$ by 2π periodicity. On the interval $[\pi, 2\pi)$ we rewrite (11) in the form

$$t(\xi)t(\xi - \pi) = t(2\xi)s(2\xi). \tag{12}$$

Using the 2π periodicity of t, let us observe that each side of (12) is a π periodic expression and, thus, this equality extends to $\cup_{k \in \mathbf{Z}}[k\pi, (k+1)\pi) = \mathbf{R}$. Then, putting $\eta = 2\xi$ and solving for $s(\eta)$ we obtain the desired equality (10) (let us observe that on several occasions here, and in the sequel, we use the fact that the reciprocal of an unimodular function is its complex conjugate). $\square$

3 The Proof of Theorem II

We begin with the characterization of the filter multipliers. Suppose, first that μ satisfies the two conditions stated in the last part of the announcement of this theorem: it is unimodular and almost everywhere equal to a 2π periodic function. We claim that there exists a unimodular function t satisfying

$$\mu(\xi) = t(2\xi)\overline{t(\xi)} \tag{13}$$

(compare with equality (9) as a motivation for this equation). Let t be any measurable unimodular function defined on the set $S = [-2\pi, -\pi) \cup [\pi, 2\pi)$. If $\xi \in 2S$, let

$$t(\xi) = t\left(\frac{\xi}{2}\right)\mu\left(\frac{\xi}{2}\right) \; ; \tag{14}$$

then (13) is clearly satisfied for $\xi \in S$. We proceed inductively: assuming t is defined on $2^j S$, $j = 1, 2, \ldots, n$, so that (13) is valid on $\cup_{j=0}^{n-1} 2^j S$, we define $t(\xi)$ by equality (14) when $\xi \in 2^{n+1}S$. If $\xi \in 2^{-1}S$ we let $t(\xi) = t(2\xi)\overline{\mu(\xi)}$. We then use this last equality to proceed inductively for the definition of t on the sets $2^n S$, $n \leq -1$. Since $\cup_{n \in \mathbf{Z}} 2^n S = \mathbf{R} \backslash \{0\}$ we obtain a desired solution of (13) for almost every point of $\mathbf{R}$ (in this case, all $\xi \neq 0$).

Suppose that m is a low pass filter defined by the scaling function φ. We then claim that $\tilde{m} = \mu m$ is the low pass filter defined by the scaling function $\tilde{\varphi}$, where $\widehat{\tilde{\varphi}} = t\hat{\varphi}$. That $\tilde{\varphi}$ is, indeed, a scaling function is an immediate consequence of Proposition 1.3: Properties (i) and (ii) are clearly satisfied since t is unimodular and they are satisfied by the scaling function φ; property (iii) is also satisfied since $\tilde{m} = \mu m$ is 2π periodic and

$$\widehat{\tilde{\varphi}}(2\xi) = t(2\xi)\hat{\varphi}(2\xi) = t(2\xi)m(\xi)\hat{\varphi}(\xi) = t(2\xi)m(\xi)\overline{t(\xi)}t(\xi)\hat{\varphi}(\xi) = \tilde{m}(\xi)\widehat{\tilde{\varphi}}(\xi) \ .$$

Conversely, if μ is a filter multiplier, the fact that it must be unimodular is an immediate consequence of the fact that any low pass filter satisfies the equality

$$|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1 \tag{15}$$

for almost all ξ (see (2.5) in Chapter 2 of [6]). For the Haar wavelet, the low pass filter is $m_h(\xi) = (1+e^{i\xi})/2$ and for the *Shannon wavelet* the low pass filter is the 2π periodic function m_s that equals the characteristic function of the interval $[-\frac{\pi}{2}, \frac{\pi}{2})$ when restricted to $[-\pi, \pi)$ (see examples B and C in chapter 2 of [6]). The fact that μm_s, being a filter, must satisfy (15) implies that $|\mu(\xi)| = 1$ on the intervals $[\pi\frac{4k-1}{2}, \pi\frac{4k+1}{2})$, $k \in \mathbf{Z}$. This last property and the fact that μm_h must satisfy (15) imply that $|\mu(\xi)| = 1$ on $[\pi\frac{4k+1}{2}, \pi\frac{4k+3}{2})$, $k \in \mathbf{Z}$. Since the union of these two collections of intervals forms a partition of $\mathbf{R}$, we see that the filter multiplier μ must be unimodular.

Since $m_h(\xi) \neq 0$ when ξ is not an odd multiple of π and μm_h, being a low pass filter, equals a 2π periodic function m a.e., we conclude that $\mu(\xi) = m(\xi)/m_h(\xi)$ also equals a 2π periodic function a.e. This establishes the characterization of filter multipliers.

We now turn to the characterization of the wavelet multipliers. Let us first show that if ν is a unimodular function such that $\nu(2\xi)/\nu(\xi) = \nu(2\xi)\overline{\nu(\xi)}$ is 2π periodic, then it is a wavelet multiplier. We must show that $\widehat{\tilde{\psi}} = \nu\hat{\psi}$ is the Fourier transform of a wavelet whenever ψ is a wavelet. We do this by showing that $\tilde{\psi}$ satisfies the properties (i), (ii) in Proposition 1.1 and $\|\tilde{\psi}\|_2 \geq 1$. Equality (i) and the last inequality are immediate since ν is unimodular and ψ satisfies these conditions (being a wavelet). Hence, the only thing needed is to show that

$$\sum_{j=0}^{\infty} \widehat{\tilde{\psi}}(2^j\xi)\overline{\widehat{\tilde{\psi}}(2^j(\xi+2q\pi))} = 0 \tag{16}$$

a.e. when q is an odd integer. The summands of this series are equal to the products

$$\nu(2^j\xi)\overline{\nu(2^j(\xi+2q\pi))}\ \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+2q\pi))} \tag{17}$$

for $j \geq 0$. When $j \geq 1$, using the unimodularity of ν and the 2π periodicity of $\nu(2\eta)\overline{\nu(\eta)}$, we have

$$\begin{gathered}\nu(2^j\xi)\overline{\nu(2^j(\xi+2q\pi))} = \\ \nu(2^j\xi)\overline{\nu(2^{j-1}\xi)}\nu(2^{j-1}\xi)\overline{\nu(2^{j-1}(\xi+2q\pi))}\cdot \\ \cdot\nu(2^{j-1}(\xi+2q\pi))\overline{\nu(2^j(\xi+2q\pi))} = \\ \nu(2^{j-1}\xi)\overline{\nu(2^{j-1}(\xi+2q\pi))}\end{gathered} \tag{18}$$

since, by the 2π periodicity of $\nu(2\eta)\overline{\nu(\eta)}$ and the unimodularity of ν, the product of first two factors in (18) is the reciprocal of the product of the last two factors. We can continue this reduction and obtain

$$\nu(2^j\xi)\overline{\nu(2^j(\xi+2q\pi))} = \nu(\xi)\overline{\nu(\xi+2q\pi)}$$

when $j \geq 1$. Thus, the series in (16) equals

$$\nu(\xi)\overline{\nu(\xi+2q\pi)} \quad \sum_{j=0}^{\infty}\hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+2q\pi))} = 0$$

a.e. and it follows that ν is, indeed, a wavelet multiplier.

If we show that a wavelet multiplier ν is unimodular, then Proposition 1.2 implies that $\tilde{\psi}$ is an MRA wavelet whenever ψ is an MRA wavelet. Consequently, the collection of wavelet multipliers is contained in the class of MRA wavelet multipliers. We shall show that if ν belongs to this last, possibly larger, class, then it does satisfy the two properties (unimodularity and the 2π periodicity of $\nu(2\xi)/\nu(\xi)$) we just used. Hence, the equality of these two classes will be established.

Let us assume, then, that ν is a wavelet multiplier and show that it must be unimodular. Let $F_n = \{\xi \in \mathbf{R} : |\nu(\xi)| \geq 1 + \frac{1}{n}\}$, $n \geq 1$. Let ψ be a wavelet such that $\{\xi : \hat{\psi}(\xi) = 0\}$ has measure 0 (for example, we can choose ψ to be the Haar wavelet). Then there exists an $\varepsilon > 0$ such that

$$|\{\xi : |\hat{\psi}(\xi)| > \varepsilon\} \cap F_n| \geq \frac{1}{2}|F_n|,$$

where $|S|$ denotes the Lebesgue measure of the measurable set S. Let $N \in \mathbf{N}$ be such that $\varepsilon(1+\frac{1}{n})^N > 1$. Then $\nu^N\hat{\psi}$ is the Fourier transform of a wavelet whose absolute value on the set $\{\xi : |\hat{\psi}(\xi)| > \varepsilon\} \cap F_n$ exceeds 1. But the

Fourier transform of a wavelet cannot exceed 1 on a set of positive measure; consequently, $|F_n| = 0$. Since $n \in \mathbf{N}$ is arbitrary, it follows that $|\nu(\xi)| \leq 1$ a.e. Moreover,

$$1 = \|\psi\|_2^2 = (1/2\pi)\int_{\mathbf{R}} |\hat{\psi}(\xi)|^2 d\xi \geq (1/2\pi)\int_{\mathbf{R}} |\nu(\xi)\hat{\psi}(\xi)|^2 d\xi = 1$$

because $(\nu\hat{\psi})^{\vee}$, being a wavelet, has L^2 norm 1. Therefore the last inequality is an equality and we can conclude that $|\nu(\xi)| = 1$ a.e. since $\hat{\psi}(\xi) \neq 0$ a.e. Let us observe that this argument also shows that an MRA wavelet multiplier must be unimodular.

We shall now show the 2π periodicity of $\nu(2\xi)/\nu(\xi)$ when ν is an MRA wavelet multiplier. It is easy to see that if ψ is an MRA wavelet then $e^{i\xi/2}|\hat{\psi}(\xi)|$ is the Fourier transform of an MRA wavelet. We can assume that ψ satisfies equality (5). Then, as we have already observed, $|\hat{\varphi}|$ is a scaling function whose associated low pass filter is $|m|$. Consequently,

$$e^{i\xi/2}\left| m\left(\frac{\xi}{2}+\pi\right)\hat{\varphi}\left(\frac{\xi}{2}\right)\right| = e^{i\xi/2}|\hat{\psi}(\xi)|$$

is the Fourier transform of an MRA wavelet. Let us choose $\psi = \psi_0$ so that $\hat{\psi}(\xi) \neq 0$ for a.e. ξ (again, the Haar wavelet provides us with such a wavelet). Let φ_0 be a scaling function, with filter m_0, such that

$$\hat{\psi}_0(\xi) = e^{i\xi/2}\,\overline{m_0\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}_0\left(\frac{\xi}{2}\right) \quad \text{and} \quad \hat{\psi}_1(\xi) = \nu(\xi)e^{i\xi/2}\left|\hat{\psi}_0(\xi)\right|.$$

Then ψ_1 is an MRA wavelet (by Proposition 1.2, since ν is unimodular) and, by Theorem I, we can find a scaling function φ_1 and accompanying filter m_1 such that

$$\nu(\xi)e^{i\xi/2}|\hat{\psi}_0(\xi)| = e^{i\xi/2}\,\overline{m_1\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}_1\left(\frac{\xi}{2}\right) \quad . \tag{19}$$

¿From (6) and the fact that $|\hat{\psi}_0(\xi)| = |\hat{\psi}_1(\xi)|$ we have $|\hat{\varphi}_0(\xi)| = |\hat{\varphi}_1(\xi)|$ and $|m_0(\xi)| = |m_1(\xi)|$ a.e. Let us now replace ξ in (19) by 2ξ and divide each side of this new equality by the corresponding sides in equality (19) to obtain

$$\frac{\nu(2\xi)\ |\hat{\psi}_0(2\xi)|}{\nu(\xi)\ |\hat{\psi}_0(\xi)|} \quad = \quad \frac{\overline{m_1(\xi+\pi)}\ \hat{\varphi}_1(\xi)}{\overline{m_1(\pi+\xi/2)}\ \hat{\varphi}_1(\xi/2)}$$

$$= \frac{\overline{m_1(\xi+\pi)}\, m_1(\xi/2)}{\overline{m_1(\pi+\xi/2)}}$$

Thus,

$$\frac{\nu(2\xi)}{\nu(\xi)} = \frac{|\hat{\psi}_0(\xi)|\, \overline{m_1(\xi+\pi)}\, m_1(\xi/2)}{|\hat{\psi}_0(2\xi)|\, \overline{m_1(\pi+\xi/2)}} =$$

$$\frac{\overline{m_1(\xi+\pi)}}{|m_0(\xi+\pi)|} \frac{|m_0(\pi+\xi/2)|}{\overline{m_1(\pi+\xi/2)}} \frac{m_1(\xi/2)}{|m_0(\xi/2)|}, \tag{20}$$

where we repeatedly used the expression of $\hat{\psi}$ in terms of m and $\hat{\varphi}$, as well as (3), to obtain the last equality; in particular, we have

$$\frac{|\hat{\psi}_0(\xi)|}{|\hat{\psi}_0(2\xi)|} = \frac{|m_0(\pi+\xi/2)|}{|m_0(\xi+\pi)|\,|m_0(\xi/2)|}.$$

The first fraction in (20) clearly defines a 2π periodic function. The product of the last two fractions in (20) equals

$$\frac{|m_1(\pi+\xi/2)|}{\overline{m_1(\pi+\xi/2)}} \frac{m_1(\xi/2)}{|m_1(\xi/2)|} = \frac{m_1(\pi+\xi/2)}{|m_1(\pi+\xi/2)|} \frac{m_1(\xi/2)}{|m_1(\xi/2)|}$$

which is clearly a 2π periodic function.

The characterization of scaling function multipliers is obtained in a similar, and simpler, way. If $t(\xi)$ is unimodular and $t(2\xi)\overline{t(\xi)}$ is 2π periodic it is immediate to verify that $t\hat{\varphi}$ satisfies the conditions of Proposition 1.3 when $\hat{\varphi}$ does. Thus, t is a scaling function multiplier. If, on the other hand, we know that t is a scaling function multiplier, the fact that it must be unimodular follows from the same argument we used for an MRA wavelet multiplier; this time we can use the Haar scaling function φ_0 instead of the Haar wavelet. We also use the fact that $\hat{\varphi}_0(\xi) \neq 0$ for almost all ξ to show the 2π periodicity of $\xi \to t(2\xi)\overline{t(\xi)}$. Since $t\hat{\varphi}_0 = \widehat{\tilde{\varphi}}$ is the Fourier transform of a scaling function, we have $\widehat{\tilde{\varphi}}(2\xi) = \tilde{m}(\xi)\widehat{\tilde{\varphi}}(\xi)$, where $\tilde{m}$ is the filter associated with $\tilde{\varphi}$. But

$$\begin{aligned} \widehat{\tilde{\varphi}}(2\xi) = t(2\xi)\hat{\varphi}_0(2\xi) &= t(2\xi)m_0(\xi)\hat{\varphi}_0(\xi) = \\ t(2\xi)m_0(\xi)\overline{t(\xi)}t(\xi)\hat{\varphi}_0(\xi) &= t(2\xi)\overline{t(\xi)}m_0(\xi)\widehat{\tilde{\varphi}}(\xi) \ . \end{aligned}$$

It follows that $\tilde{m}(\xi) = t(2\xi)\overline{t(\xi)}m_0(\xi)$ and, consequently,

$$t(2\xi)\overline{t(\xi)} = \frac{\tilde{m}(\xi)}{m_0(\xi)}$$

(since the functions we are using are non-zero *a.e.* and t is unimodular). But the last quotient is 2π periodic. $\square$

Let us observe that the three types of multipliers we have characterized are related. An examination of our arguments shows that if $t(\xi)$ is a scaling function multiplier that produces the scaling function $\tilde{\varphi}$ from the scaling function φ, then $\mu(\xi) = t(2\xi)\overline{t(\xi)}$ is the multiplier that gives us the low pass filter $\tilde{m}$ defined by $\tilde{\varphi}$ from the low pass filter m associated with the scaling function φ. The low pass filters are uniquely determined, via equality (3), by the scaling functions. A low pass filter, however, can arise from different scaling functions. This is reflected by the fact that equation (13) has an infinitude of solutions. Each of these solutions is a scaling function multiplier.

If ν is a 2π periodic wavelet function multiplier, then each of the solutions t of equation (11) (with $\nu = s$) is a scaling function multiplier. There are several other relations between these multipliers and the roles they play in the structure of the sets $\mathcal{W}_{\psi_0}$, $\mathcal{S}_{\psi_0}$ and $\mathcal{M}_{\psi_0}$. These questions will be the subject of a future study.

4 The proof of Theorem III

We have already shown that $\mathcal{W}_{\psi_0} = \mathcal{S}_{\psi_0}$ (see equality (7)). Thus, to establish Theorem III we must show that $\mathcal{M}_{\psi_0}$ coincides with either $\mathcal{W}_{\psi_0}$ or $\mathcal{S}_{\psi_0}$. The inclusion $\mathcal{M}_{\psi_0} \subseteq \mathcal{W}_{\psi_0}$ is immediate since wavelet multiplier functions are unimodular.

We shall show that $\mathcal{S}_{\psi_0} \subseteq \mathcal{M}_{\psi_0}$. Suppose $\psi_1 \in \mathcal{S}_{\psi_0}$. Let φ_j be the scaling function such that

$$\hat{\psi}_j(\xi) = e^{i\xi/2}\, \overline{m_j\left(\pi + \frac{\xi}{2}\right)}\hat{\varphi}_j\left(\frac{\xi}{2}\right) \quad ,$$

$j = 0, 1$ (Theorem I guarantees the existence of these scaling functions and associated filters). As we have observed, $|\hat{\varphi}_0(\xi)| = |\hat{\varphi}_1(\xi)|$ a.e. Let $\tilde{\varphi}$ be the scaling function satisfying $\hat{\tilde{\varphi}} = |\hat{\varphi}_j(\xi)|$ (j is either 0 or 1); then $\tilde{m} = |m_j|$ is the associated low pass filter and let $\tilde{\psi}$ be the wavelet satisfying

$$\hat{\tilde{\psi}}(\xi) = e^{i\xi/2}\, \tilde{m}\left(\frac{\xi}{2}+\pi\right) \hat{\tilde{\varphi}}\left(\frac{\xi}{2}\right) \quad .$$

We shall show that there exists a wavelet multiplier function ν_j such that $\hat{\psi}_j = \nu_j \hat{\tilde{\psi}}$ for $j = 0$ and $j = 1$. Then $\nu = \nu_1 \overline{\nu_0}$ is also a wavelet multiplier function (this is immediate from the characterization given in Theorem II) and $\hat{\psi}_1 = \nu \hat{\psi}_0$. This, then, would establish the desired inclusion $\mathcal{S}_{\psi_0} \subseteq \mathcal{M}_{\psi_0}$.

In order to construct the wavelet multiplier ν_1 we first assume that

$$E = \{\xi : \hat{\varphi}_0(\xi) \neq 0\} = \{\xi : \hat{\tilde{\varphi}}(\xi) \neq 0\} = \{\xi : \hat{\varphi}_1(\xi) \neq 0\} = \mathbf{R} \quad .$$

In this case

$$t(\xi) = \frac{\hat{\tilde{\varphi}}(\xi)}{\hat{\varphi}_1(\xi)}$$

is a well defined unimodular function. Moreover, it follows from our assumption and (3) that neither $m_1(\xi)$ nor $\tilde{m}(\xi)$ is ever 0. Let $\mu = \text{sgn } m_1$ (that is, $\mu(\xi) = m_1(\xi)/\tilde{m}(\xi)$). We claim that

$$t(2\xi)\overline{t(\xi)} = \overline{\mu(\xi)} \quad . \tag{21}$$

To establish this equality we first observe that

$$\begin{aligned} \hat{\tilde{\varphi}}(2\xi) = t(2\xi)\hat{\varphi}_1(2\xi) &= t(2\xi)m_1(\xi)\hat{\varphi}_1(\xi) = \\ t(2\xi)\overline{t(\xi)}m_1(\xi)t(\xi)\hat{\varphi}_1(\xi) &= t(2\xi)\overline{t(\xi)}m_1(\xi)\hat{\tilde{\varphi}}(\xi) \quad . \end{aligned}$$

From this it follows that $\tilde{m}(\xi) = t(2\xi)\overline{t(\xi)}m_1(\xi)$ and (21) is an immediate consequence of this last equality. Moreover, $t(2\xi)\overline{t(\xi)} = \overline{\mu(\xi)}$, as the ratio of two nonzero 2π periodic functions, is also 2π periodic.

We claim that $\nu_1(\xi) = \overline{\mu(\pi+\xi/2)t(\xi/2)}$ is the desired wavelet multiplier function. It is clear that it is unimodular; furthermore, using (21), we have

$$\nu_1(2\xi)\overline{\nu_1(\xi)} = \overline{\mu(\pi+\xi)t(\xi)}\mu\left(\pi+\frac{\xi}{2}\right)t\left(\frac{\xi}{2}\right) = \overline{\mu(\pi+\xi)}\mu\left(\frac{\xi}{2}\right)\mu\left(\pi+\frac{\xi}{2}\right) \quad .$$

But the last expression is 2π periodic since μ has this property. It follows from Theorem II that ν_1 is a wavelet multiplier function. Furthermore, we claim that

$$\hat{\psi}_1 = \nu_1 \hat{\tilde{\psi}} \quad . \tag{22}$$

Indeed,

$$\hat{\psi}_1(\xi) = e^{i\xi/2}\,\overline{m_1\left(\pi+\frac{\xi}{2}\right)}\hat{\varphi}_1\left(\frac{\xi}{2}\right) =$$

$$e^{i\xi/2}\left\{\overline{m_1\left(\pi+\frac{\xi}{2}\right)}\mu\left(\pi+\frac{\xi}{2}\right)\right\}\left\{\overline{\mu\left(\pi+\frac{\xi}{2}\right)t\left(\frac{\xi}{2}\right)}\right\}\left\{t\left(\frac{\xi}{2}\right)\hat{\varphi}_1\left(\frac{\xi}{2}\right)\right\} =$$

$$e^{i\xi/2}\,\overline{\tilde{m}\left(\pi+\frac{\xi}{2}\right)}\nu_1(\xi)\widehat{\tilde{\varphi}}\left(\frac{\xi}{2}\right) = \nu_1(\xi)\widehat{\tilde{\psi}}(\xi)\ \ .$$

We have established Theorem III when $\hat{\varphi}_1(\xi) \neq 0$ for all ξ. When this is not the case, $t(\xi) = \widehat{\tilde{\varphi}}(\xi)/\hat{\varphi}_1(\xi)$ is only defined on a set whose complement may have positive measure; when this is the case we need to find an appropriate extension of the unimodular function $t(\xi)$ for all $\xi \in \mathbf{R}$. We shall do this so that (21) is satisfied by this extension when μ is the 2π periodic unimodular function satisying $\mu(\xi)\tilde{m}(\xi) = m_1(\xi)$ and $\mu(\xi) = 1$ when $m_1(\xi) = 0$ (we can assume that $\widehat{\tilde{\varphi}}(\xi) = |\hat{\varphi}_1(\xi)|$ and $\tilde{m}(\xi) = |m_1(\xi)|$ for all ξ). The argument that established (22) with $\nu_1(\xi) = \overline{\mu(\pi+\xi/2)t(\xi/2)}$ can then be carried out in order to finish the proof of Theorem III.

Thus, we shall now show how to construct this extension of $t(\xi)$. Let

$$F \equiv \{\xi \in \mathbf{R} : \hat{\varphi}_1(2^{l+1}\xi) = m_1(2^l\xi)\hat{\varphi}_1(2^l\xi) \text{ for all } \ l \in \mathbf{Z}\}.$$

If we let

$$F_l \equiv \{\xi \in \mathbf{R} : \hat{\varphi}_1(2^{l+1}\xi) = m_1(2^l\xi)\hat{\varphi}_1(2^l\xi)\}$$

for each $l \in \mathbf{Z}$, then $F = \cap_{l\in\mathbf{Z}}F_l$ and $\mathbf{R}\backslash F = \cup_{l\in\mathbf{Z}}(\mathbf{R}\backslash F_l)$. Observe that $F_l = 2^{-l}F_0$ and, since $|\mathbf{R}\backslash F_0| = 0$, we have $|\mathbf{R}\backslash F_l| = 0$ for all $l \in \mathbf{Z}$ and it follows that

$$|\mathbf{R}\backslash F| = 0\ \ . \tag{23}$$

It is also immediate that

$$F = 2^l F \quad \text{for all } \ l \in \mathbf{Z}. \tag{24}$$

Let $E = \{\xi \in F : \hat{\varphi}_1(\xi) \neq 0\}$. If $2\xi \in E$ then, by the definition of E, $2\xi \in F$ and, by (24), we conclude that $\xi \in F$. In particular, $\hat{\varphi}_1(2\xi) = m_1(\xi)\hat{\varphi}_1(\xi)$; this equality and $\hat{\varphi}_1(2\xi) \neq 0$ (since $2\xi \in E$) imply $\hat{\varphi}_1(\xi) \neq 0$. Thus, $\xi \in E$; that is, $2^{-1}E \subseteq E$. It follows that

$$2^n E \subseteq 2^{n+1}E \quad n = 0, 1, 2, \dots\ \ . \tag{25}$$

Suppose that K is a measurable subset of $F\backslash \cup_{n\geq 0} 2^n E = \cap_{n\geq 0}(F\backslash 2^n E)$. We claim that

$$\chi_K(\xi)\hat{\varphi}_1(2^{-n}\xi) \equiv 0 \quad \text{for all } \; n \geq 0. \tag{26}$$

It is clear that this equality is true when $\xi \notin K$. If $\xi \in K$ then $\xi \in F$ and, by (24), $2^n\xi \in F$ for all $n \in \mathbf{Z}$; moreover, $2^{-n}\xi \notin E$ for all $n \geq 0$. Consequently, by the definition of E, $\hat{\varphi}_1(2^{-n}\xi) = 0$ for all $n \geq 0$. This proves (26).

The Fourier transform of $2^{n/2}\varphi_1(2^n x - k)$ is $2^{-n/2}e^{-ik2^{-n}\xi}\hat{\varphi}_1(2^{-n}\xi)$, k, $n \in \mathbf{Z}$. Hence, it follows from (26) that $\chi_K \perp \hat{V}_n$ for all $n \in \mathbf{Z}$. This fact, together with (2.4) imply that $|K| = 0$ and, consequently, $|F_n\backslash\cup_{n\geq 0}2^n E| = 0$. Since

$$\mathbf{R}\backslash \cup_{n\geq 0} 2^n E \;\subseteq\; (\mathbf{R}\backslash F) \cup (F\backslash \cup_{n\geq 0} 2^n E) \quad ,$$

the last equality and (23) imply

$$\left|\mathbf{R}\backslash \cup_{n\geq 0} 2^n E\right| = 0 \;\; . \tag{27}$$

We are now ready to construct the appropriate extension of $t(\xi)$. Let $\Delta_0 = E$ and $\Delta_n = 2^n E\backslash 2^{n-1}E$ for $n \geq 1$. Thus, if $H = \cup_{n\geq 0}2^n E$, then

$$\mathbf{R} = (\mathbf{R}\backslash H) \cup (\cup_{n\geq 0}\Delta_n) \quad ,$$

where $\Delta_m \cap \Delta_n = \emptyset$ when $m \neq n$, $(\mathbf{R} - H) \cap \Delta_n = \emptyset$ for $n \geq 0$ and $|\mathbf{R} - H| = 0$ (by (25) and (27)). Let μ be the 2π periodic, unimodular measurable function we defined in the paragraph that follows (22). We define $t_0(\xi)$, for $\xi \in \Delta_0 = E$ as the ratio $\hat{\hat{\varphi}}(\xi)/\hat{\varphi}_1(\xi)$. Then, for $n \geq 1$ we define $t_{n+1}(\xi)$ for $\xi \in \Delta_{n+1}$ by letting

$$t_{n+1}(\xi) = \overline{\mu(\xi/2)}t_n(\xi/2)$$

assuming t_n is defined on Δ_n (observe that $\Delta_{n+1} = 2\Delta_n$ for $n \geq 1$). Then $t(\xi)$ is defined on $H = \cup_{n\geq 0}2^n E = \cup_{n\geq 0}\Delta_n$ by putting $t(\xi) = t_n(\xi)$ when $\xi \in \Delta_n$. Since $\mathbf{R}\backslash H$ has measure 0 we only need to establish (21) on H; that is

$$\overline{t(2\xi)}t(\xi) = \mu(\xi) \tag{28}$$

for $\xi \in H$ with μ satisfying $\mu(\xi)\tilde{m}(\xi) = m_1(\xi)$ and $\mu(\xi) = 1$ when $m_1(\xi) = 0$. As was the case before, μ is a.e. equal to a 2π periodic measurable function. Also observe that $H = 2H$.

Suppose ξ and 2ξ belong to $\Delta_0 = E$. Then,

$$\hat{\tilde{\varphi}}(2\xi) = t_0(2\xi)\hat{\varphi}_1(2\xi) = t_0(2\xi)m_1(\xi)\hat{\varphi}_1(\xi) = t_0(2\xi)\overline{t_0(\xi)}m_1(\xi)\hat{\tilde{\varphi}}(\xi) \ .$$

Since $\hat{\tilde{\varphi}}(2\xi) = \tilde{m}(\xi)\hat{\tilde{\varphi}}(\xi)$ and all the values involved are non-zero, $\tilde{m}(\xi) = t_0(2\xi)\overline{t_0(\xi)}m_1(\xi)$ and we must have $\overline{t(2\xi)}t(\xi) = \overline{t_0(2\xi)}t_0(\xi) = \mu(\xi)$ in this case. Now suppose $\xi \in E$ but $2\xi \notin E$, so that $2\xi \in 2E \backslash E = \Delta_1$. Then the definition of $t = t_1$ on Δ_1 in terms of t_0 and μ show that (28) is satisfied for these values of ξ. We can now carry out an induction argument, similar to the ones we have already given, that shows that (28) is satisfied for all $\xi \in H$ as the disjoint union of the sets Δ_n. As we observed above, we can then establish (22) by the same argument we used above in order to finish the proof of Theorem III. □

5 The Proof of Theorem IV

The proof of this connectivity result will consist of two parts. The first is devoted to showing that the class $\mathcal{M}_{\psi_0}$ is arcwise connected. In the second part we show that if ψ_0 and ψ_1 are two MRA wavelets then an appropriate element of $\mathcal{M}_{\psi_0}$ can be connected by a continuous arc with a particular element of the class $\mathcal{M}_{\psi_1}$. It is clear that this, then, proves Theorem IV.

Suppose $\psi_1 \in \mathcal{M}_{\psi_0}$. Then $\hat{\psi}_1 = \nu\hat{\psi}_0$, where ν is a wavelet multiplier. Thus, by Theorem III

$$\mu(\xi) = \nu(2\xi)\overline{\nu(\xi)} \tag{29}$$

is unimodular and is equal a.e. to a 2π periodic function. For simplicity, let us assume that this holds for all ξ. We can write $\mu(\xi) = e^{i\beta(\xi)}$, where $0 \leq \beta(\xi) < 2\pi$ for all ξ. We thus obtain a unique 2π periodic phase β for the function μ. We can also write $\nu(\xi) = e^{i\lambda(\xi)}$. Our task is to find an appropriate choice for $\lambda(\xi)$ so that

$$\nu_t(\xi) \equiv e^{it\lambda(\xi)}, \tag{30}$$

$t \in [0,1]$, is a wavelet multiplier and the function $\theta : t \longrightarrow \psi_t(\xi) := (\nu_t(\xi)\hat{\psi}_0(\xi))^\vee$ is a continuous map from $[0,1]$ to $L^2(\mathbf{R})$ whose values are wavelets that, for $t = 0$ and $t = 1$, equal ψ_0 and ψ_1 respectively.

We do this by means of an argument that is a variant of one we used above (compare with the construction of the function $t(\xi)$ at the end of the last section). Let $E = [-4\pi, -2\pi) \cup [2\pi, 4\pi)$. On E we select λ so that $0 \leq \lambda(\xi) < 2\pi$. If $\xi \in 2E$ we let

$$\lambda(\xi) = \lambda(\xi/2) + \beta(\xi/2) \ . \tag{31}$$

Then, making use of (29), we have

$$e^{i\lambda(\xi)} = e^{i\lambda(\xi/2)} e^{i\beta(\xi/2)} = \nu(\xi/2)\mu(\xi/2) = \nu(\xi) \ .$$

We then continue inductively: assuming that $\lambda(\xi)$ is defined on $2^k E$, $0 \leq k \leq n$, we let $\lambda(\xi)$, for $\xi \in 2^{n+1}E$, be defined by equality (31). In this way we have selected a phase $\lambda(\xi)$ of $\nu(\xi) = e^{i\lambda(\xi)}$ for $\xi \in \cup_{n\geq 0} 2^n E$ in such a way that

$$\lambda(2\xi) - \lambda(\xi) = \beta(\xi) \ . \tag{32}$$

If $\xi \in 2^{-1}E$ we let $\lambda(\xi) = \lambda(2\xi) - \beta(\xi)$ and use this equality inductively to obtain $\lambda(\xi)$ defined on $\cup_{n\geq 1} 2^{-n}E$. This gives us a function λ defined on $\mathbf{R}\backslash\{0\}$ satisfying (32) for all $\xi \neq 0$.

It follows from Theorem II that ν_t is a wavelet multiplier for each $t \in [0,1]$: it is clearly a unimodular function and, by (32),

$$\frac{\nu_t(2\xi)}{\nu_t(\xi)} = e^{it[\lambda(2\xi)-\lambda(\xi)]} = e^{it\beta(\xi)}$$

which is 2π periodic since β has this property. Thus, $\psi_t = (\nu_t \hat{\psi}_0)^\vee \in \mathcal{M}_{\psi_0}$.

Finally, we claim that $\theta(t) = \psi_t$ is a continuous mapping from $[0,1]$ to $L^2(\mathbf{R})$. This is the case since

$$|\hat{\psi}_t(\xi) - \hat{\psi}_s(\xi)|^2 = |e^{it\lambda(\xi)} - e^{is\lambda(\xi)}|^2 \, |\hat{\psi}_0(\xi)|^2 \ \leq \ 4|\hat{\psi}_0(\xi)|^2 \ ,$$

and, by the Lebesgue dominated convergence theorem and, then, Plancherel's theorem, we have, for each $s \in [0,1]$,

$$\lim_{t\to s} \|\psi_t - \psi_s\|_2 = 0 \ .$$

This shows that each of the classes $\mathcal{M}_{\psi_0}$ is arcwise connected.

We now pass to the second part of the proof. Since, by Theorem III, $\mathcal{M}_{\psi_0} = \mathcal{S}_{\psi_0}$, to show that any two MRA wavelets are connected by a continuous path it suffices to choose ψ_0 to be the Shannon wavelet and select an appropriate element ψ_1 in any other class $\mathcal{S}_\psi$ where ψ is an MRA wavelet, and construct such a path that connects ψ_0 to ψ_1. The appropriate $\psi_1 \in \mathcal{S}_\psi$ is chosen so that it is associated with a scaling function φ_1 such that $\hat{\varphi}_1 \geq 0$. By the definition of $\mathcal{S}_\psi$ this can always be done and we can assume that

$$\hat{\psi}_1(\xi) = e^{i\xi/2}\,\overline{m_1(\pi + \xi/2)}\hat{\varphi}_1(\xi/2) \ ,$$

where m_1 is the low pass filter associated with φ_1. We recall that the Shannon wavelet ψ_0 satisfies the analogous equality

$$\hat{\psi}_0(\xi) = e^{i\xi/2}\,\overline{m_0(\pi + \xi/2)}\hat{\varphi}_0(\xi/2) \ ,$$

where $\hat{\varphi}_0(\xi) = \chi_{[-\pi,\pi)}(\xi)$ and the associated filter m_0 is the 2π periodic function that equals $\chi_{[-\pi/2,\pi/2)}$ on the period interval $[-\pi,\pi)$.

We shall construct a family of "intermediate" filters $\{m_t\}$, $t \in [0,1]$, that connects m_0 with m_1 and then use these filters to define a family of scaling functions $\{\varphi_t\}$ from which we will obtain the desired path connecting ψ_0 to ψ_1. This family is defined on $[-\pi,\pi)$ by letting m_t satisfy

$$m_t(\xi) \ = \ \begin{cases} (1-t)m_0(\xi) + t m_1(\xi) & , \quad \xi \in [-\frac{\pi}{2},\frac{\pi}{2})\setminus[-(1-t)\frac{\pi}{2},\,(1-t)\frac{\pi}{2}) \\ 1 & , \quad \xi \in [-(1-t)\frac{\pi}{2},\,(1-t)\frac{\pi}{2}) \\ \sqrt{1 - m_t^2(\xi+\pi)} & , \quad \xi \in [-\pi,-\frac{\pi}{2}) \\ \sqrt{1 - m_t^2(\xi-\pi)} & , \quad \xi \in [\frac{\pi}{2},\pi) \end{cases}$$

and, then, m_t is extended to $\mathbf{R}$ so that it is 2π periodic.

We thus, obtain a non-negative 2π periodic function m_t such that

$$m_t^2(\xi) + m_t^2(\xi+\pi) = 1 \quad \text{for all } \xi \in \mathbf{R} \text{ and } t \in [0,1]. \tag{33}$$

Moreover, $m_t(\xi) \equiv 1$ on $[-(1-t)\frac{\pi}{2},(1-t)\frac{\pi}{2})$. At $t=0$ we obtain the Shannon filter and at $t=1$ we have the filter defined by φ_1. Let

$$\hat{\varphi}_t(\xi) = \prod_{j=1}^{\infty} m_t(2^{-j}\xi)$$

for $\xi \in \mathbf{R}$. This product is well defined since $0 \leq m_t(\xi) \leq 1$. Moreover,

$$\hat{\varphi}_t(\xi) = 1 \quad \text{when } \xi \in [-(1-t)\pi,(1-t)\pi) \quad \text{and } 0 < t < 1. \tag{34}$$

It is also clear that $\hat{\varphi}_t(2\xi) = m_t(\xi)\hat{\varphi}_t(\xi)$ for all $\xi \in \mathbf{R}$. Thus, $\hat{\varphi}_t$ satisfies properties (ii)and (iii) of Proposition 1.3. We shall show that $\hat{\varphi}_t \in L^2(\mathbf{R})$ and its inverse Fourier transfom φ_t satisfies property (i) of this proposition. Thus φ_t is a scaling function. The argument we use to establish this is well known in wavelet theory (see, for example, the proof of Proposition 3.9 in Chapter 2 of [6]); however, our hypotheses are somewhat different and, for the sake of completeness, we include it here.

For $k \geq 1$ let

$$\mu_{t,k}(\xi) = \chi_{[-\pi,\pi)}(2^{-k}\xi) \prod_{j=1}^{k} m_t(2^{-j}\xi) \ .$$

For $n \in \mathbf{Z}$ and $k \geq 2$ let

$$I^n_{t,k} = \int_{\mathbf{R}} [\mu_{t,k}(\xi)]^2 e^{-in\xi} d\xi \ = \ \int_{-2^k\pi}^{2^k\pi} m_t^2(2^{-k}\xi) \prod_{j=1}^{k-1} m_t^2(2^{-j}\xi)\, e^{-in\xi} d\xi \ .$$

Observe that $\prod_{j=1}^{k-1} m_t^2(2^{-j}\xi)$ is $2^k\pi$ periodic. Hence, using this fact and (33) we have:

$$I^n_{t,k} = \int_0^{2^k\pi} [m_t^2(2^{-k}\xi) + m_t^2(2^{-k}\xi - \pi)] \prod_{j=1}^{k-1} m_t^2(2^{-j}\xi)\, e^{-in\xi} d\xi =$$

$$\begin{aligned}
\int_{-2^{k-1}\pi}^{2^{k-1}\pi} \prod_{j=1}^{k-1} m_t^2(2^{-j}\xi)\, e^{-in\xi} d\xi \ &= \ \int_{\mathbf{R}} [\mu_{t,k-1}(\xi)]^2\, e^{-in\xi} d\xi = \\
I^n_{t,k-1} \ &= \ \ldots = I^n_{t,1} = \\
\int_{\mathbf{R}} [\mu_{t,1}(\xi)]^2\, e^{-in\xi} d\xi \ &= \ \int_{-2\pi}^{2\pi} m_t^2(\xi/2)\, e^{-in\xi} d\xi = \\
\int_0^{2\pi} [m_t^2(\xi/2) + m_t^2(\xi/2 - \pi)]\, e^{-in\xi} d\xi \ &= \ \int_0^{2\pi} e^{-in\xi} d\xi = 2\pi\delta_{n,0} \ .
\end{aligned}$$

Clearly, $\lim_{k\to\infty} \mu_{t,k}(\xi) = \hat{\varphi}_t(\xi)$ for all $\xi \in \mathbf{R}$ and, by the above equalities, $I^0_{t,k} = \|\mu_{t,k}\|_2^2 = 2\pi$. Thus, an application of Fatou's lemma shows that $\varphi_t \in L^2(\mathbf{R})$ with $\|\hat{\varphi}_t\|_2^2 \leq 2\pi$.

Suppose $\xi \in [-\pi, \pi)$, then $\xi/2 \in [-\pi/2, \pi/2)$ and, thus, $m_t(2^{-j}\xi) \geq 1 - t$ for $j \geq 1$. If $t < 1$ let us choose a k_0 sufficiently large so that $2^{-k_0} \leq 1 - t$.

Hence, if $\xi \in [-\pi, \pi)$ and $k \geq k_0$, it follows from (34) that $\hat{\varphi}_t(2^{-k}\xi) = 1$. Therefore,

$$\hat{\varphi}(\xi) = \prod_{j=1}^{k_0} m_t(2^{-j}\xi) \prod_{j=k_0+1}^{\infty} m_t(2^{-j}\xi) \geq$$
$$(1-t)^{k_0} \hat{\varphi}_t(2^{-k_0}\xi) = (1-t)^{k_0}$$

if $\xi \in [-\pi, \pi)$. Thus, we obtain

$$\chi_{[-\pi,\pi)}(\xi) \leq \frac{\hat{\varphi}_t(\xi)}{(1-t)^{k_0}} \tag{35}$$

for all $\xi \in \mathbf{R}$. From this we conclude that

$$\mu_{t,k}(\xi) = \chi_{[-\pi,\pi)}(2^{-k}\xi) \prod_{j=1}^{k} m_t(2^{-j}\xi) \leq$$

$$\frac{\hat{\varphi}_t(2^{-k}\xi)}{(1-t)^{k_0}} \prod_{j=1}^{k} m_t(2^{-j}\xi) = \frac{\hat{\varphi}_t(\xi)}{(1-t)^{k_0}} .$$

Since, as we just showed, $\hat{\varphi}_t \in L^2(\mathbf{R})$, we can apply Lebesgue's Dominated Convergence Theorem to obtain

$$\int_{\mathbf{R}} [\hat{\varphi}_t(\xi)]^2 e^{-in\xi} d\xi = \lim_{k\to\infty} \int_{\mathbf{R}} [\mu_{t,k}(\xi)]^2 e^{-in\xi} d\xi = 2\pi\delta_{n,o} .$$

Therefore, $\{\varphi_t(\cdot - k)\}$, $k \in \mathbf{Z}$ is an orthonormal system and we have shown that φ_t is a scaling function since it satisfies all the hypotheses of Proposition 1.3.

The proof of Theorem IV will be concluded once we show that $t \to \psi_t$ is a continuous mapping from $[0,1]$ to $L^2(\mathbf{R})$, where

$$\hat{\psi}_t(\xi) = e^{i\xi/2} m_t\left(\frac{\xi}{2} + \pi\right) \hat{\varphi}_t\left(\frac{\xi}{2}\right) .$$

We begin by examining the continuity of the map $t \longrightarrow \hat{\varphi}_t$. Let us fix a point $s \in [0,1]$ and list those points $\xi \in \mathbf{R}$ such that $t \longrightarrow m_t(\xi)$ is *not* continuous at s. If $0 < s < 1$ there are, at most, four such points within $[-\pi, \pi)$: $\pm\pi\frac{1-s}{2}$ and $\pm\pi\frac{1+s}{2}$ (when $s = 0$ or $s = 1$, there are, at most, two such points; moreover, for these two values of s, we are only interested in continuity from the right at 0 and from the left at 1). Because of the 2π periodicity of m_s we must also consider the $2k\pi$ translates of these points, $k \in \mathbf{Z}$. We now pass to the points of discontinuity at s of the map $t \longrightarrow m_t(2^{-j}\xi)$, $j \geq 1$. These values make up the set

$$\mathcal{D}_s = \left\{ \pm 2^j \pi \left(\frac{1-s}{2} + 2k \right),\ \pm 2^j \pi \left(\frac{1+s}{2} + 2l \right) : \quad k, l \in \mathbf{Z}\ , j \geq 1 \right\} \ .$$

Let $\mathcal{U} = \{\xi \in \mathbf{R} : \ \lim_{j\to\infty} \hat{\varphi}_1(2^{-j}\xi) = 1\}$. We then have $|\mathbf{R}\backslash\mathcal{U}| = 0 = |\mathcal{D}_s|$.

Lemma 5.1 *If* $\xi \in \mathcal{U}\backslash\mathcal{D}_s$ *then* $\lim_{t\to s} \hat{\varphi}_t(\xi) = \hat{\varphi}_s(\xi)$.

Proof. If $\eta \in [-\pi/2, \pi/2)$ then $m_t(\eta)$ is either 1 or $(1-t)m_0(\eta) + tm_1(\eta)$. In either case, $m_t(\eta) \geq m_1(\eta)$. From this it follows that

$$\hat{\varphi}_t(\eta) \geq \hat{\varphi}_1(\eta) \ \text{ when } \ \ \eta \in [-\pi, \pi) \ . \tag{36}$$

For a fixed $\xi \in \mathcal{U}\backslash\mathcal{D}_s$ and $\varepsilon \in (0,1)$ we can find $M = M(\varepsilon, \xi)$ such that

$$\hat{\varphi}_1(2^{-n}\xi) \geq 1 - \varepsilon \ \text{ and } |2^{-n}\xi| < \pi$$

if $n \geq M$. Hence,

$$|\hat{\varphi}_t(\xi) - \hat{\varphi}_s(\xi)| = \left| \prod_{j=1}^{\infty} m_t\left(\frac{\xi}{2^j}\right) - \prod_{j=1}^{\infty} m_s\left(\frac{\xi}{2^j}\right) \right| =$$

$$\left| \left[\prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right) \right] \hat{\varphi}_t\left(\frac{\xi}{2^M}\right) - \left[\prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right) \right] \hat{\varphi}_s\left(\frac{\xi}{2^M}\right) \right| =$$

$$\left| \left[\hat{\varphi}_t\left(\frac{\xi}{2^M}\right) - 1 \right] \prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right) - \left[\hat{\varphi}_s\left(\frac{\xi}{2^M}\right) - 1 \right] \prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right) + \right.$$

$$\left. + \prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right) - \prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right) \right| \leq 2\varepsilon + \left| \prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right) - \prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right) \right| \ .$$

Because of our choice of ξ, $t \longrightarrow m_t(2^{-j}\xi)$ is continuous at s for each $j \geq 1$; therefore, there exists $\delta = \delta(\varepsilon)$ such that

$$\left| \prod_{j=1}^{M} m_t\left(\frac{\xi}{2^j}\right) - \prod_{j=1}^{M} m_s\left(\frac{\xi}{2^j}\right) \right| < \varepsilon$$

if $|t-s| < \delta$. Hence, $|\hat{\varphi}_t(\xi) - \hat{\varphi}_s(\xi)| < 3\varepsilon$ if $|t-s| < \delta$ and the lemma is proved. $\square$

Since $t \longrightarrow m_t(\xi/2 + \pi)$ is continuous at s whenever $\xi/2+\pi$ is not a translate by $2k\pi$, $k \in \mathbf{Z}$, of the points of the form $\pm\pi(1 \pm s)/2$ and $\xi \in \mathcal{U}$ if and only if $\xi/2 \in \mathcal{U}$ it follows from Lemma 5.1 that

$$\lim_{t \longrightarrow s} \hat{\psi}_t(\xi) = \hat{\psi}_s(\xi) \tag{37}$$

for a.e. $\xi \in \mathbf{R}$. Since $\|\hat{\psi}_t\|_2^2 = 2\pi = \|\hat{\psi}_s\|_2^2$ and m_t, m_s, $\hat{\varphi}_t$, and $\hat{\varphi}_s$ are non negative, an application of Schwarz's inequality gives us

$$0 \leq \int_{\mathbf{R}} m_t\left(\frac{\xi}{2}+\pi\right) \hat{\varphi}_t\left(\frac{\xi}{2}\right) m_s\left(\frac{\xi}{2}+\pi\right) \hat{\varphi}_s\left(\frac{\xi}{2}\right) d\xi = \langle \hat{\psi}_t, \hat{\psi}_s \rangle \leq 2\pi \ .$$

We claim that

$$\lim_{t \to s} \langle \hat{\psi}_t, \hat{\psi}_s \rangle = 2\pi \ . \tag{38}$$

If this were not the case, there would exist a sequence $\{t_n\} \subseteq [0,1]$ and an $\varepsilon > 0$ such that $\lim_{n\to\infty} t_n = s$ and

$$\int_{\mathbf{R}} m_{t_n}\left(\frac{\xi}{2}+\pi\right) \hat{\varphi}_{t_n}\left(\frac{\xi}{2}\right) m_s\left(\frac{\xi}{2}+\pi\right) \hat{\varphi}_s\left(\frac{\xi}{2}\right) d\xi \leq 2\pi - \varepsilon$$

for $n = 1, 2, 3, \ldots$. Let us choose a compact subset K of $\mathbf{R}$ such that

$$\int_K |\hat{\psi}_s(\xi)|^2 d\xi > 2\pi - \varepsilon \ .$$

Then, by (37) and the Dominated Convergence theorem (recall that $|\hat{\psi}(\xi)| \leq 1$ when ψ is a wavelet) we have

$$2\pi - \varepsilon < \int_K |\hat{\psi}_s(\xi)|^2 d\xi = \lim_{n\to\infty} \int_K \hat{\psi}_{t_n}(\xi)\, \overline{\hat{\psi}_s(\xi)}\, d\xi \leq$$

$$\limsup_{n\to\infty} \int_{\mathbf{R}} m_{t_n}\left(\frac{\xi}{2}+\pi\right) \hat{\varphi}_{t_n}\left(\frac{\xi}{2}\right) m_s\left(\frac{\xi}{2}+\pi\right) \hat{\varphi}_s\left(\frac{\xi}{2}\right) d\xi \ \le 2\pi - \varepsilon$$

which is a contradiction. Hence, from (38)

$$\|\hat{\psi}_t - \hat{\psi}_s\|_2^2 = \|\hat{\psi}_t\|_2^2 + \|\hat{\psi}_s\|_2^2 - \langle \hat{\psi}_t, \hat{\psi}_s\rangle - \langle \hat{\psi}_s, \hat{\psi}_t\rangle \longrightarrow 4\pi - 2\pi - 2\pi = 0$$

as $t \longrightarrow s$. This establishes the desired continuity of the path we constructed that connects ψ_0 to ψ_1 and Theorem IV is proved. □

6 Further observations and description of the consortium

The results in this paper have been obtained in a genuine collaboration by members of the Wutam Consortium. There are and will be articles that elaborate some of the results announced here. We emphasize, however, that in this and subsequent reports by the Wutam Consortium, this collaboration will continue and each member of the consortium should consider these articles as part of their original scientific contribution.

Perhaps the first results on the connectivity of wavelets are contained in a paper by Aline Bonami, Silvain Durand and Guido Weiss [2]. These authors restricted their attention to wavelets produced by very smooth filters. The filters m they considered are elements of an infinite dimensional C^∞ manifold of 2π periodic functions m satisfying equality (15) and $m(0) = 1$. Their construction of a path joining two such filters is very different from the one presented in this paper. The continuity of the path is defined in terms of the topology introduced on the manifold. This does produce a continuous path of very smooth Fourier transforms of scaling functions, where the topology of the image is defined in terms of the L^2 norms of the functions involved and their derivatives. There is, however, a very important (and somewhat subtle) difference when all this is transferred to the Fourier transforms of wavelets defined in terms of the filters and scaling functions that have been constructed. There is a topological impediment that makes it impossible, in general, to obtain a corresponding continuous path of wavelets. In order to obtain an insight into this situation let us examine equality (5). The unimodular function $s(\xi)$, in this situation, must also be smooth and, hence, must belong to a homotopy class corresponding to an integer, the "winding number" of the path that has values in the circle of radius 1 about the origin in the complex plane. One can show that if we have a continuous path of wavelets, in the smooth sense we are considering, the corresponding path $t \longrightarrow s_t(\xi)$, $t \in [0,1]$, must be a jointly continuous function in the variables t and ξ. Thus, if s_0 and s_1 belong to different homotopy classes, they cannot be connected by an arc $t \longrightarrow s_t$ having this joint continuity property. A simple example of this situation is provided by trying to join the Haar wavelet ψ_0 to its translate by 1, $\psi_1 = \psi_0(\cdot - 1)$. If $\hat{\psi}_0$ satisfies equality (4), then ψ_1 satisfies (5) with $s(\xi) = e^{-i\xi}$. Since this last function and the function that is identically equal to 1 are not in the same homotopy class, there is no path, of the type we described, that joins them. These questions, as well as an extension of the results by Bonami, Durand and Weiss can be found in the Ph.D. thesis of Gustavo Garrigós, a student of Weiss at Washington University.

Another interesting connectivity result was obtained by Darrin Speegle. He showed (in his thesis as a student at Texas A&M) that all *Minimally Supported Frequency* (MSF) wavelets are connected. These are the wavelets whose Fourier transform has an absolute value that equals the characteristic function of a set $K \subseteq \mathbf{R}$. Such a set must have measure equal to 2π. The support of the Fourier transform of any wavelet cannot have measure smaller than 2π; this is the motivation for so naming the above class. The MSF wavelets are of interest for many reasons. The well known Journé wavelet (see page 64 of [6]) was the first example of a non-MRA wavelet; it is an MSF wavelet. The Shannon wavelet is another example of a member of this class; it is, of course, an MRA wavelet. Auscher [1] and Lemarié [8] have, independently, shown that if one makes very mild assumptions about the Fourier transform of a wavelet (continuity and a decrease at ∞) it must be an MRA wavelet. These facts tend to indicate that "most" wavelets are either MRA wavelets or MSF wavelets. Thus, if one joins Theorem IV with Speegle's result it is not unreasonable to conjecture that the class of all wavelets is connected. We shall return to these considerations below. An elegant proof of the connectivity of a class of MSF wavelets can be found in [7]. It should also be noted that some notions of the connectivity of wavelets formed some of the motivation for the results obtained in [5].

The notion of a wavelet multiplier function were introduced in a paper of X. Dai, Qing Gu, David Larson and R. Liang [3] and in the Memoir [5] by Dai and Larson. The proof we presented of the characterization of these functions given in Theorem II is a variant of the one first produced by these four authors. The consideration of the other multipliers considered in Theorem III is naturally motivated by the study of wavelet multipliers. Further results related to these multipliers, as well as the structure of the classes $\mathcal{M}_{\psi_0}$, $\mathcal{W}_{\psi_0}$ and $\mathcal{S}_{\psi_0}$ can be found in a paper by Papadakis, Šikić and Weiss ([9]). We have already mentioned the fact that the equality $\mathcal{M}_\psi = \mathcal{W}_\psi$ may be false when ψ is not an MRA wavelet. Gu has obtained an example of two wavelets ψ_1, ψ_2 such that $\hat{\psi}_1 \neq \nu\hat{\psi}_2$ for all wavelet multiplier functions ν. In order to put these facts into some sort of perspective we first remind the reader that if ψ is an MSF wavelet, then $e^{i\lambda(\xi)}|\hat{\psi}(\xi)|$ is the Fourier transform of a wavelet for all real valued measurable functions $\lambda(\xi)$ (see the "remark" on pages 349-50 of [6]). Thus, $\mathcal{M}_\psi = \mathcal{W}_\psi$ when ψ is an MSF wavelet. In this paper we show that this equality holds for all MRA wavelets. These observations should be kept in mind in connection with our statement that "most wavelets are MSF or MRA wavelets;" Gu's example shows that there are other wavelets. At present, it is not clear where they "fit" with respect to the general connectivity question.

This particular exposition was written by Weiss who owes very special thanks to Garrigós, Paluszyński and Šikić for many discussions concerning the form that this exposition should take. They also read this manuscript as it was evolving and made several very appropriate suggestions and corrections. Theorems I and III will be elaborated in the collaboration [9]. The elegant argument used in the proof of Lemma 2.1 is mainly due to three students who took a course in the theory of wavelets given by Weiss: Marcin Bownik, Wojtek Czaja and Ziemovit Rzeszotnik. The proof of the lemma was assigned as an exercise in the course; their solution was certainly better than Weiss' original one. One can see from this solution the "inherent" discontinuity of the solution, $t(\xi)$ of the functional equation (10). This is related to the homotopy impediment we discussed four paragraphs above.

We shall now list the members of the Wutam Consortium in two groups, the one associated with Texas $A\&M$ and the one associated with Washington University. We shall give the present affiliation of each individual.

From Texas A&M

Xingde Dai, University of North Carolina at Charlotte;

Qing Gu, Texas A&M University;
Deguang Han, Texas A&M University;
David Larson, Texas A&M University;
Rufeng Liang, University of North Carolina at Charlotte;
Shijin Lu, Texas A&M University;
Darrin Speegle, St. Louis University.

From Washington University

Gustavo Garrigós, Washington University;
Eugenio Hernández, Universidad Autónoma de Madrid;
Maciej Paluszyński, Wroclaw University;
Manos Papadakis, Washington University;
Hrvoje Šikić, Washington University and the University of Zagreb;
David Weiland, University of Texas;
Guido Weiss, Washington University.

Not all fourteen of these individuals were directly involved in the research we described in this paper. This is, however, the first of what we plan to be a series of expositions of the collaborative research conducted by this group. Other results have already been obtained and will be described in these future publications. Each member of the Consortium has already made a contribution to the series. As stated above, it is only fair that each one of these researchers be considered as a collaborator for this article. We believe that these results, that are so closely connected, are best presented as a whole, rather as a number of papers whose inter-relations are not clearly explained.

X.Dai and D.Speegle obtained their Ph.D. degrees at Texas A&M, R.Liang is a student of Dai, Q.Gu, D.Han, and S.Lu are students working with Prof.Larson at Texas A&M. G.Garrigós is a student at Washington University, Hernández, Paluszyński and Weiland were students of Prof.Weiss at Washington University, Papadakis and Šikić are visiting faculty members at Washington University.

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Texas *A&M* University and Washington University in St. Louis