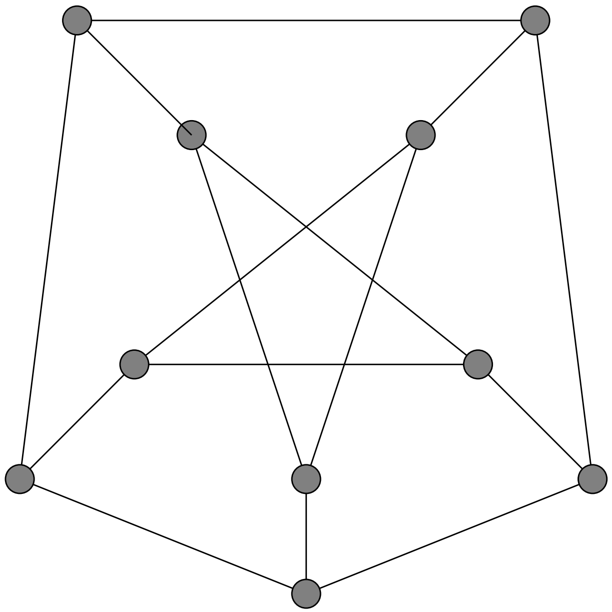
Algorithms

Graphs

Graphs



Definition: A graph is a collection of edges and vertices. Each edge connects two vertices.

Graphs

Vertices: Nodes, points, computers, users, items, . . .

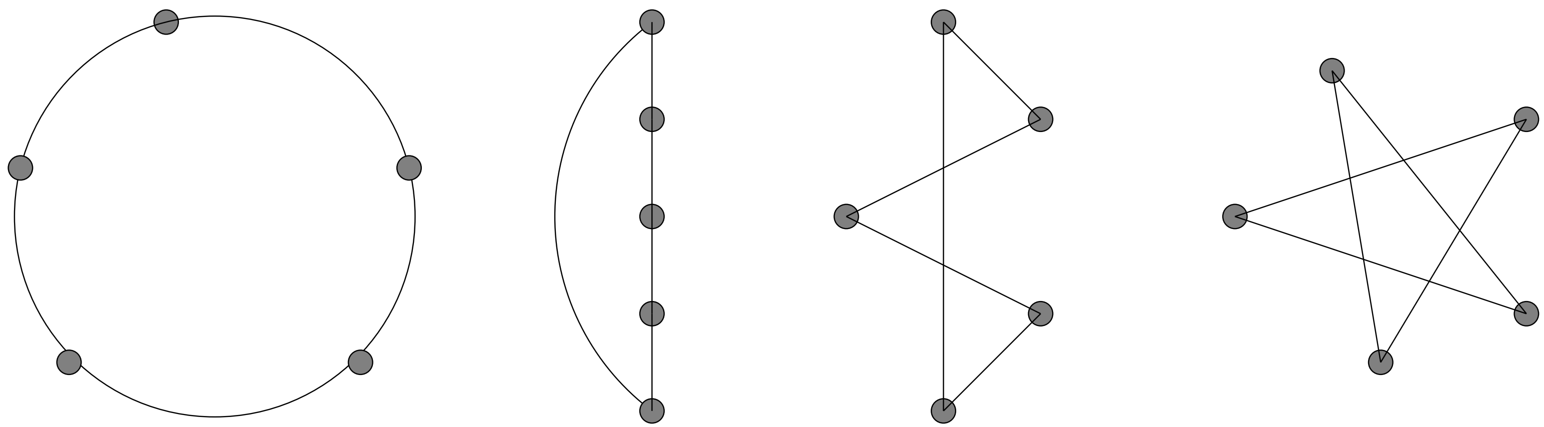
Edges: Arcs, links, lines, cables, . . .

Applications: Communication, Transportation, Databases, Electronic Circuits, . . .

An alternative definition: A graph is a collection of subsets of size 2 from the set $\{1, \ldots, n\}$. A hyper-graph is a collection of subsets of any size from the set $\{1, \ldots, n\}$.

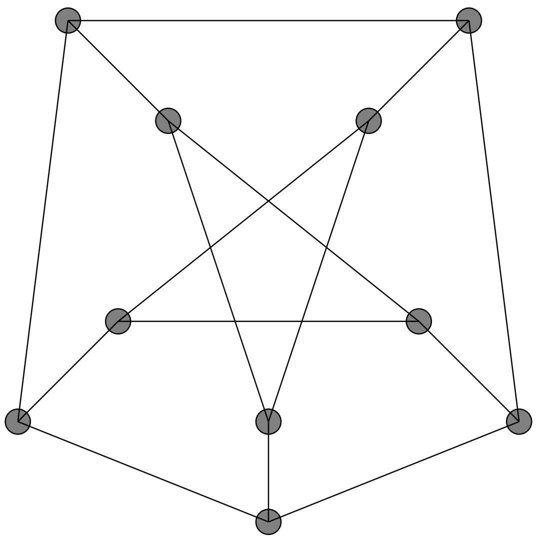
Drawing Graphs

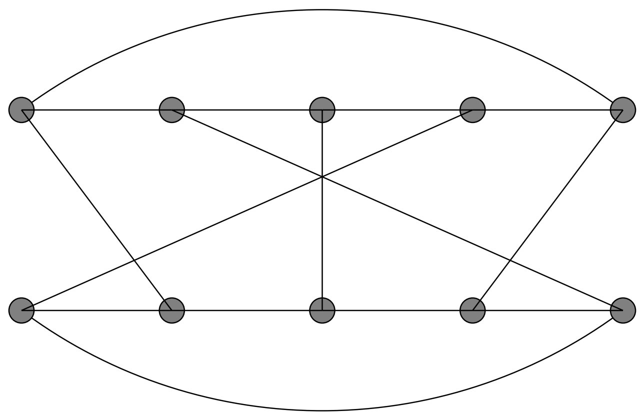
4 possible drawings illustrating the same graph:



Drawing Graphs

2 drawings representing the same graph:

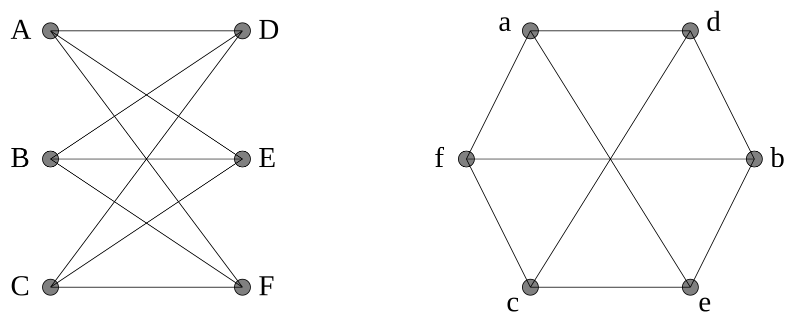




Graph Isomorphism

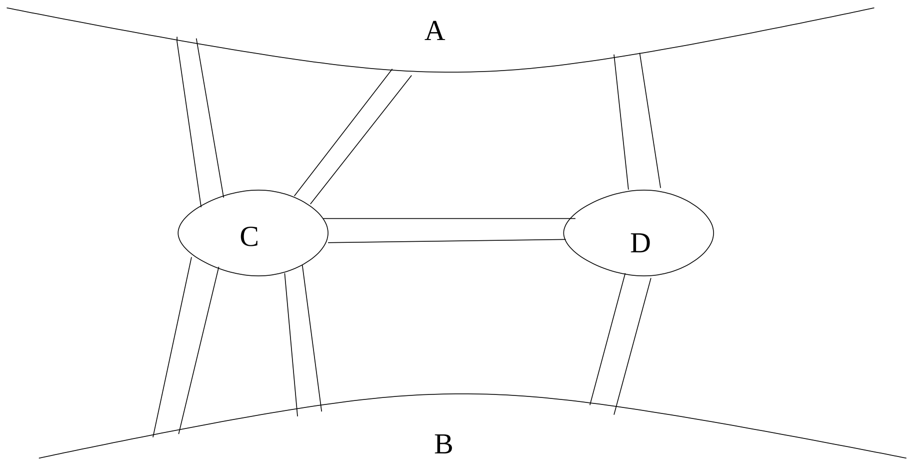
Graph G_1 and graph G_2 are isomorphic if there is one-one correspondence between their vertices such that:

number of edges joining any two vertices of G_1 is equal to number of edges joining the corresponding vertices of G_2.



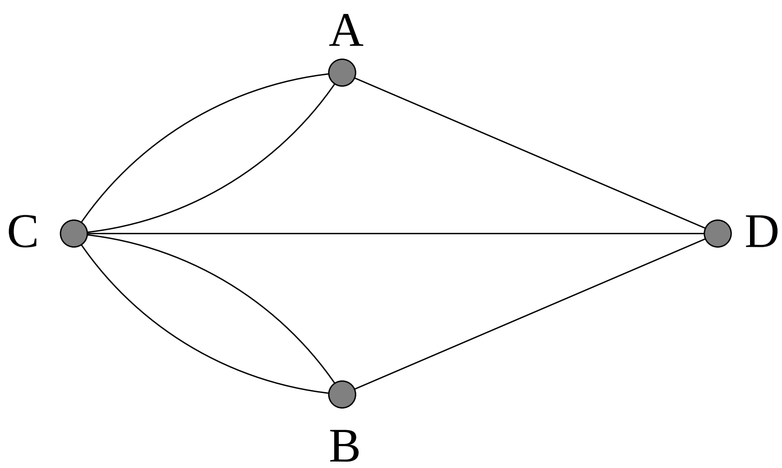
$$a \leftrightarrow A \quad b \leftrightarrow B \quad c \leftrightarrow C \quad d \leftrightarrow D \quad e \leftrightarrow E \quad f \leftrightarrow F$$

The Bridges of Königsberg



Is it possible to traverse each of the 7 bridges of this town exactly once, starting and ending at any point?

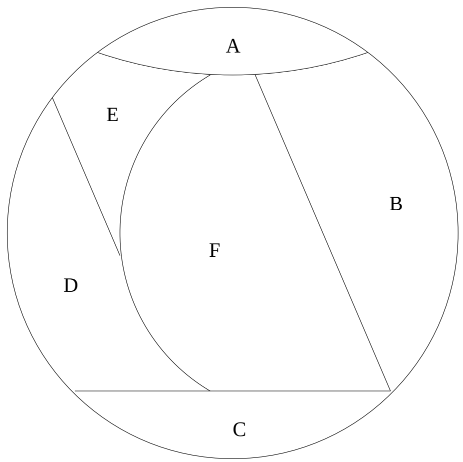
The Bridges of Königsberg



Is it possible to traverse each of the edges of this graph exactly once, starting and ending at any vertex?

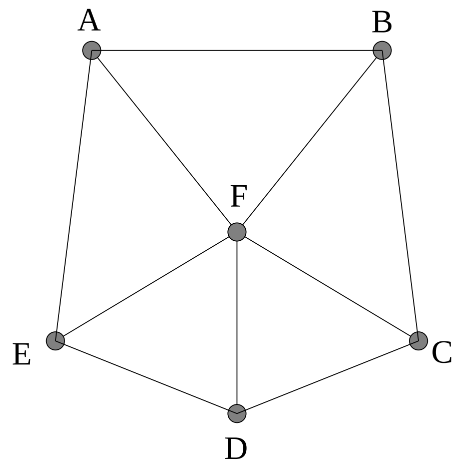
Does a graph have an Euler tour?

The Four Coloring Problem



Is it possible to color a map with at most 4 colors such that neighboring countries get different colors?

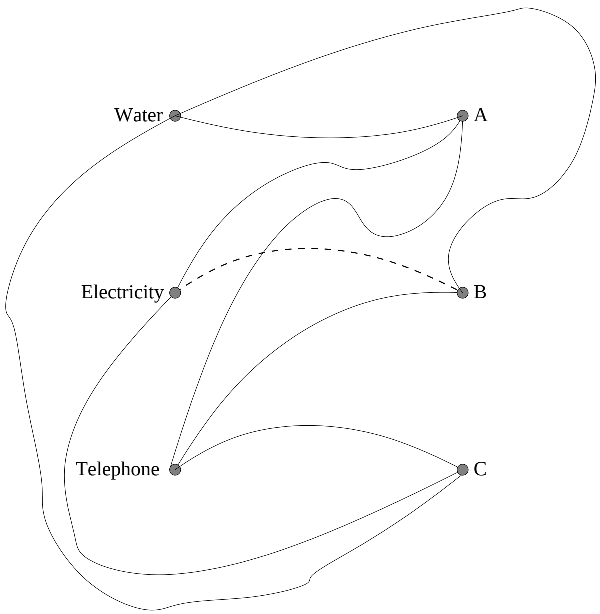
The Four Coloring Problem



Is it possible to color the vertices of this graph with at most 4 colors?

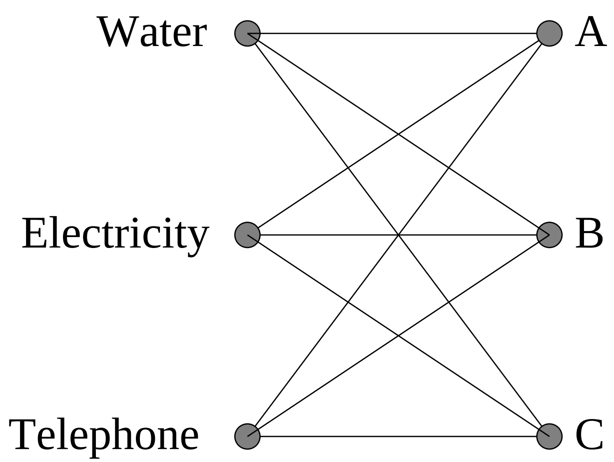
Is it possible to color every planar graph with at most 4 colors?

The Three Utilities Problem



Is it possible to connect the houses $\{A, B, C\}$ with the utilities {Water, Electricity, Telephone} such that cables do not cross?

The Three Utilities Problem



Is it possible to draw the vertices and edges of this graph such that edges do not cross?

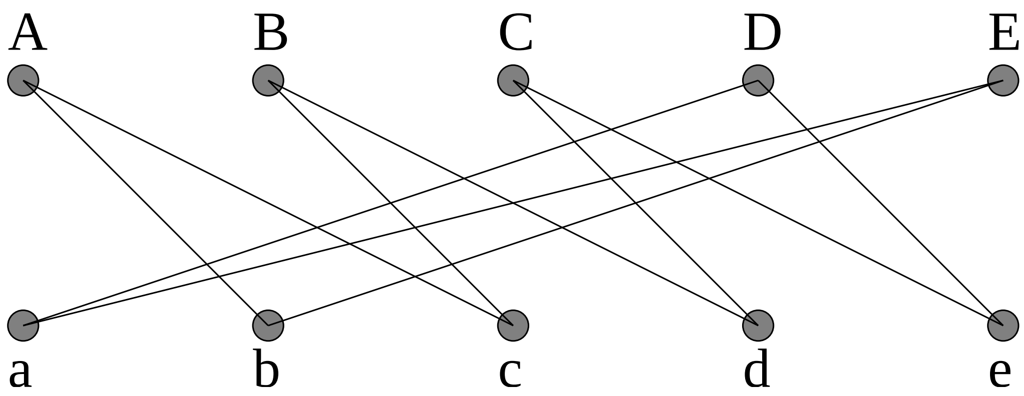
Which graphs are planar?

The Marriage Problem

Anna loves:	Bob and Charlie
Betsy loves:	Charlie and David
Claudia loves:	David and Edward
Donna loves:	Edward and Albert
Elizabeth loves:	Albert and Bob

Under what conditions a collection of girls each loves several boys can be married so that each girl marries a boy she loves?

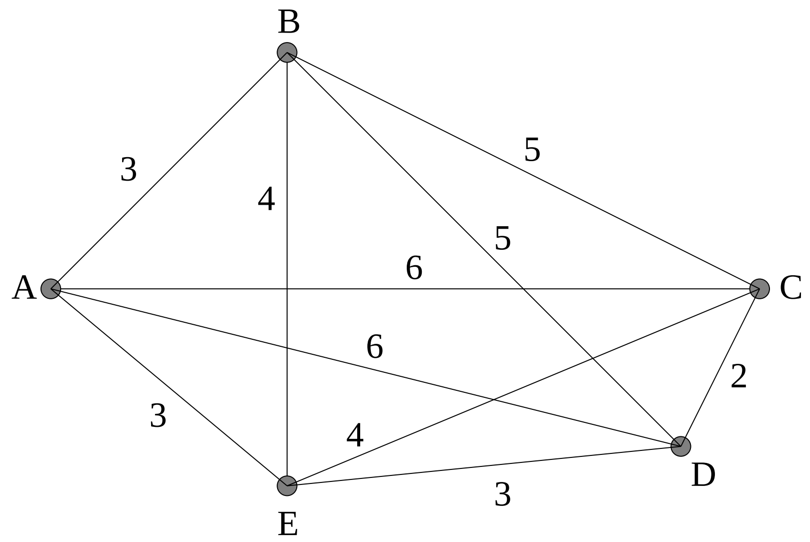
The Marriage Problem



Find in this graph a set of disjoint edges that cover all the vertices in the top side.

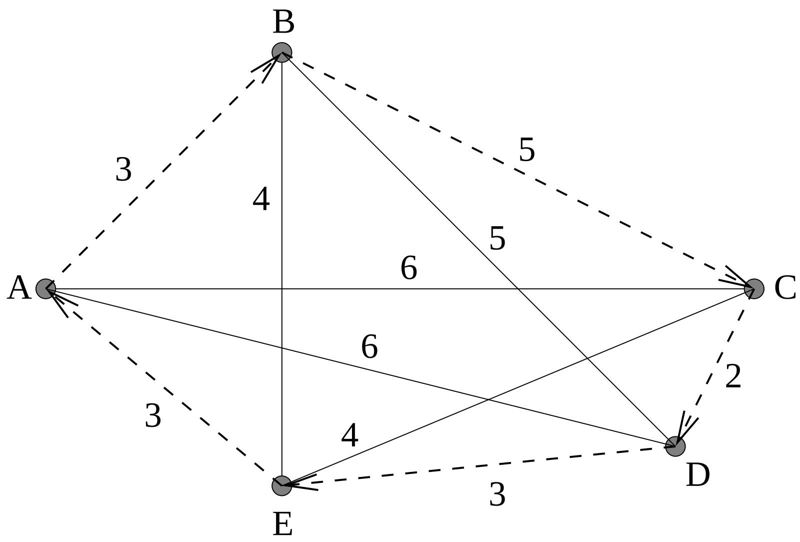
Does a (bipartite) graph have a perfect matching?

The Travelling Salesperson Problem



A salesperson wants to sell products in the above 5 cities $\{A, B, C, D, E\}$ starting at A and ending at A while travelling as little as possible.

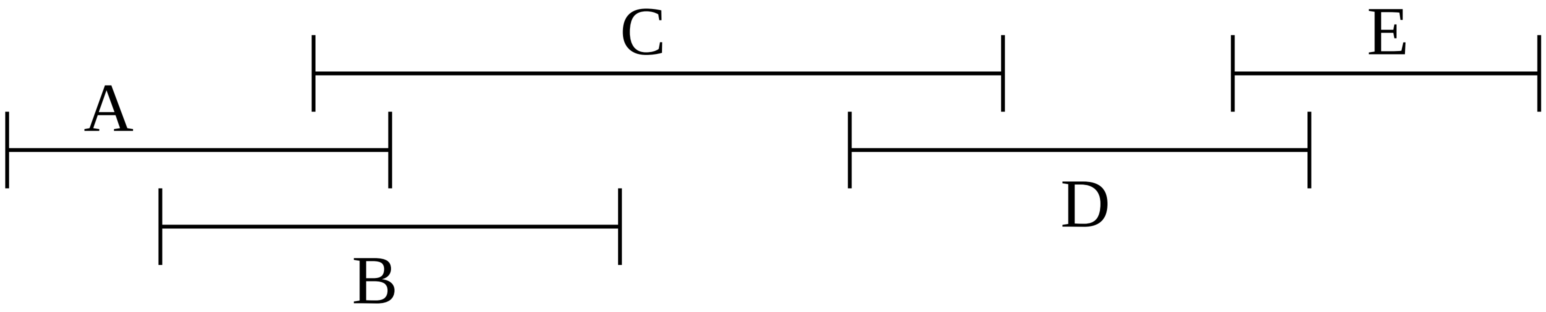
The Travelling Salesperson Problem



Find the shortest path in this graph that visits each vertex at least once and starts and ends at vertex A.

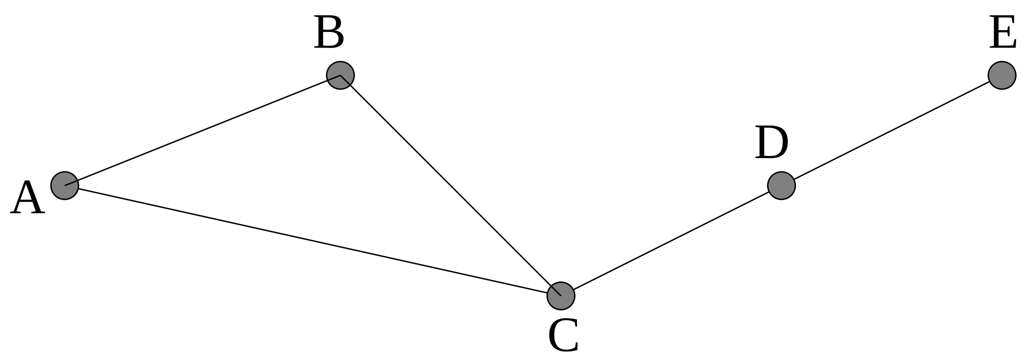
Find the shortest Hamiltonian cycle in a graph.

The Activity Center Problem



What is the maximal number of activities that can be served by a single server?

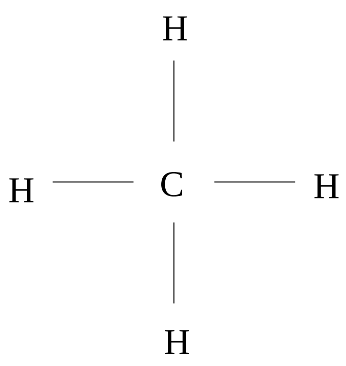
The Activity Center Problem



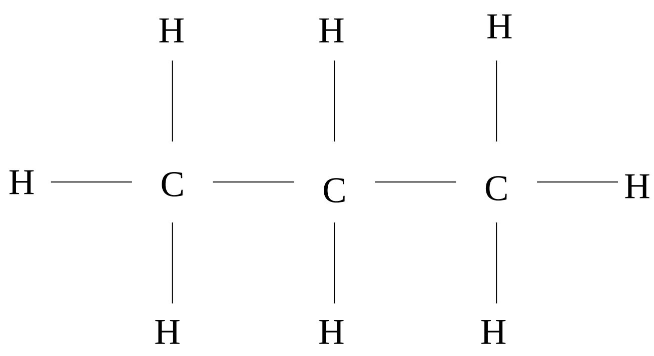
What is the maximal number of vertices in this graph with no edge between any two of them?

Find a maximum independent set in a graph.

Chemical Molecules



Methane



Propane

In the C_xH_y molecule, y hydrogen atoms are connected to x carbon atoms. A hydrogen atom can be connected to exactly one carbon atom. A carbon atom can be connected to four other atoms either hydrogen or carbon.

Chemical Molecules

How many possible structures exist for the molecule C_4H_{10}?

How many non-isomorphic connected graphs exist with x vertices of degree 4 and y vertices of degree 1?

Is there a (connected) graph whose degree sequence is $d_1 \geq \cdots \geq d_n$? How many non-isomorphic such graphs exist?

Some Notations

- $G = (V, E)$ - a graph G.
- $V = \{1, \ldots, n\}$ - a set of vertices.
- $E \subseteq V \times V$ - a set of edges.
- $e = (u, v) \in E$ - an edge.
- $|V| = V = n$ - number of vertices.
- $|E| = E = m$ - number of edges.

Directed and Undirected Graphs

In undirected graphs: $(u, v) = (v, u)$.

In directed graphs (D-graphs): $(u \rightarrow v) \neq (v \rightarrow u)$.

The underlying undirected graph $G' = (V', E')$ of a directed graph $G = (V, E)$:

- ⋆ Has the same set of vertices: $V = V'$.
- ⋆ Has all the edges of G without their direction.
  - $(u \rightarrow v)$ becomes (u, v).

Undirected Edges

- ⋆ Vertices u and v are the endpoints of the edge (u, v).
- ⋆ Edge (u, v) is incident with vertices u and v.
- ⋆ Vertices u and v are neighbors if edge (u, v) exists.
  - – u is adjacent to v and v is adjacent to u.
- ⋆ Vertex u has degree d if it has d neighbors.
- ⋆ Edge (v, v) is a (self) loop edge.
- ⋆ Edges $e_1 = (u, v)$ and $e_2 = (u, v)$ are parallel edges.

Directed Edges

- ⋆ Vertex u is the origin (initial) and vertex v is the destination (terminal) of the directed edge $(u \rightarrow v)$.
- ⋆ Vertex v is the neighbor of vertex u if the directed edge $(u \rightarrow v)$ exists.
  - v is adjacent to u but u is not adjacent to v.
- ⋆ Vertex u has
  - out-degree d if it has d neighbors.
  - in-degree d if it is the neighbor of d vertices.

Weighted Graphs

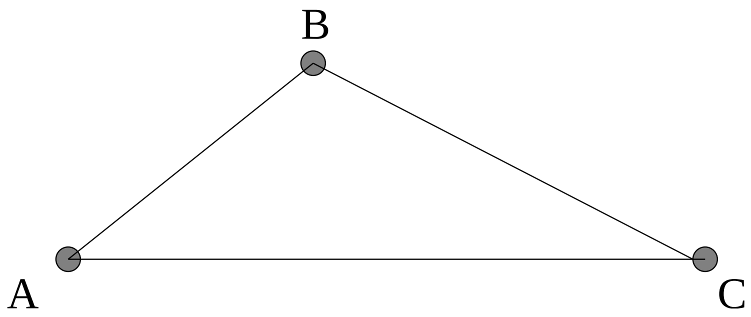
In Weighted graphs there exists a weight function:

– $w : E \rightarrow \Re$.

– w : weight, distance, length, time, cost, capacity, ...

– Weights could be negative.

The Triangle Inequality



$$w(AC) \leq w(AB) + w(BC)$$

⋆ Sometimes weights obey the triangle inequality
- Distances in the plane.

Simple Graphs

- ⋆ A simple directed or undirected graph is a graph with no parallel edges and no self loops.
- ⋆ In a simple directed graph both edges: $(u \longrightarrow v)$ and $(v \longrightarrow u)$ could exist (they are not parallel edges).

Number of Edges in Simple Graphs

- ⋆ A simple undirected graph has at most $m = \binom{n}{2}$ edges.

- ⋆ A simple directed graph has at most $m = n(n-1)$ edges.

- ⋆ A dense simple (directed or undirected) graph has many edges: $m = \Theta(n^2)$.

- ⋆ A sparse (shallow) simple (directed or undirected) graph has few edges: $m = \Theta(n)$.

Labelled and Unlabelled Graphs

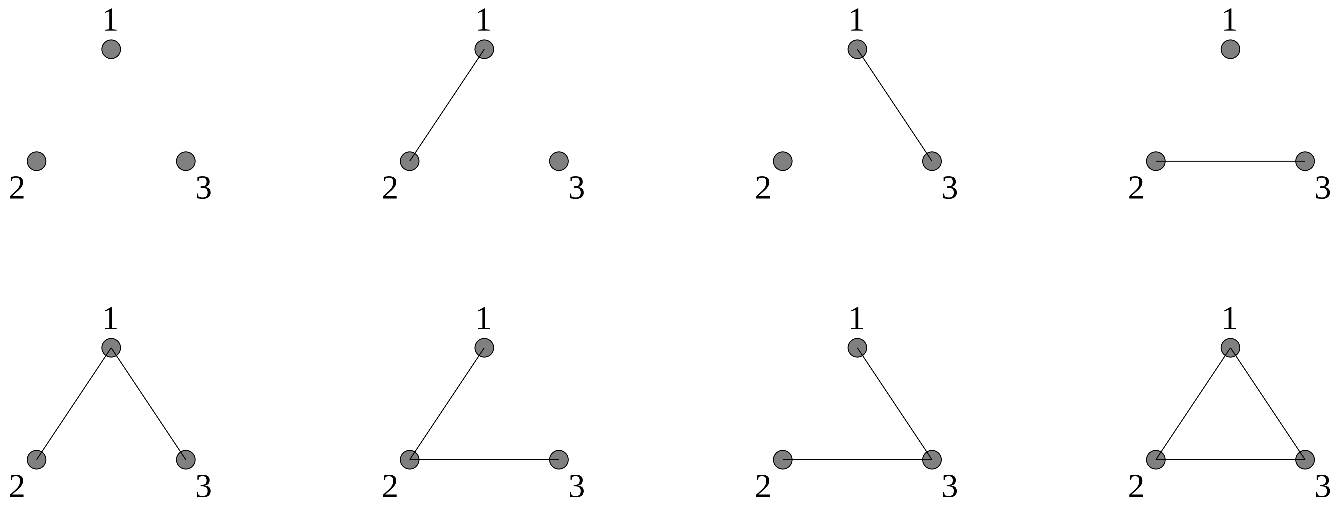
In a labelled graph each vertex has a unique label (ID):

– Usually the labels are: $1, \ldots, n$.

Observation: There are $2^{\binom{n}{2}}$ non-isomorphic labelled graphs with n vertices.

Proof: Each possible edge exists or does not exist.

Labelled Graphs



The 8 labelled graphs with $n = 3$ vertices.

Unlabelled Graphs

The 4 unlabelled graph with $n = 3$ vertices.

Paths and Cycles

- ⋆ An undirected or directed path $\mathcal{P} = \langle v_0, v_1, \ldots, v_k \rangle$ of length k is an ordered list of vertices such that (v_i, v_{i+1}) or $(v_i \rightarrow v_{i+1})$ exists for $0 \leq i \leq k-1$ and all the edges are different.
- ⋆ An undirected or directed cycle $\mathcal{C} = \langle v_0, v_1, \ldots, v_{k-1}, v_0 \rangle$ of length k is an undirected or directed path that starts and ends with the same vertex.
- ⋆ In a simple path, directed or undirected, all the vertices are different.
- ⋆ In a simple cycle, directed or undirected, all the vertices except $v_0 = v_k$ are different.

Special Paths and Cycles

⋆ An undirected or directed Euler path (tour):

– a path that traverses all the edges.

⋆ An undirected or directed Euler cycle (circuit):

– a cycle that traverses all the edges.

⋆ An undirected or directed Hamiltonian path (tour):

– a simple path that visits all the vertices.

⋆ An undirected or directed Hamiltonian cycle (circuit):

– a simple cycle that visits all the vertices.

Connected Graphs

Connectivity: In connected undirected graphs there exists a path between any pair of vertices.

Observation: In a simple connected undirected graph there are at least $m = n - 1$ edges.

Strong connectivity: In a strongly connected directed graph there exists a directed path from u to v for any pair of vertices u and v.

Observation: In a simple strongly connected directed graph there are at least $m = n$ edges.

Weakly Connected Directed Graphs

Definition I: In a weakly connected directed graph there exists a directed path either from u to v or from v to u for any pair of vertices u and v.

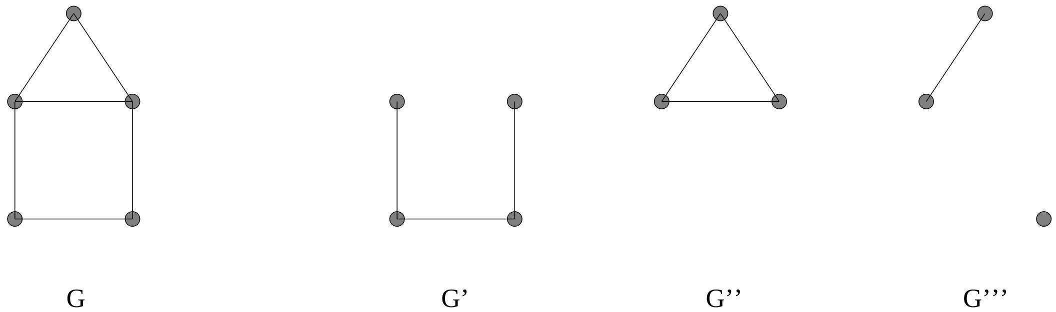
Definition II: In a weakly connected directed graph there exists a path between any pair of vertices in the underlying undirected graph.

Observation: The definitions are not equivalent.

Sub-Graphs

A (directed or undirected) Graph $G' = (V', E')$ is a sub-graph of a (directed or undirected) graph $G = (V, E)$ if:

– $V' \subseteq V$ and $E' \subseteq E$.



G', G'', G''' are sub-graphs of G

Connected Components - Undirected Graphs

- ⋆ A connected sub-graph G' is a connected component of an undirected graph G if there is no connected sub-graph G'' of G such that G' is also a subgraph of G''.
- ⋆ A connected component G' is a maximal sub-graph with the connectivity property.
- ⋆ A connected graph has exactly one connected component.

Connected Components - Directed Graphs

- ⋆ A strongly connected directed sub-graph G' is a strongly connected component of a directed graph G if there is no strongly connected directed sub-graph G'' of G such that G' is also a subgraph of G''.
- ⋆ A strongly connected component G' is a maximal sub-graph with the strong connectivity property.
- ⋆ A strongly connected graph has exactly one strongly connected component.

Counting Edges

Theorem: Let G be a simple undirected graph with n vertices and k connected components then:

$$n - k \leq m \leq \frac{(n-k)(n-k+1)}{2} .$$

Corollary: A simple undirected graph with n vertices is connected if it has m edges for:

$$m > \frac{(n-1)(n-2)}{2}$$

Assumptions

Unless stated otherwise, usually a graph is:

- Simple.
- Undirected.
- Connected.
- Unweighted.
- Unlabelled.

Forests and Trees

Forest: A graph with no cycles.

Tree: A connected graph with no cycles.

By definition:

– A tree is a connected forest.
– Each connected component of a forest is a tree.

Trees

Theorem: An undirected and simple graph is a tree if:

- It is connected and has no cycles.
- It is connected and has exactly $m = n - 1$ edges.
- It has no cycles and has exactly $m = n - 1$ edges.
- It is connected and deleting any edge disconnects it.
- Any 2 vertices are connected by exactly one path.
- It has no cycles and any new edge forms one cycle.

Corollary: The number of edges in a forest with n vertices and k trees is $m = n - k$.

Rooted and Ordered Trees

Rooted trees:

⋆ One vertex is designated as the root.

⋆ Vertices with degree 1 are called leaves.

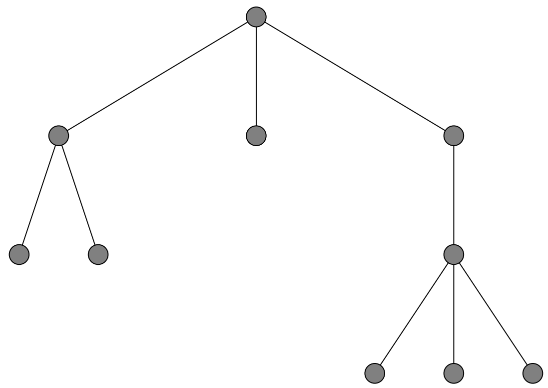
⋆ Non-leaves vertices are internal vertices.

⋆ All the edges are directed from the root to the leaves.

Ordered trees:

⋆ Children of an internal parent vertex are ordered.

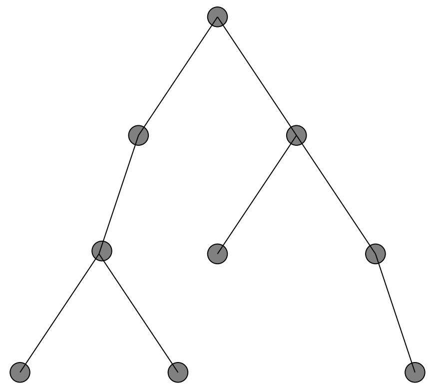
Drawing Rooted Trees



⋆ Parents above children.

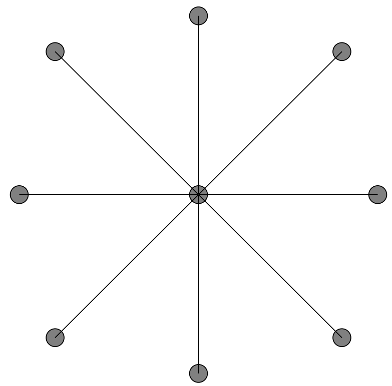
⋆ Older children to the left of younger children.

Binary Trees



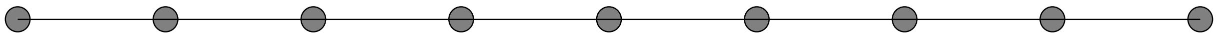
Binary trees: The root has degree either 1 or 2, the leaves have degree 1, and the degree of non-root internal vertices is either 2 or 3.

Star Trees



Star: A rooted tree with 1 root and $n-1$ leaves. The degree of one vertex (the root) is $n-1$ and the degree of any non-root vertex is 1.

Path Trees

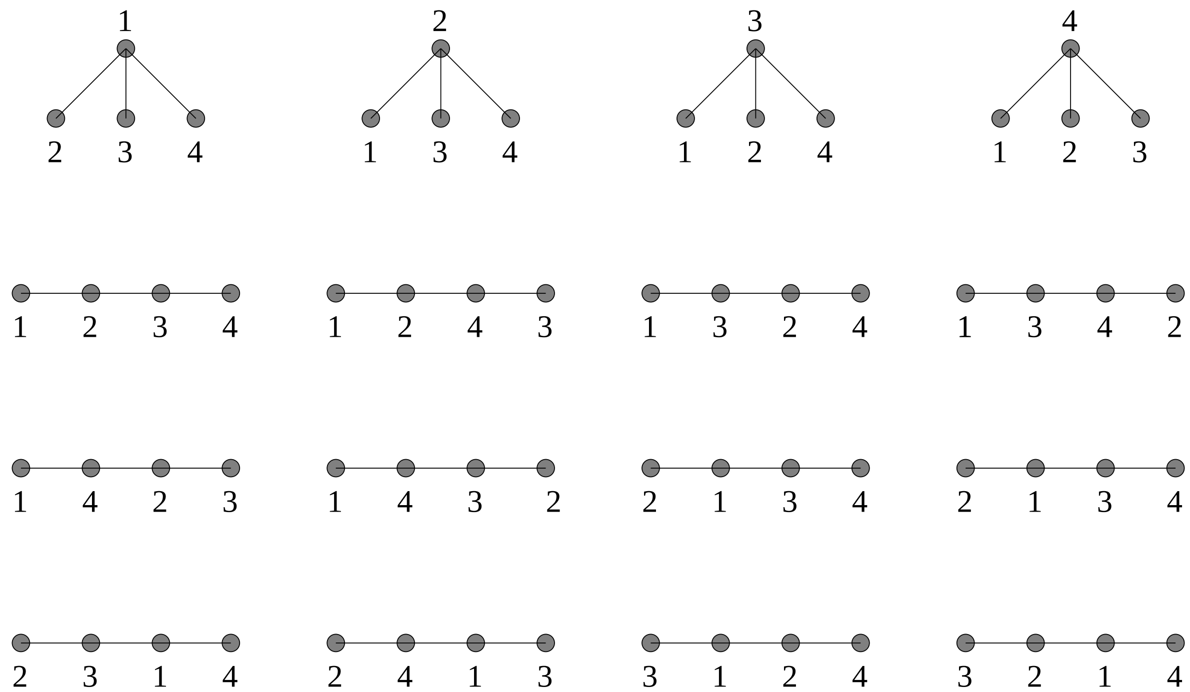


Path: A tree with exactly 2 leaves.

Claim I: The degree of a non-leave vertex is exactly 2.

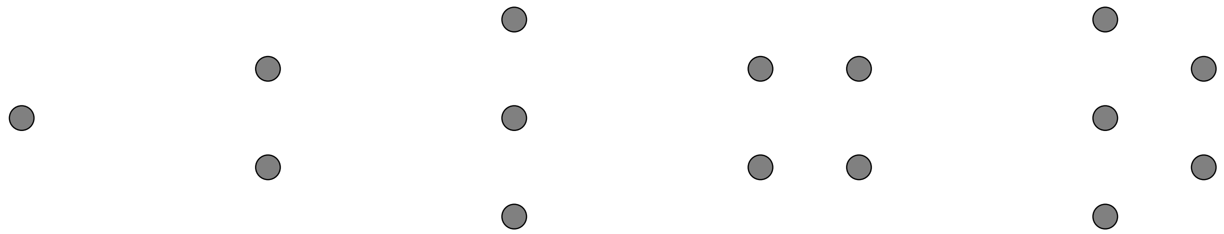
Claim II: The path is the only tree with exactly 2 leaves.

Counting Labelled Trees



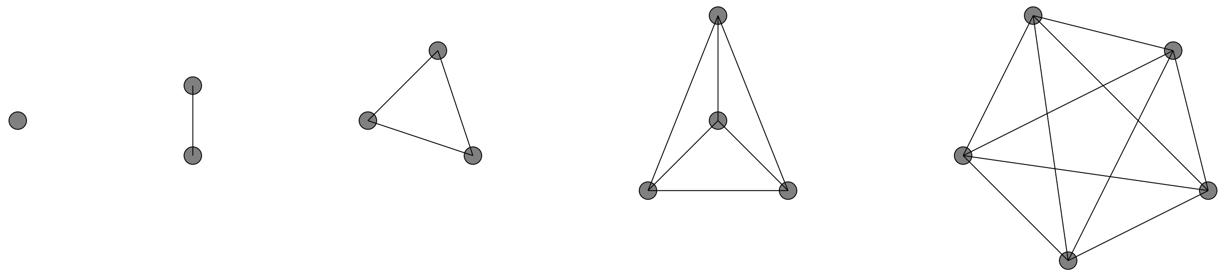
Theorem: There are n^{n-2} distinct labelled n vertices trees.

Null Graphs



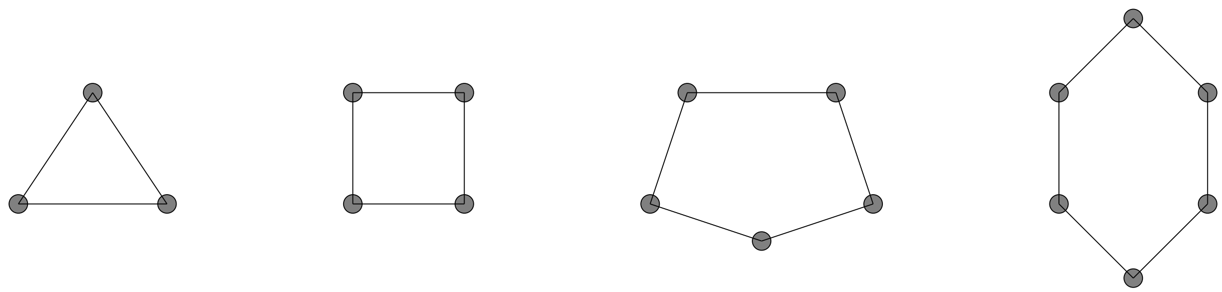
- ⋆ Null graphs are graphs with no edges.
- ⋆ The null graph with n vertices is denoted by N_n.
- ⋆ In null graphs $m = 0$.

Complete Graphs



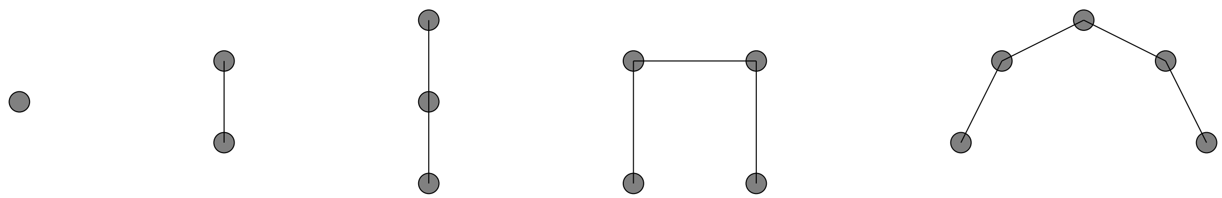
- ⋆ Complete graphs (cliques) are graphs with all possible edges.
- ⋆ The complete graph with n vertices is denoted by K_n.
- ⋆ In complete graphs $m = \binom{n}{2} = \frac{n(n-1)}{2}$.

Cycles



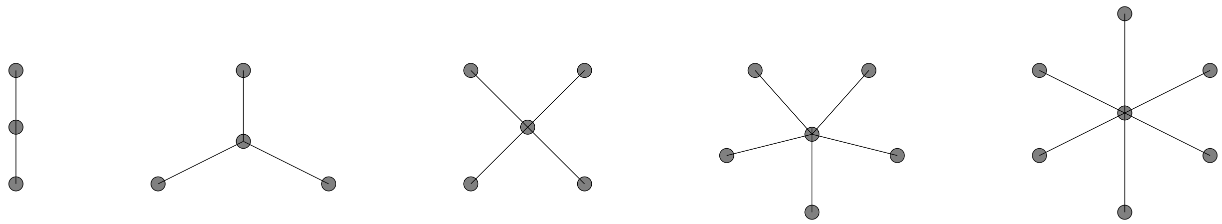
- Cycles (rings) are connected graphs in which all vertices have degree 2 ($n \geq 3$).
- The cycle with n vertices is denoted by C_n.
- In cycles $m = n$.

Paths



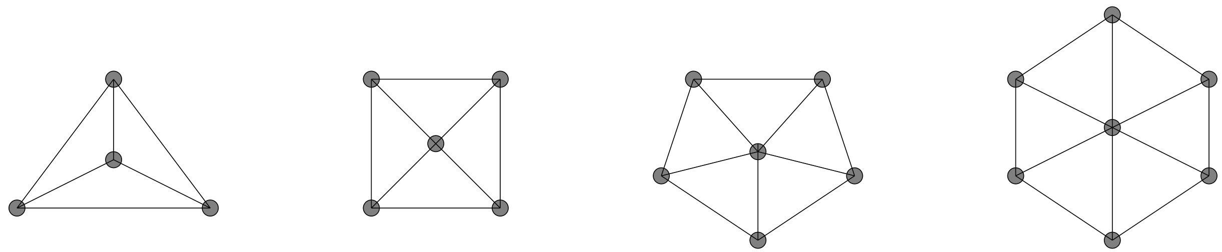
- Paths are cycles with one edge removed.
- The path with n vertices is denoted by P_n.
- In paths $m = n - 1$.

Stars



- ⋆ Stars are graphs with one root and $n-1$ leaves.
- ⋆ The star with n vertices is denoted by S_n.
- ⋆ In stars $m = n - 1$.

Wheels

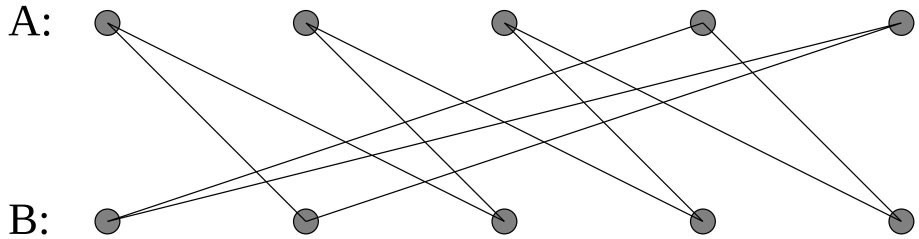


- ⋆ Wheels are stars in which all the $n-1$ leaves form a cycle C_{n-1} $(n \geq 4)$.

- ⋆ The wheel with n vertices is denoted by W_n.

- ⋆ In wheels $m = 2n - 2$.

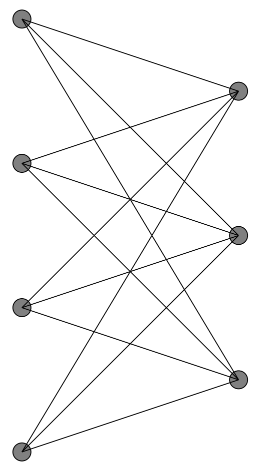
Bipartite Graphs



Bipartite graphs $V = A \cup B$: each edge is incident to one vertex from A and one vertex from B.

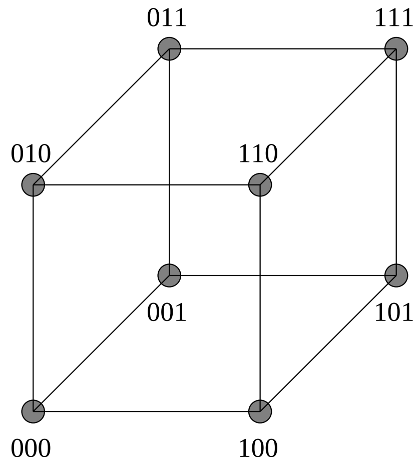
Observation: A graph is bipartite iff each cycle is of even length.

Complete Bipartite Graphs



Complete bipartite graphs $K_{r,c}$: All possible $r \cdot c$ edges exist.

Cubes



- ⋆ There are $n = 2^k$ vertices representing all the binary sequences of length k.
- ⋆ Two vertices are connected by an edge if their corresponding sequences differ by exactly one bit.

Cubes

Observation: Cubes are bipartite graphs.

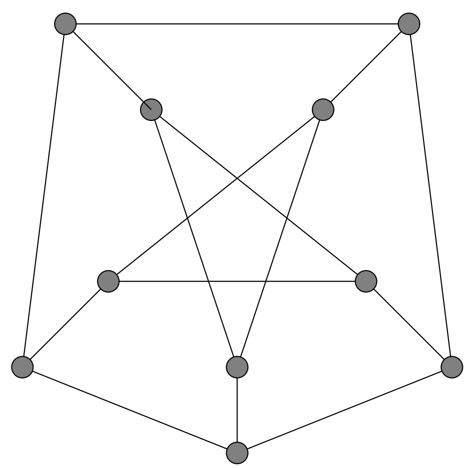
Proof:

- ⋆ A: The vertices with even number of 1 in their binary representation.
- ⋆ B: The vertices with odd number of 1 in their binary representation.
- ⋆ Any edge connects 2 vertices one from the set A and one from the set B.

d-regular Graphs

In d-regular graphs, the degree of each vertex is exactly d.

In d-regular graphs, $m = \frac{d \cdot n}{2}$.



The Petersen Graph: a 3-regular graph.

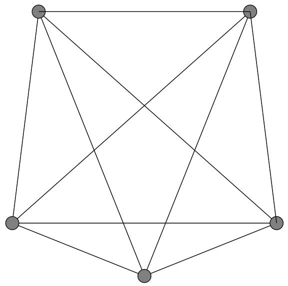
Planar Graphs

Definition: Planar graphs are graphs that can be drawn on the plane such that edges do not cross each other.

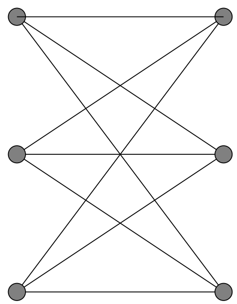
Theorem: A graph is planar if and only if it does not have sub-graphs homeomorphic to K_5 and $K_{3,3}$.

Theorem: Every planar graph can be drawn with straight lines.

Non-Planar Graphs



K_5: the complete graph with 5 vertices.



$K_{3,3}$: the complete $\langle 3, 3 \rangle$ bipartite graph.

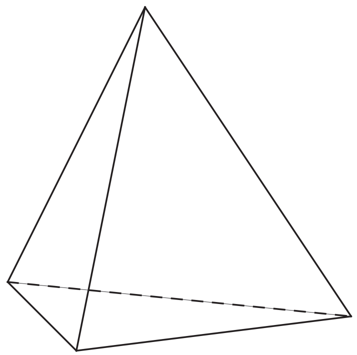
Platonic Graphs

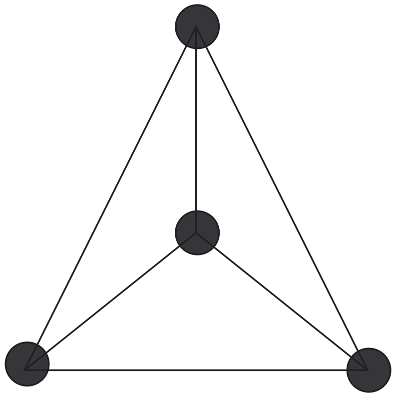
Graphs that are formed from the vertices and edges of the five regular (Platonic) solids:

- Tetrahedron: 4 vertices 3-regular graph.
- Octahedron: 6 vertices 4-regular graph.
- Cube: 8 vertices 3-regular graph.
- Icosahedron: 12 vertices 5-regular graph.
- Dodecahedron: 20 vertices 3-regular graph.

Observation: The platonic graphs are d-regular planar graphs.

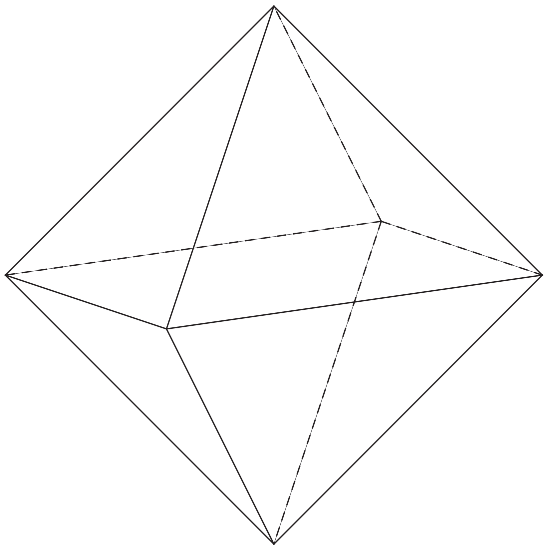
The Tetrahedron

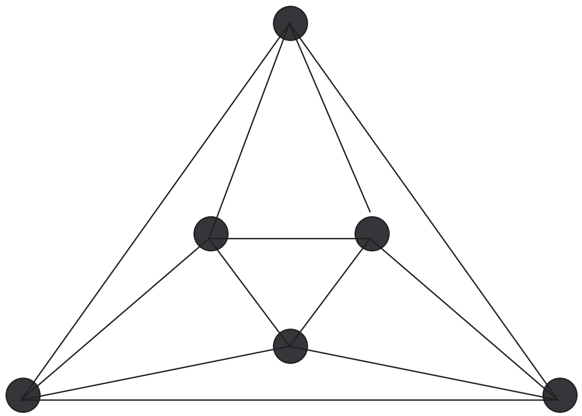




4 vertices; 6 edges; 4 faces; degree 3

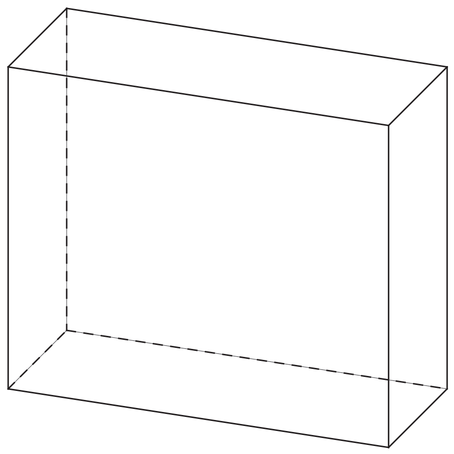
The Octahedron

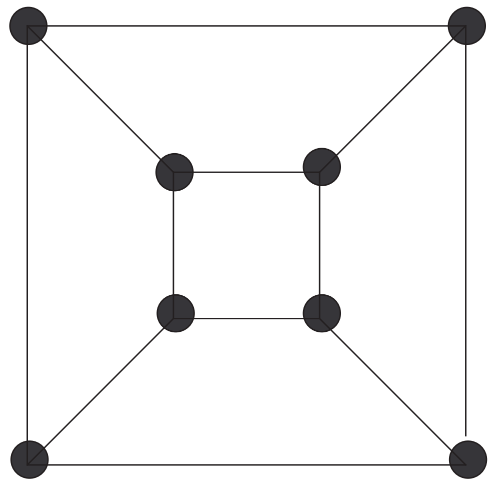




6 vertices; 12 edges; 8 faces; degree 4

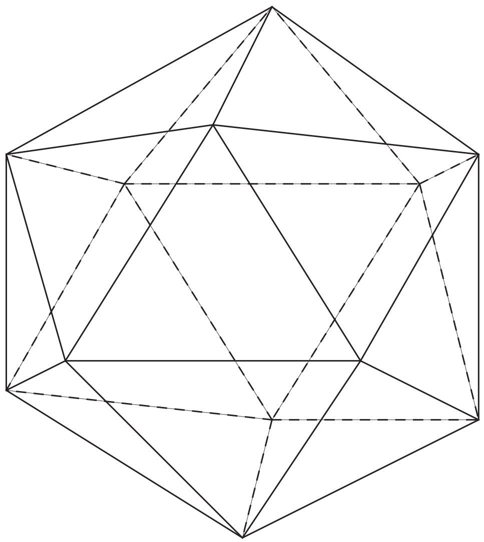
The Cube

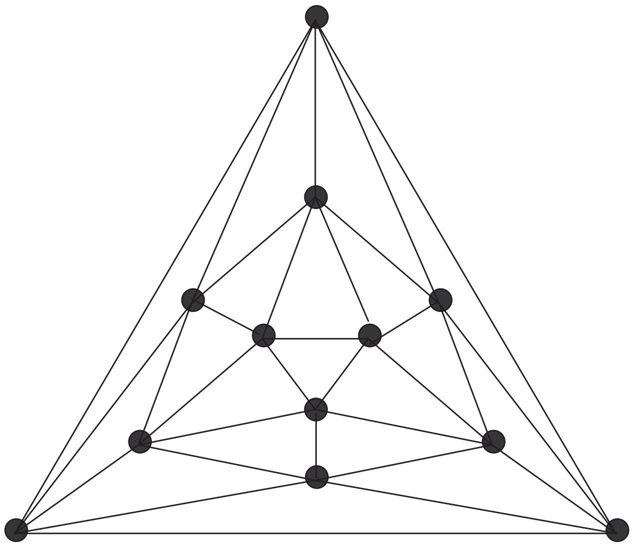




8 vertices; 12 edges; 6 faces; degree 3

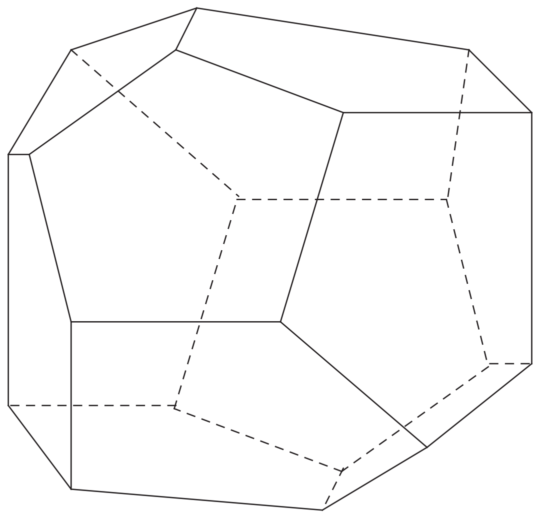
The Icosahedron

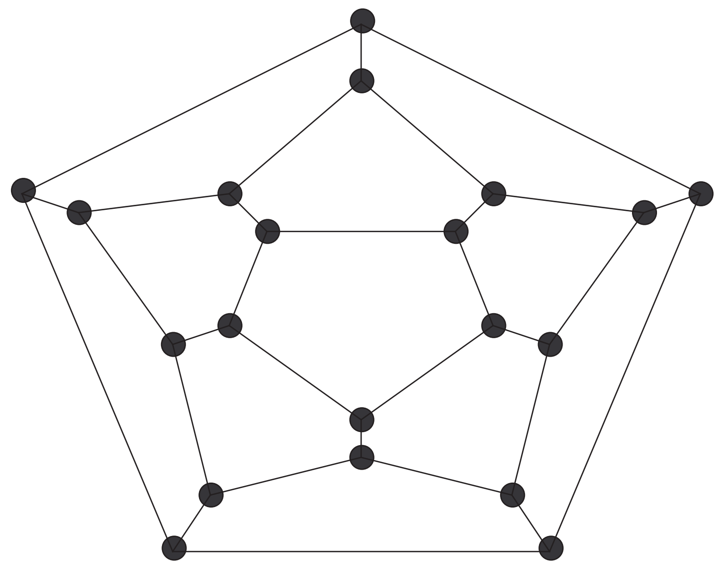




12 vertices; 30 edges; 20 faces; degree 5

The Dodecahedron





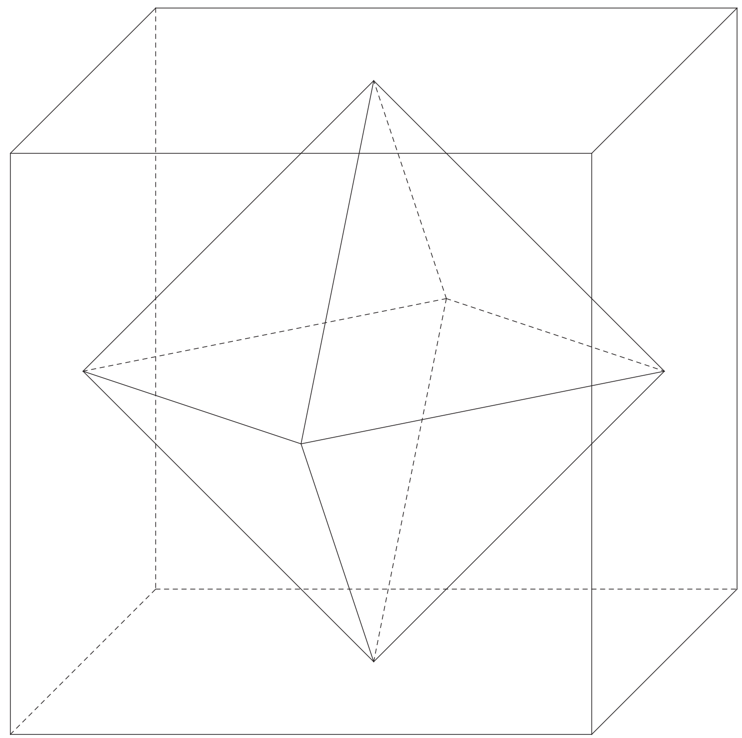
20 vertices; 30 edges; 12 faces; degree 3

Dual Planar Graphs

In the dual planar graph G^* of a planar graph G vertices correspond to faces of G and two vertices in G^* are joined by an edge if the corresponding faces in G share an edge.

- The Octahedron is the dual graph of the Cube.
- The Cube is the dual graph of the Octahedron.
- The Icosahedron is the the dual graph of the Dodecahedron.
- The Dodecahedron is the the dual graph of the Icosahedron.
- The Tetrahedron is the dual graph of itself.

Duaity of the Cube and the Octahedron



Random Graphs

Definition I:

⋆ Each edge exists with probability $0 \leq p \leq 1$.

⋆ Observation: Expected number of edges is $E(m) = p\binom{n}{2}$.

Definition II:

⋆ A graph with m edges that is selected randomly with a uniform distribution over all graphs with m edges.

Interval Graphs

- ⋆ Vertices represent intervals on the x-axis.
- ⋆ An edge indicates that two intervals intersect.

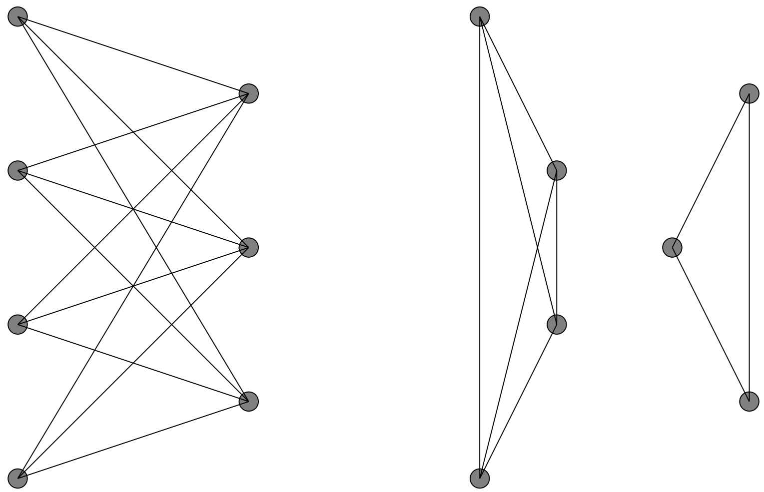
Complement Graphs

- ⋆ $\tilde{G} = (\tilde{V}, \tilde{E})$ is the complement graph of $G = V, E)$ if:
  - – $V = \tilde{V}$ and $(x, y) \in E \leftrightarrow (x, y) \notin \tilde{E}$.
- ⋆ A graph G is self-complementary if it is isomorphic to $\tilde{G}$.
- ⋆ Lemma: At least one of G and $\tilde{G}$ is connected.

Complement Graphs – Observation

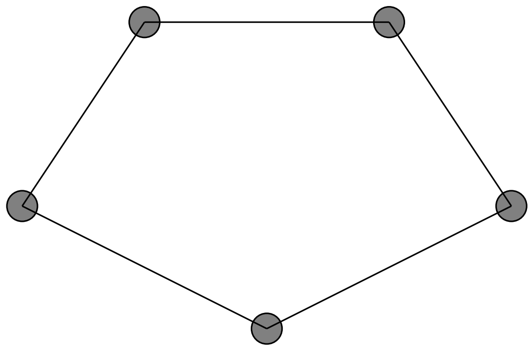
$$N_n = \tilde{K_n}.$$

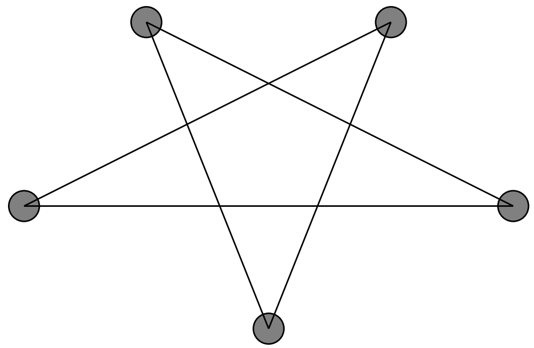
Complement Graphs – Observation



$$\tilde{K_{r,s}} = K_r \cup K_s.$$

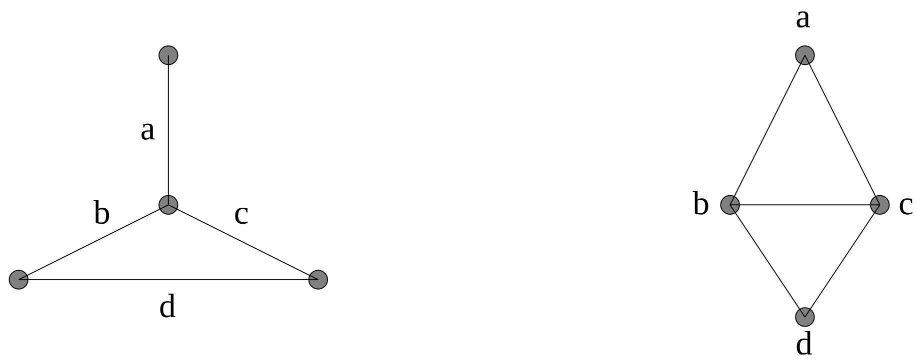
Complement Graphs – Observation





$$C_5 = \tilde{C}_5.$$

Line Graphs

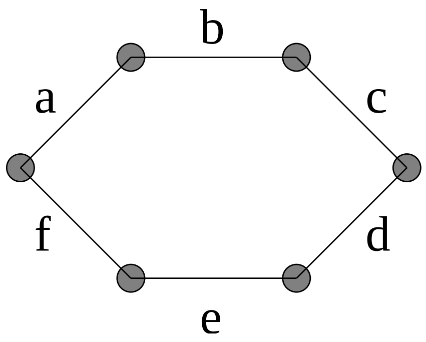


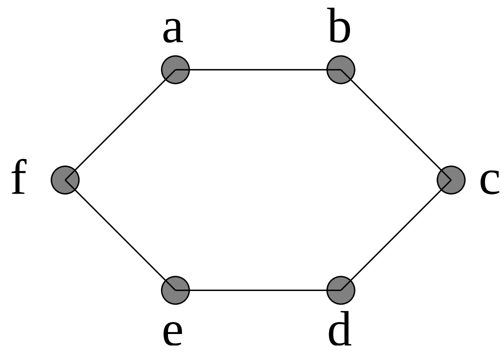
In the line graph $L(G) = (E, F)$ of $G = (V, E)$ vertices correspond to edges of G and two vertices in $L(G)$ are joined by an edge if the corresponding edges in G share a vertex.

$$(e_i, e_j) \in F \text{ iff } e_i = (x, y) \text{ and } e_j = (y, z) \text{ for } x, y, z \in V.$$

Observation: $L(L(G)) = G$ is a wrong statement.

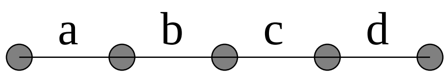
Line Graphs – Observation

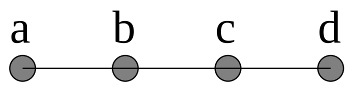




$$L(C_n) = C_n.$$

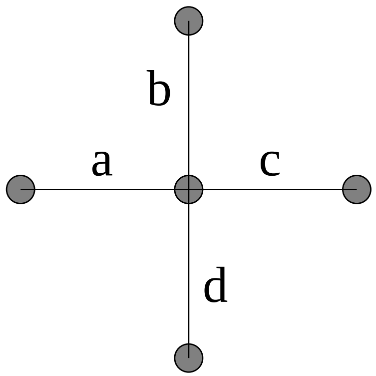
Line Graphs – Observation

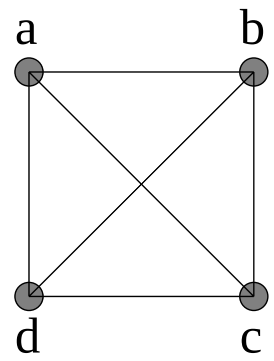




$$L(P_n) = P_{n-1}.$$

Line Graphs – Observation





$$L(S_n) = K_{n-1}.$$

Social Graphs

Definition: The social graph contains all the friendship relations (edges) among n people (vertices).

I: In any group of $n \geq 2$ people, there are 2 people with the same number of friends in the group.

II: There exists a group of 5 people for which no 3 are mutual friends and no 3 are mutual strangers.

III: Every group of 6 people contains either three mutual friends or three mutual strangers.

Data structure for Graphs

- ⋆ Adjacency lists: $\Theta(m)$ memory.
- ⋆ An adjacency Matrix: $\Theta(n^2)$ memory.
- ⋆ An incident matrix: $\Theta(n \cdot m)$ memory.

The Adjacency Lists Representation

- ⋆ Each vertex is associated with a linked list consisting of all of its neighbors.

- ⋆ In a directed graph there are 2 lists:
  – an incoming list and an outgoing list.

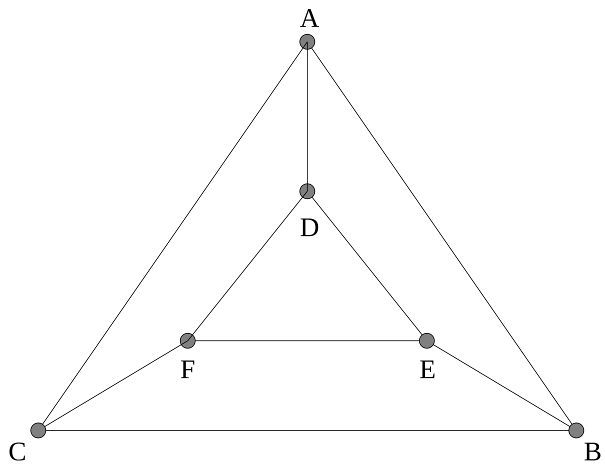
- ⋆ In a weighted graph each record in the list has an additional field for the weight.

Memory: $\Theta(n+m)$.

– Undirected graphs: $\sum_v Deg(v) = 2m$

– Directed graphs: $\sum_v OutDeg(v) = \sum_v InDeg(v) = m$

Example – Adjacency Lists



$$\begin{array}{lcl} A & \longrightarrow & (B, C, D) \\ B & \longrightarrow & (A, C, E) \\ C & \longrightarrow & (A, B, F) \\ D & \longrightarrow & (A, E, F) \\ E & \longrightarrow & (B, D, F) \\ F & \longrightarrow & (C, D, E) \end{array}$$

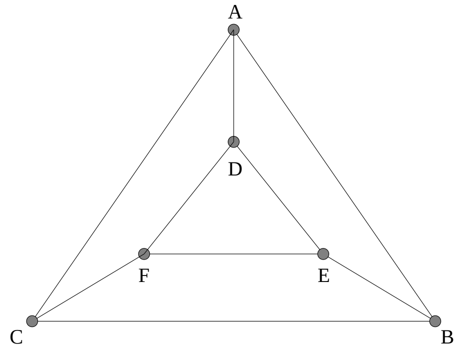
The Adjacency Matrix Representation

- $\star$ A matrix A of size $n \times n$:
  - $A[u, v] = 1$ if (u, v) or $(u \rightarrow v)$ is an edge.
  - $A[u, v] = 0$ if (u, v) or $(u \rightarrow v)$ is not an edge.
- $\star$ In simple graphs: $A[u, u] = 0$
- $\star$ In undirected graphs: $A[u, v] = A[v, u]$
- $\star$ In weighted graphs: $A[u, v] = w(u, v)$

Memory: $\Theta(n^2)$.

– Independent of m that could be much smaller than $\Theta(n^2)$.

Example – Adjacency Matrix



	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	1	0	1	0
C	1	1	0	0	0	1
D	1	0	0	0	1	1
E	0	1	0	1	0	1
F	0	0	1	1	1	0

The Incident Matrix Representation

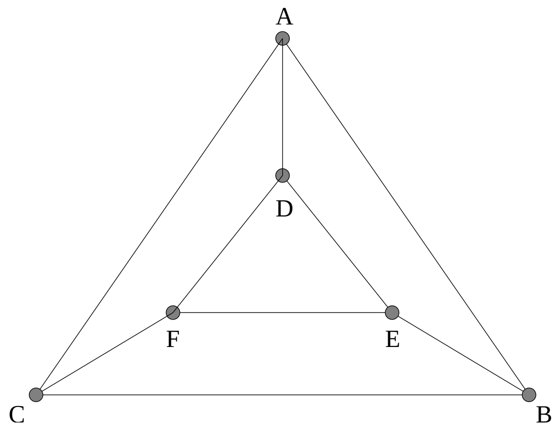
- ⋆ A matrix A of size $n \times m$:
  - $A[v, e] = 1$ if undirected edge e is incident with v.
  - $A[u, e] = -1$ and $A[v, e] = 1$ for a directed edge $u \rightarrow v$.
  - Otherwise $A[v, e] = 0$.

- ⋆ In simple graphs all the columns are different and each contains exactly 2 non-zero entries.

- ⋆ In weighted undirected graphs: $A[v, e] = w(e)$ if edge e is incident with vertex v.

Memory: $\Theta(n \cdot m)$.

Example – Incident Matrix



	(A, B)	(A, C)	(A, D)	(B, C)	(B, E)	(C, F)	(D, E)	(D, F)	(E, F)
A	1	1	1	0	0	0	0	0	0
B	1	0	0	1	1	0	0	0	0
C	0	1	0	1	0	1	0	0	0
D	0	0	1	0	0	0	1	1	0
E	0	0	0	0	1	0	1	0	1
F	0	0	0	0	0	1	0	1	1

Which Data Structure to Choose?

- Adjacency matrices are simpler to implement and maintain.
- Adjacency matrices are better for dense graphs.
- Adjacency lists are better for sparse graphs.
- Adjacency lists are better for algorithms whose complexity depends on m.
- Incident matrices are usually not efficient for algorithms.

Graphic Graphs

- ⋆ The degree d_x of vertex x in graph G is the number of neighbors of x in G.

- ⋆ The hand-shaking Lemma: $\sum_{i=1}^{n} d_i = 2m$.

- ⋆ Corollary: Number of odd degree vertices is even.

- ⋆ The degree sequence of G is $S = (d_1, \ldots, d_n)$.

- ⋆ A sequence $S = (d_1, \ldots, d_n)$ is graphic if there exists a graph with n vertices whose degree sequence is S.

Non-Graphic Graphs

⋆ $(3, 3, 3, 3, 3, 3, 3)$ is not graphic (equivalently, there is no 7-vertex 3-regular graph).
  – Since $\sum_{i=1}^{n} d_i$ is odd.

⋆ $(5, 5, 4, 4, 0)$ is not graphic.
  – Since there are 5 vertices and therefore the maximum degree could be at most 4.

⋆ $(3, 2, 1, 0)$ is not graphic.
  – Since there are 3 positive degree vertices and only one vertex with degree 3.

Graphic Graphs – Observations

I In a graphic sequence $S = (d_1 \geq \cdots \geq d_n)$ $d_1 \leq n - 1$.

II In a graphic sequence $S = (d_1 \geq \cdots \geq d_n)$ $d_{d_1+1} > 0$.

III The sequence $(0, 0, \ldots, 0)$ of length n is graphic. Since it represents the null graph N_n.

Transformation

Let $S = (d_1 \geq \cdots \geq d_n)$, then

$$f(S) = (d_2 - 1 \geq \cdots \geq d_{d_1+1} - 1, d_{d_1+2} \geq \cdots \geq d_n).$$

Example:

$$S = (5, 4, 3, 3, 2, 1, 1, 1)$$

$$f(S) = (3, 2, 2, 1, 0, 1, 1)$$

Lemma

⋆ $S = (d_1 \geq \cdots \geq d_n)$ is graphic iff $f(S)$ is graphic.

⇐ To get a graphic representation for S, add a vertex of degree d_1 to the graphic representation of $f(S)$ and connect this vertex to all vertices whose degrees in $f(S)$ are smaller by 1 than those in S.

⇒ To get a graphic representation for $f(S)$, omit a vertex of degree d_1 from the graphic representation of S. Make sure (how?) that this vertex is connected to the vertices whose degrees are $d_2, \ldots, d_{d_1+1}$.

Algorithm

Graphic($S = (d_1 \geq \cdots \geq d_n \geq 0)$)

case $d_1 \geq n$ return FALSE (* Obs. I *)

case $d_{d_1+1} = 0$ return FALSE (* Obs. II *)

case $d_1 = 0$ return TRUE (* Obs. III *)

otherwise return Graphic(Sort($f(S)$)) (* Lemma *)

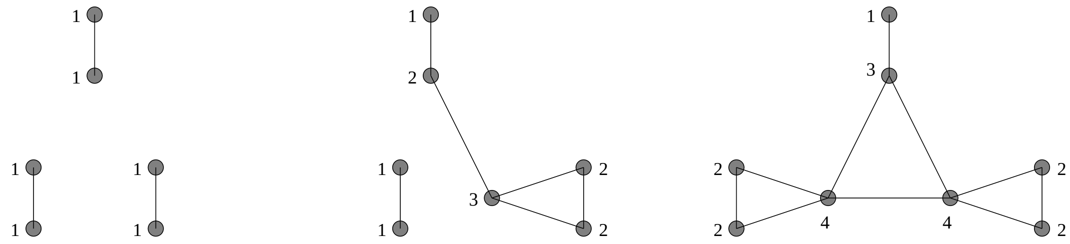
Algorithm

- Complexity:
  - $O(m)$ for the transformations since $\sum_{i=1}^{n} d_i = 2m$.
  - $O(n^2)$ for the sorting (merging n times).

- Constructing the graph for $S = (d_1 \geq \cdots \geq d_n \geq 0)$: Follow the "$\Leftarrow$" part of the proof of the lemma starting with the sequence $(0, \ldots, 0)$ and ending with S.

Example

4	4	3	2	2	2	2	1	I
	3	2	1	1	2	2	1	
	3	2	2	2	1	1	1	II
		1	1	1	1	1	1	
		1	1	1	1	1	1	III
			0	1	1	1	1	
			1	1	1	1	0	IV
				0	1	1	0	
				1	1	0	0	V
					0	0	0	VI



III II I