PETTIS INTEGRABILITY OF WEAKLY CONTINUOUS FUNCTIONS AND BAIRE MEASURES

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ABSTRACT

We analyse the Pettis integrability of weakly continuous bounded functions defined on a completely regular space S and taking values in a Banach space. We prove that the set of Baire measures with respect to which such functions are universally Pettis integrable is precisely the space $M_g(S)$ of Grothendieck measures introduced by Wheeler. This leads us to prove that $M_g(S)$ is $\sigma(M_g(S), C_b(S))$-sequentially complete, and we obtain a characterization in βS of the measures in $M_g(S)$. We also obtain analogous results for the space of separable measures $M_\infty(S)$.

Let S be a Hausdorff completely regular space and let $C_b(S)$ be the space of all bounded continuous real-valued functions on S. Wheeler [**12**] introduces the space of Baire measures $M_g(S)$ as the completion of the space $L(S)$ in the Mackey topology $\tau(L(S), C_b(S))$, where $L(S)$ is the linear span of all atomic measures δ_S determined by S. The characterization problem for $M_g(S)$ in terms of its measure representation in the Stone–Čech compactification βS of S was proposed by Wheeler in [**12**]. A possible connection between the space $M_g(S)$ and the Pettis integral was also suggested by him [**11**].

The results of this paper show a link between the theories of vector integration and Baire measures. Let X be a Banach space. We characterize $M_g(S)$ as the space of Baire measures with respect to which all weakly continuous bounded functions $S \to X$ are universally Pettis integrable. This result is the basis of our proof that $M_g(S)$ is $\sigma(M_g(S), C_b(S))$-sequentially complete, and of our characterization of the space $M_g(S)$ in terms of βS, giving an answer to the problem of Wheeler. This has been done by analysing the Pettis integrability of weakly continuous bounded functions $f: S \to X$ in terms of their natural extensions $f_\beta: \beta S \to X^{**}$.

Using similar ideas, but dealing with norm-continuous functions, we shall obtain analogous results about the space of separable measures $M_\infty(S)$, completing previous results of Haydon [**5**] and Koumoullis [**8**].

1. *Terminology*

Throughout the paper S denotes a Hausdorff completely regular space, and $B_a(S)$ is the Baire σ-algebra on S, that is, the σ-algebra generated by the family $\mathscr{Z}(S)$ of zero sets of S. We denote by h_β the unique continuous extension of $h \in C_b(S)$ to βS. The usual spaces of σ-additive, τ-additive and tight measures on $B_a(S)$ are denoted by $M_\sigma(S)$, $M_\tau(S)$ and $M_t(S)$, respectively. If $\mu \in M_\sigma(S)$ and $h \in C_b(S)$, we write $I_\mu(h)$ to denote $\int h\, d\mu$.

Every member μ of $M_\sigma(S)$ induces a regular Borel measure $\hat{\mu}$ on the Borel subsets $B_0(\beta S)$ of βS. This measure $\hat{\mu}$ is the Borel measure representing the functional

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$h_\beta \mapsto I_\mu(h)$. If E is a Baire subset of βS, then $E \cap S$ is a Baire subset of S and $\hat{\mu}(E) = \mu(E \cap S)$.

Let X be a Banach space with dual X^*, and let μ be a probability measure in $M_\sigma(S)$. A function $f: S \to X$ is weakly measurable if $\langle x^*, f \rangle$ is measurable for all $x^* \in X^*$. Two weakly measurable functions $f, g: S \to X$ are said to be weakly equivalent if and only if $\langle x^*, f \rangle = \langle x^*, g \rangle$ almost everywhere for all $x^* \in X^*$. A weakly measurable function f is Pettis integrable if, for each E in $B_a(S)$, there is an element x_E in X such that $\langle x^*, x_E \rangle = \int_E \langle x^*, f \rangle \, d\mu$ for all $x^* \in X^*$; in that case we write $x_E = \int_E f \, d\mu$. If f is a bounded weakly measurable function, we denote by (D)-$\int_E f \, d\mu$ the Dunford integral of f over E, this is, the member x_E^{**} of X^{**} defined by $\langle x^*, x_E^{**} \rangle = \int_E \langle x^*, f \rangle \, d\mu$.

The space of all X-valued Pettis integrable functions on S is denoted by $P(\mu, X)$. The subspace of $P(\mu, X)$ whose elements are the functions which are weakly equivalent to Bochner measurable functions is denoted by $P^*(\mu, X)$. The space of Bochner integrable functions is denoted in the usual way by $L^1(\mu, X)$.

We write $C_b^w(S, X)$ for the space of bounded weakly continuous X-valued functions on S and $C_b(S, X)$ for the space of norm-continuous X-valued functions on S. In this paper we characterize these Baire measures μ on $B_a(S)$ which are such that $C_b^w(S, X)$ (or $C_b(S, X)$) is included in $P(\mu, X)$, $P^*(\mu, X)$ or $L^1(\mu, X)$ for every Banach space X.

For information about Baire measures we refer to the recent survey of Wheeler **[12]**. Basic references for vector integration are **[2, 10]**.

2. *Integrability of a bounded weakly continuous function via its canonical extension to βS*

The Pettis integrability of a bounded weakly measurable function $f: S \to X$ is analysed in [9] via the Stonian transform $\hat{f}: K \to X^{**}$, defined on the Stone representation space K of the measure algebra $B_a(S)/\mu^{-1}(0)$. Given f in $C_b^w(S, X)$, it seems more appropriate to consider the canonical extension $f_\beta: \beta S \to X^{**}$, defined in a similar way:

$$\langle x^*, f_\beta(\alpha) \rangle = \langle x^*, f \rangle_\beta(\alpha) \quad \text{for all } x^* \in X^* \text{ and } \alpha \in \beta S.$$

This representation of f will lead us to give $\hat{\mu}$-characterizations on βS of the Baire measures with respect to which all bounded weakly continuous functions are universally integrable.

If $M \subset X^{**}$, then throughout the paper, $\overline{\text{co}}\,(M)$ denotes the weak* closed convex hull of M. Let $\tilde{X}$ be the family of elements x^{**} in X^{**} such that x^{**} is in the weak* closure of a countable subset of X. The following result of Talagrand will be used later.

THEOREM 1 [9, p. 482]. *Let $f: S \to X$ be bounded and weakly measurable. Then f is Pettis integrable if and only if*

$$\overline{\text{co}}\,\{(1/\mu(B)) \cdot x_B^{**} : B \subset A, \mu(B) > 0\} \cap \tilde{X} \neq \varnothing$$

for every Baire set $A \subset S$ with $\mu(A) > 0$.

THEOREM 2. *Let μ be a probability measure in $M_\sigma(S)$ and suppose that $f \in C_b^w(S, X)$. The following are equivalent:* (a) $f \in P(\mu, X)$; (b) *if K is a compact subset of βS and $\hat{\mu}(K) > 0$ then $\overline{\text{co}}\, f_\beta(K) \cap X \neq \varnothing$.*

Proof. (a) $\Rightarrow$ (b) Suppose that f is Pettis integrable and let K be a compact subset of βS such that $\hat{\mu}(K) > 0$. There is a decreasing sequence V_n of cozero sets in βS such that $K \cap V_n \neq \emptyset$ and $\hat{\mu}(V_n)$ converges to $\hat{\mu}(K)$. Since $U_n = V_n \cap S$ are Baire subsets of S, we can consider the sequence $x_n = \int_{U_n} f\,d\mu$ in X. Then we have

$$\langle x^*, x_n \rangle = \int_{V_n} \langle x^*, f \rangle_\beta \, d\hat{\mu} \quad \text{for each } x^* \in X^*.$$

Since f is bounded, the sequence x_n is weak* convergent to the element $x^{**} \in X^{**}$ defined by $\langle x^*, x^{**} \rangle = \int_K \langle x^*, f \rangle \, d\mu$. As the range of the indefinite Pettis integral of f is weakly compact, x^{**} is a member of X. Then $\hat{\mu}(K) \cdot x^{**}$ is an element of $\overline{\mathrm{co}}\, f_\beta(K) \cap X$.

(b) $\Rightarrow$ (a) Let A be a Baire subset of S such that $\mu(A) > 0$, and suppose that $H \subset A$ is a zero set with $\mu(H) > 0$. There exists a zero set $\hat{H} \subset \beta S$ with $\hat{H} \cap S = H$. The regular measure on $\hat{H}$ induced by $\hat{\mu}$ has non-empty support $K \subset \hat{H}$, and $\hat{\mu}(K) = \hat{\mu}(\hat{H}) > 0$. If (b) holds, then $\overline{\mathrm{co}}\, f_\beta(K) \cap X \neq \emptyset$. By Theorem 1 it will be enough to prove that

$$f_\beta(K) \subset M(A) = \overline{\mathrm{co}}\, \{(1/\mu(B)) \cdot x_B^{**} : B \subset A, \mu(B) > 0\}.$$

For $\alpha \in K$ and $x^* \in X^*$, let $t = \langle x^*, f \rangle_\beta(\alpha)$. The subsets $\hat{Z}_n = \hat{H} \cap \{\xi \in \beta S : |\langle x^*, f \rangle_\beta(\xi) - t| \leqslant (1/n)\}$ are neighbourhoods of α in $\hat{H}$; we therefore have $\hat{\mu}(\hat{Z}_n) > 0$. For the sequence of zero sets $Z_n = \hat{Z}_n \cap S$ we have that $\mu(Z_n) = \hat{\mu}(\hat{Z}_n) > 0$, and it can be proved easily that

$$\lim_n \frac{1}{\mu(Z_n)} \cdot \int_{Z_n} \langle x^*, f \rangle \, d\mu = t.$$

Since $Z_n \subset H \subset A$, the sequence $x_n^{**} = (1/\mu(Z_n)) \cdot (D)\text{-}\int_{Z_n} f\,d\mu$ is in $M(A)$, and $\langle x^*, x_n^{**} \rangle$ converges to $\langle x^*, f_\beta(\alpha) \rangle$. Then we have

$$\langle x^*, f_\beta(\alpha) \rangle \leqslant \sup\{\langle x^*, x^{**} \rangle : x^{**} \in M(A)\}$$

for each $\alpha \in K$ and $x^* \in X^*$. Hence $f_\beta(K) \subset M(A)$; this completes the proof.

THEOREM 3. *Let μ be a probability measure in $M_\sigma(S)$ and suppose that $f \in C_b^w(S, X)$. The following are equivalent: (a) $f \in P^*(\mu, X)$; (b) if K is a compact in βS and $\hat{\mu}(K) > 0$ then $f_\beta(K) \cap X \neq \emptyset$; (c) f_β is essentially valued in X.*

Proof. Let λ be the distribution μf^{-1} induced on the Baire subsets $B_a(X)$ of (X, weak), and λ_1 the distribution $\hat{\mu} f_\beta^{-1}$ induced on the Borel subsets $B_0(X^{**})$ of $(X^{**}, \text{weak*})$. If E is a Baire subset of $(X^{**}, \text{weak*})$, then $f_\beta^{-1}(E)$ is a Baire subset of βS; therefore $\lambda_1(E) = \lambda(E \cap X) = \mu(S \cap f_\beta^{-1}(E))$.

(a) $\Rightarrow$ (b) From [4], λ is tight. If $\varepsilon > 0$ and H is a weak compact subset of X such that $\lambda(Z) < \varepsilon$ for every zero subset Z of $X \backslash H$, then $\lambda_1(Z^{**}) = \lambda(Z^{**} \cap X) < \varepsilon$ for every zero subset Z^{**} of $X^{**} \backslash H$, and so $\lambda_1(X^{**} \backslash H) \leqslant \varepsilon$. If $K \subset \beta S$ is a compact subset and $\hat{\mu}(K) > 0$, then $f_\beta(K)$ is weak* compact in X^{**} and $\lambda_1(f(K)) > 0$. Since λ_1 is regular for the weak compact subsets of X, we have that $f_\beta(K) \cap X \neq \emptyset$.

(b) $\Rightarrow$ (c) Let $D = \{\alpha \in \beta S : f_\beta(\alpha) \in X\}$ and let K be a compact subset of $\beta S \backslash D$. If (b) holds, then $\hat{\mu}(K) = 0$. If $\hat{\mu}^*$ and λ_1^* are the exterior measures associated with $\hat{\mu}$ and λ_1, respectively, then $\hat{\mu}^*(D) = 1$, and thus $\lambda_1^*(X) = 1$. Since λ_1 is a regular Borel measure and X is universally Borel measurable in $(X^{**}, \text{weak*})$ [4, p. 670], there is a Borel subset $A \subset X$ such that $\lambda_1(A) = \lambda_1^*(X)$. Therefore $f_\beta(\alpha) \in A$ almost everywhere with respect to $\hat{\mu}$, and f_β is essentially valued in X.

(c) $\Rightarrow$ (a) If (c) holds, then $\lambda_1^*(X) = 1$, and the compact regular Borel measure λ_1 induces, on the Borel subsets of (X, weak), a Borel measure ν in the natural way. If E is a Borel subset of (X^{**}, weak^*) then $\nu(E \cap X) = \lambda_1(E)$. Since $B_a(X)$ and $B_a(X^{**})$ are the σ-algebras generated by X^* [**4**, p. 668], we have that $B_a(X)$ coincides with the family $\{E \cap X : E \in B_a(X^{**})\}$. Thus $\nu(A) = \lambda(A)$ for every subset A of X. Since X is universally Borel measurable in (X^{**}, weak^*) the measure ν is compact regular, and thus λ is a tight Baire measure. Now, an appeal to [**4**] shows that f is weakly equivalent to a bounded Bochner measurable function; therefore (a) holds, and the proof is complete.

3. *Universal integrability of bounded weakly continuous functions*

Let $M_\infty(S)$ and $M_g(S)$ be the spaces of separable and Grothendieck measures, respectively; these were introduced by Dudley [**3**] and Wheeler [**12**], respectively. Let $M_\infty(S)$ be the linear subspace of $M_\sigma(S)$ of all measures $\mu \in M_\sigma(S)$ such that for each continuous pseudometric d on S, there is a d-closed d-separable subset Z of S with $|\mu|(Z) = |\mu|(S)$. Denoting by $\mathscr{E}$ the collection of all equicontinuous, absolutely convex, compact subsets of $C_b(S)$ endowed with the topology t_p of pointwise convergence, measures μ in $M_\infty(S)$ can be characterized in functional terms, like measures μ in $M_\sigma(S)$, such that the restriction of I_μ to each $H \in \mathscr{E}$ is continuous in the topology t_p [**12**, p. 120].

Let $\mathscr{H}$ be the family of all absolutely convex, t_p-compact subsets of $C_b(S)$ (then each $H \in \mathscr{H}$ is uniformly bounded [**12**]). The subspace $M_g(S)$ of $M_\sigma(S)$ is formed by those Baire measures μ in $M_\sigma(S)$ such that the restriction of I_μ to each $H \in \mathscr{H}$ is continuous in the topology t_p. Since $\mathscr{E} \subset \mathscr{H}$ we have that $M_g(S) \subset M_\infty(S)$. It is also verified that $M_\tau(S) \subset M_g(S)$, and both inclusions can be strict [**12**].

Throughout, every set $H \in \mathscr{H}$ will be considered as a compact space with the t_p-topology; $C(H)$ will be the Banach space of t_p-continuous real functions $\psi : H \to R$ with the supremum norm.

LEMMA 4. *If $H \in \mathscr{H}$, the function $F_H : S \to C(H)$, defined by $F_H(s)(h) = h(s)$ for $h \in H$, is bounded and weakly continuous. If $H \in \mathscr{E}$, then F_H is norm-continuous.*

Proof. Since H is uniformly bounded, it follows that F_H is bounded. Let λ be a regular probability on the Borel subsets of H; then the function $g(s) = \int_H h(s)\, d\lambda(h)$ is the pointwise limit of a net in the convex hull of H, so that g is a member of H. Therefore, F_H is weakly continuous. If H is equicontinuous it is obvious that F_H is norm-continuous.

In that what follows we give characterizations of $M_\infty(S)$ and $M_g(S)$ in terms of the universal Pettis integrability of bounded norm-continuous and weakly continuous Banach valued functions. Simultaneously, characterizations of such spaces on βS are given by our previous results concerning the canonical extension to βS. To accomplish this, we introduce the following families of compact subsets of βS.

Let $\mathscr{K}_\infty(S)$, $\mathscr{K}_g(S)$ *be the families of compact subsets K of βS such that there is a Banach space X and a function $f \in C_b(S, X)$, $C_b^w(S, X)$, respectively, with $f_\beta(K) \cap X = \varnothing$, $\overline{\text{co}}\, f_\beta(K) \cap X = \varnothing$, respectively.*

Lemma 4 allows us to give an internal characterization of $\mathscr{K}_\infty(S)$ in terms of the family $\mathscr{E}$. If $\alpha \in \beta S$ and $h \in C_b(S)$, we write $I_\alpha(h) = h_\beta(\alpha)$.

PROPOSITION 5. *Assume that K is a compact subset of βS. Then $K \in \mathscr{K}_\infty(S)$ if and only if there is $H \in \mathscr{E}$ such that the restriction of I_α to H is not t_p-continuous for each $\alpha \in K$.*

Proof. Let K be a compact subset of βS such that there is a Banach space X and a function $f \in C_b(S, X)$ with $f_\beta(K) \cap X = \varnothing$. Then $H = \{\langle x^*, f\rangle : \| x^* \| \leqslant 1\}$ is a member of $\mathscr{E}$. For each $\alpha \in K$, $f_\beta(\alpha) \notin X$. By a well-known result, the linear form $x^* \to \langle x^*, f\rangle_\beta(\alpha)$ is not weak* continuous on the unit ball of X^*. Therefore, for all $\alpha \in K$, the restriction of I_α to H is not continuous.

Conversely, suppose that H is a member of $\mathscr{E}$ such that the condition of the proposition holds. We consider the Banach space $X = C(H)$ and the continuous function $f = F_H$ given in Lemma 4. Then $H = \{\langle x^*, f\rangle : \| x^* \| \leqslant 1\}$, and a standard argument based in the compactness of H leads us to conclude that the restriction of I_α to H is continuous whenever $f_\beta(\alpha) \in X$. Therefore, $f_\beta(\alpha) \notin X$ for every $\alpha \in K$, and so $K \in \mathscr{K}_\infty$.

The following characterization of the space $M_\infty(S)$ completes previous results of Haydon [**5**, Proposition 2.1] and Koumoullis [**8**, p. 473].

THEOREM 6. *Let μ be a probability in $M_\sigma(S)$. The following are equivalent:*

(a) $C_b(S, X) \subset L^1(\mu, X)$ *for all Banach spaces X;*

(b) $C_b(S, X) \subset P(\mu, X)$ *for all Banach spaces X;*

(c) $\mu \in M_\infty(S)$;

(d) $\hat{\mu}(K) = 0$ *for each $K \in \mathscr{K}_\infty$.*

Proof. (a) $\Rightarrow$ (b) This implication is clear.

(b) $\Rightarrow$ (c) If (b) holds and $H \in \mathscr{E}$, then we can consider $X = C(H)$ and F_H as given by Lemma 4. Since F_H is Pettis integrable, if $\psi = \int_S F_H \, d\mu$ and $e_h \in C(H)^*$ is the evaluation in $h \in H$, we have

$$\psi(h) = \langle e_h, \psi\rangle = \int_S \langle e_h, F_H\rangle \, d\mu = \int_S h \, d\mu$$

for all $h \in H$. Thus, the restriction of I_μ to H is in $C(H)$. Therefore μ is a member of $M_\infty(S)$.

(c) $\Rightarrow$ (a) Given $f \in C_b(S, X)$, the distribution μf^{-1} induced on the Borel sets of $(X, \| \|)$ is supported by a closed separable subspace Y of X. Then $S_0 = f^{-1}(Y)$ is a Baire subset of S that $\mu(S_0) = 1$, so that f is Bochner measurable and (a) holds.

(a) $\Leftrightarrow$ (d) From Theorem 3, (d) holds if any only if $C_b(S, X) \subset P^*(\mu, X)$ for all Banach spaces X. Since (a) is equivalent to (b) and $L^1(\mu, X) \subset P^*(\mu, X) \subset P(\mu, X)$, we conclude that (a) and (d) are equivalent.

REMARK. The above theorem allows us to say that all continuous bounded functions with values in Banach spaces are Bochner integrable with respect to every σ-additive Baire measure, provided that there are no real-valued measurable cardinals

[**12**, 6.9]. If c is a real-valued measurable cardinal, $S = [0, 1]$ with the discrete topology, μ is a probability on the power set of S which vanishes on all singletons, and $X = \ell^2[0, 1]$, then the function $f: S \to X$ defined by $f(k) = e_k$ (the k-th unit vector) is a member of $C_b(S, X)$ which is not Bochner integrable, since its range is not essentially separable.

It can be shown that the family $\mathcal{K}_\infty$ agrees with that used by Koumoullis in [**8**]. Thus, Theorem 6 provides another proof of the Koumoullis characterization of $M_\infty(S)$.

THEOREM 7. *Let μ be a probability in $M_\sigma(S)$. The following are equivalent:*

(a) $C_b^w(S, X) \subset P(\mu, X)$ *for all Banach spaces* X;

(b) $\mu \in M_g(S)$;

(c) $\hat{\mu}(K) = 0$ *for each* $K \in \mathcal{K}_g$.

Proof. (a) $\Rightarrow$ (b) Let H be a member of $\mathcal{H}$. The function F_H as given by Lemma 4 is bounded and weakly continuous. If $\psi = \int_S F_H \, d\mu$, we can proceed as in the proof of Theorem 6 to show that $\psi \in C(H)$ is the restriction of I_μ to H. Then (b) holds.

(b) $\Rightarrow$ (a) For each Baire subset B of S, the Baire measure $\mu_B(E) = \mu(E \cap B)$ is an element of $M_g(S)$ [**12**, p. 122]. Since $H = \{\langle x^*, f\rangle : \| x^* \| \leqslant 1\}$ is a member of $\mathcal{H}$, the linear form x_B^{**} defined by

$$x^* \longmapsto \int_B \langle x^*, f\rangle \, d\mu = \int_S \langle x^*, f\rangle \, d\mu_B$$

is weak* continuous on the unit ball of X^*. This implies that x_B^{**} is weak* continuous; then x_B^{**} is a member of X. Therefore, (a) holds.

(a) $\Leftrightarrow$ (c) This is a consequence of Theorem 2.

COROLLARY 8. *Let $C \subset M_g(S)$ be a countable subset whose members are positive measures. If $\nu \in M_\sigma(S)$ is in the closure of C for $\sigma(M_\sigma(S), C_b(S))$, then $\nu \in M_g(S)$.*

Proof. Let X be an arbitrary Banach space and suppose that $f \in C_b^w(S, X)$. If we can show that f is Pettis ν-integrable, an appeal to Theorem 7 will allow us to conclude that $\nu \in M_g(S)$. Let Z be a zero subset of S with $\nu(Z) > 0$, and let $\{\psi_n\}$ be a sequence in $C_b(S)$ which decreases pointwise to the characteristic function of Z. Let $C = \{\mu_k : k \in \mathbb{N}\}$. By Theorem 7, $x_{n,k} = \int_S \psi_n \cdot f \, d\mu_k$ is in X for all k and n in $\mathbb{N}$. For each n, the element of X^{**} defined by $x_n^{**} = (D)\text{-}\int_S \psi_n \cdot f \, d\nu$ is in the weak* closure of the set $\{x_{n,k} : k \in \mathbb{N}\}$. Since the sequence x_n^{**} is weak* convergent to the element $x_Z^{**} = (D)\text{-}\int_Z f \, d\nu$, we see that x_Z^{**} is in the weak* closure of a countable subset of X. For each Baire subset $A \subset S$ with $\nu(A) > 0$ there is a zero set $Z \subset A$ such that $\nu(Z) > 0$. Using Theorem 1, we conclude that f is Pettis ν-integrable.

COROLLARY 9. *The space $M_g(S)$ is sequentially complete for the topology $\sigma(M_g(S), C_b(S))$.*

Proof. Suppose that μ_n is a sequence in $M_g(S)$ such that $\int \psi \, d\mu_n$ is a convergent sequence for all ψ in $C_b(S)$. Since $M_\sigma(S)$ is sequentially complete for the $\sigma(M_\sigma(S),$

$C_b(S))$ topology [12, p. 162], there is a measure μ in $M_\sigma(S)$ such that $\int \psi\, d\mu_n$ converges to $\int \psi\, d\mu$ for all $\psi \in C_b(S)$. Since $\{\mu_n : n \in \mathbb{N}\}$ is relatively compact in $\sigma(M_\sigma(S), C_b(S))$, it follows that $\{\mu_n^+ : n \in \mathbb{N}\}$ and $\{\mu_n^- : n \in \mathbb{N}\}$ are also relatively compact in this topology [12, p. 142]. Thus, there are ν_1 and ν_2, cluster points in the above topology of the sequences μ_n^+ and μ_n^- respectively, such that $\mu = \nu_1 - \nu_2$. From Corollary 8, ν_1 and ν_2 belong to $M_g(S)$, and so μ is in $M_g(S)$.

REMARK. Taking norm-continuous functions in the proof of Corollary 8, the same arguments provide a new proof of the sequential completeness of $M_\infty(S)$ endowed with the $\sigma(M_\infty(S), C_b(S))$ topology [12, p. 162].

In the next corollary we denote by $B_1^w(S, X)$ the family of bounded functions $f: S \to X$ such that there is a sequence f_n in $C_b^w(S, X)$ for which

$$\langle x^*, f(t) \rangle = \lim_n \langle x^*, f_n(t) \rangle \quad \text{for all } t \in S \text{ and } x^* \in X^*. \tag{*}$$

COROLLARY 10. *If μ is a probability in $M_g(S)$, then $B_1^w(S, X) \subset P(\mu, X)$ for all Banach spaces X.*

Proof. Given $f \in B_1^w(S, X)$, let f_n be a sequence in $C_b^w(S, X)$ such that (*) holds. For each zero subset Z of S with $\mu(Z) > 0$, let ψ_k be a sequence in $C_b(S)$ which decreases pointwise to the characteristic function of Z. If $x_{n,k} = \int_S \psi_k \cdot f_n\, d\mu$, similar reasoning to that in the proof of Corollary 8 shows that f is Pettis integrable.

In [11] Wheeler introduces the Baire measures space $Z(S)$, defined in a similar way to $M_g(S)$, but related to the family $\mathscr{H}_0$ of uniformly bounded and t_p-compact subsets $H \subset C_b(S)$, rather than $\mathscr{H}$. If S is a K_R space then $M_g(S) = Z(S)$ by [12, Theorem 13.10]. In this case $Z(S)$ is sequentially complete for the $\sigma(Z(S), C_b(S))$ topology.

Now we characterize the subspace $M_g^*(S)$ of $M_\sigma(S)$ whose positive cone $M_g^*(S)^+$ is composed of the measures $\mu \in M_\sigma(S)^+$ such that $C_b^w(S, X) \subset P^*(\mu, X)$ for each Banach space X. It is convenient to introduce the following definition. Given $\mu \in M_\sigma(S)^+$, we say that H satisfies the condition C_μ if there is a Borel set $B \subset \beta S$ with $\hat{\mu}(B) = \hat{\mu}(\beta S)$ such that $G: h \to h_{\beta|B}$ $(h \in H)$ is continuous for the pointwise topologies in H and $G(H)$.

It is easy to prove that if H satisfies C_μ, then the restriction of I_μ to H is continuous for the pointwise topology. Hence, the set of the measures μ such that each $H \in \mathscr{H}$ satisfies $C_{|\mu|}$ is a vector subspace $M_g^*(S)$ of $M_g(S)$. The characterization in βS of this new space is obtained by considering the family $\mathscr{K}_g^*$ of compact subsets K of βS for which there exists a Banach space and a function $f \in C_b^w(S, X)$ such that $f_\beta(K) \cap X = \varnothing$. Proceeding as in the proof of Proposition 5, we can give an internal characterization of $\mathscr{K}_g^*$ in terms of the family $\mathscr{H}$. The compact set $K \subset \beta S$ is in $\mathscr{K}_g^*$ if and only if there exists $H \in \mathscr{H}$ such that for each $\alpha \in K$ the restriction of I_α to H is not continuous.

THEOREM 11. *Let μ be a probability in $M_\sigma(S)$. The following are equivalent:*

(a) *$C_b^w(S, X) \subset P^*(\mu, X)$ for all Banach spaces X;*

(b) *$\mu \in M_g^*(S)$;*

(c) *$\hat{\mu}(K) = 0$ for each $K \in \mathscr{K}_g^*$.*

Proof. (a)$\Leftrightarrow$(c) This follows from Theorem 3.

(a)$\Rightarrow$(b) Let H be a member of $\mathscr{H}$. If $X = C(H)$ and $f = F_H$ are as in Lemma 4, then $H = \{\langle x^*, f\rangle : \| x^* \| \leqslant 1\}$. From Theorem 3 there is a Borel subset B of βS such that $\hat{\mu}(B) = 1$ and $f_\beta(B) \subset X$. If $G(h) = h_{\beta|B}$, then $G(H)$ is t_p-compact in $C_b(B)$. If $h_j = \langle x_j^*, f\rangle$ is a net in H, t_p-convergent to $h = \langle x^*, f\rangle$, then $G(h)$ is the only t_p-cluster point of $G(h_j)$; hence $G(h_j)$ is t_p-convergent to $G(h)$. Therefore, every $H \in \mathscr{H}$ satisfies C_μ, and μ is in $M_g^*(S)$.

(b)$\Rightarrow$(c) Given $K \in \mathscr{K}_g^*$, there is $H \in \mathscr{H}$ such that, for every $\alpha \in K$, the restriction of I_α to H is not continuous. If we suppose that (b) holds, there is a Borel subset B of βS with $\hat{\mu}(B) = 1$ such that $G(h) = h_{\beta|B}$ is continuous on H for the t_p-topology. Therefore, for each $\alpha \in B$, the restriction of I_α to H is continuous. Thus, $K \cap B = \varnothing$, so that $\hat{\mu}(K) = 0$ and (c) holds.

From Theorem 12 and the characterization of Knowles of $M_\tau(S)$ in terms of βS [**12**, p. 124] we see that $M_\tau(S) \subset M_g^*(S)$. If S is not a μ-space [**12**], the above inclusion is strict. It is enough to note that $S = \beta S \cap M_\tau(S) \nsubseteq \beta S \cap M_g(S)$ [**12**, p. 123], for $\beta S \cap M_g(S)$ coincides with $\beta S \cap M_g^*(S)$ by Theorems 7 and 11 and [**9**, Theorem 5.9, p. 494].

PROBLEM. The similarities between the functional characterizations of $M_\infty(S)$ and $M_g(S)$, and between the representations on βS of $M_\infty(S)$ and $M_g^*(S)$, lead us to pose the following question. Is it possible to distinguish between $M_g(S)$ and $M_g^*(S)$?

A natural condition for $\mu \in M_g(S)$ is that for each $H \in \mathscr{H}$ there is a Baire set $E \subset S$ such that $|\mu|(E) = |\mu|(S)$ and $H_{|E} = \{h_{|E} : h \in H\}$ is t_p-metrizable. The subspace of measures with this property is denoted by $M^*(S)$.

Let μ be a Baire positive measure and let $\tilde{\mu}$ be its completion; μ is said to be completion regular if each Borel set is $\tilde{\mu}$-measurable. If $\mu \in M_\tau(S)$ is completion regular, and we take a Baire set E contained in the support of μ with $\mu(E) = \mu(S)$, then the metrizability theorem of [6, p. 171] proves that $H_{|E}$ is t_p-metrizable for each $H \in \mathscr{H}$, and so μ is in $M^*(S)$. A lifting on $\mathscr{L}^\infty(\tilde{\mu})$ is a multiplicative linear mapping $\rho : \mathscr{L}^\infty(\tilde{\mu}) \to \mathscr{L}^\infty(\tilde{\mu})$ which satisfies (a) $\rho(f) = f$ almost everywhere, and (b) if $f \geqslant g$ almost everywhere, then $\rho(f) \geqslant \rho(g)$ at all points of S. The existence of liftings for all complete positive measures is proved in [7]. The lifting ρ is said to be *almost strong* if there is a Baire set E with $\mu(E) = \mu(S)$ such that $\psi_{|E}$ coincides with $\rho(\psi)_{|E}$ for each $\psi \in C_b(S)$. In [1] it is proved that μ is τ-additive and completion regular when there is an almost strong lifting ρ on $\mathscr{L}^\infty(\tilde{\mu})$. Such liftings satisfy the following property.

(P) For each $H \in \mathscr{H}$ there is a Baire set $E \subset S$ such that $\mu(E) = \mu(S)$ and $\rho(\psi)_{|E} = \psi_{|E}$ for each $\psi \in H$.

THEOREM 12. *Let μ be a probability in $M_\sigma(S)^+$. The following are equivalent:*

(a) *$\mu \in M^*(S)$;*

(b) *$C_b^w(S, X) \subset L^1(\mu, X)$ for each Banach space X;*

(c) *every lifting ρ on $\mathscr{L}^\infty(\tilde{\mu})$ satisfies* (P).

Proof. (a) ⇒ (b) Let $\mu \in M^*(S)$ and $f \in C_b^w(S, X)$. If $H = \{\langle x^*, f\rangle : \| x^* \| \leqslant 1\}$, then there is a Baire subset $E \subset S$ such that $\mu(E) = \mu(S)$ and $H_{|E}$ is t_p-metrizable. Consequently, the Banach space $C(H_{|E})$ is separable.

Let F be the set of equivalence classes of E under the following relation: $s \sim s'$ if and only if $h(s) = h(s')$ for each $h \in H$. If we endow F with the initial topology for its natural injection into $C(H_{|E})$, then f induces a continuous function $\tilde{f}: F \to X$ defined by $\tilde{f}(\tilde{s}) = f(s)$, where $\tilde{s}$ is the equivalence class of s. Since F is separable, $f(E) = \tilde{f}(F)$ is separable also, and f is in $L^1(\mu, X)$.

(b) ⇒ (c) Let ρ be a lifting on $\mathscr{L}^\infty(\tilde{\mu})$. For $H \in \mathscr{H}$, if we consider $X = C(H)$ and $f = F_H$ as in Lemma 4, then $H = \{\langle x^*, f\rangle : \| x^* \| \leqslant 1\}$. Since f is Bochner measurable, if $\rho(f): S \to X^{**}$ is defined by $\langle \rho(f), x^*\rangle = \rho(\langle x^*, f\rangle)$, then [**10**, Theorem 3.4.4] proves that modifying $\rho(f)$ on a null set gives a Bochner measurable function scalarly equivalent to f. Therefore, f and $\rho(f)$ coincide almost everywhere. Then there exists a Baire subset E of S with $\mu(S) = \mu(E)$ such that $\langle x^*, f\rangle_{|E} = \rho(\langle x^*, f\rangle)_{|E}$ for each $x^* \in X^*$.

(c) ⇒ (a) This is a consequence of the metrizability theorem of [**6**].

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