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ON THE SUBLINEAR FUNCTIONAL ASSOCIATED TO A FAMILY OF INVARIANT MEANS

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In this paper we give conditions under which we can obtain explicit analytic expressions of the fundamental sublinear functional ρ_K of the family of means $K=[G]\cap J$ where $[G]$ is a subset of G-invariant means, G a semigroup of operators on l^∞ and J a saturated set of means. Such conditions allow us to assure that $[G]\cap J\neq\phi$.

We characterize the set of the almost convergent sequences related to the family K by means of the same functional ρ_K. We obtain also ρ_K in terms of ρ_J when ρ_J is known. This allow us to give different expressions of the same functional ρ_K when the family J changes and from which several examples are given.

INTRODUCTION AND NOTATIONS

Let us denote by l_∞ the space of the real bounded sequences $x=(x(n))$ with the supremum norm $||x||=\sup_n|x(n)|$. We will call $A:l^\infty \longrightarrow l^\infty$ an operator on l^∞ if it is a continuous linear mapping and matrix operator if it could be defined in terms of and infinite matrix (a_{nk}) holding $\sup_n \sum_k |a_{nk}|=||A||<+\infty$.

Let e be the sequence (1,1,...). An element $\mu\in(l_\infty)^*$ holding $\mu(x)\geqslant 0$ if $x\geqslant 0$ and $\mu(e)=1$ is called a mean on l^∞. In the following, M will denote the set of such means and G a semigroup of operators on l^∞. If μ is an element of M such that $\mu(Lx)=\mu(x)$ for every $L\in G$ and every $x\in l_\infty$, we will say that μ is a G-invariant mean and denote by $[G]$ the subset of M composed of such means. The generalized limits of Banach are the invariant means with respect to the semigroup generated by the shift operator $S(x)(n)=x(n+1)$. We will say that a mean $\mu\in M$ is a generalized limit if it satisfies that $\mu(x)=\lim_n x(n)$ for each convergent sequence $x=(x(n))$.

We will denote by $\mathcal{L}$ the set of all generalized limits and by B the subset of $\mathcal{L}$ composed of the generalized limits of Banach. Another remarkable subset of $\mathcal{L}$ is the family $\mathcal{L}_0$ of the generalized limits associated to the ultrafilters that contain the filter of Frechet of $\mathbb{N}$.

It is easy to see that $B\cap\mathscr{L}_o=\phi$ and it is well known that $\mathscr{L}_o$ is the set of the extreme points of $\mathscr{L}$.

It is also known that the invariant means with respect to certain semigroups of operators play an important role in the theory of summability of bounded sequences through the concept of bounded almost convergent sequences and different extensions (see [6,Definition 4.1], [13, III and IV]), all of which are obtained when we particularize the family J in the following general definition.

Definition 1.-*Given a non empty family of means* $J\subset M$, *if* x *is a bounded sequence such that* $\mu(x)=\alpha$ *for every* $\mu\in J$, *we will say that* x *is* J-*convergent to* α (*or almost convergent to* α *with respect to* J). *Then we will write* $J\text{-}\lim_n x(n)=\alpha$.

As an example, the B-convergent sequences are the almost convergent sequences introduced by G. Lorentz [11, p. 169], and it is easy to prove that the $\mathscr{L}$-convergent sequences are the convergent in the usual sense. Obviously, the M-convergent sequences are the constants.

We associate the functional on l^∞, $\rho_J(x)=\sup\{\mu(x):\ \mu\in J\}$, with each non empty family of means $J\subset M$. ρ_J (or simply ρ if J is clear in the context) is continuous, convex and positive homogeneous and we will call it the fundamental sublinear functional of the family J or sublinear functional associated with J.

This functional does not change if J is replaced by its convex and weak* closed hull. If J is weak* closed and convex, it follows immediately from a separation theorem that $J=\{\mu\in(l^\infty)^*:\ \mu(x)\leqslant\rho(x)$ for every $x\in l^\infty\}$.

For some of the latter results it is convenient to take into account that $\rho_{\mathscr{L}}$ is given by $\rho_{\mathscr{L}}(x)=\overline{\lim_n}\, x(n)$. In fact, it is clear that $\mu(x)\leqslant\overline{\lim_n}\, x(n)$ for each $\mu\in\mathscr{L}$ and for some $\mu\in\mathscr{L}_o$ the equality holds. It is also obvious that $\rho_M(x)=\sup_n x(n)$.

One of the first problems to consider when we study the different notions of J-convergent sequences is the obtention of explicit analytic expressions of the fundamental sublinear functional ρ_J. That is interesting about this problem is the fact that each time we obtain a concrete analytic expression of ρ_J, we have a characterization of the J-convergent sequences. It is sufficient for this to take into account that a bounded sequence x is J-convergent to zero if and only if $\rho_J(x)=-\rho_J(-x)=0$.

As an example, different analytic expressions of $w=\rho_B$ are known for the family B. One of the simplest is

$$w(x)=\lim_n \sup_k (x(k)+...+x(k+n))/(n+1)$$ (see [14, Corollary 22])

The known characterization of G. Lorentz of the almost convergent sequences is obtained immediately from this analytic expression of w. A bounded sequence x is B-convergent to α if and only if $w(x-\alpha e)=-w(\alpha e-x)=0$. That condition is equivalent to

$$\lim_k (x(k+1)+...+x(k+n))/(n+k)=\alpha \text{ uniformly in } n\in\mathbb{N}.$$

Given a non empty family of means on l^∞ of which a concrete analytic expression of a fundamental linear functional ρ_J is known and given a semigroup of operators on l^∞, G, the object of this paper is to give conditions under which we can obtain explicit analytic expressions for the fundamental sublinear functional ρ_K of the family $K=[G]\cap J$ and assure that in J,G-invariant means exist.

The conditions that relate the semigroup G to the family J are of ergodic nature and are inspired in the results of Duran [6, Theorem 3.4 and Corollaries] and especially in the notion of the ergodic semigroup of Eberlein . They are established in an explicit way in Definition 3. The main result is Theorem 4. Apart from other applications, the different analytic expressions of the sublinear functional of Banach w and at the same time the different characterizations of the usual notions of almost convergent sequences are obtained.

MAIN RESULT

First, it is convenient to make some remarks about the concept of J-convergent sequence that will be of interest later. Let us denote with J^* the intersection of M and the weak* closed subspace of $(l^\infty)^*$ generated by J. It is clear that $J\subset J^*$ and J^* is convex and weak* compact. When $J=J^*$ we will say that J is saturated. It is easy to prove that J^* is saturated and if $x\in l^\infty$ is J-convergent to α , then x is J^*-convergent to α and viceversa. (It is sufficient to recall that the weak* closed subspaces F of a dual satisfy $F=({}^\perp F)^\perp$). It is not difficult to prove that in the cases of major interest $J=\mathcal{L}$,B or more generally in J=[G], the family J is always saturated. In the following it will not be restrictive to suppose that J is non empty and saturated.

When J_o, $J\subset M$ are non empty and J is saturated, the characterization of the weak* closed subspaces of $(l^\infty)^*$ allows us to prove that the condition $J_o \subset J$ is equivalent to the fact that the J_o-convergence is an extension of the J-convergence. This observation leads to a useful characterization of the operators L on l^∞ whose adjoint L^t satisfies $L^t(J) \subset J$.

An operator L will be called J-regular if it transforms each J-convergent sequence x into a J-convergent sequence Lx with $J\text{-}\lim_n (Lx)(n) = J\text{-}\lim_n x(n)$ and J-positive if $L^t(J) \subset M$.

Proposition 2.- If J is saturated, a necessary and sufficient condition for the operator L to verify $L^t(J) \subset J$ is that L be J-regular and J-positive.

Proof.-

The necessity is evident. The sufficiency follows from the previous observations. If L is J-regular, then $\mu(e)=1$ for each $\mu \in L^t(J)$ If L is also J-positive then $L^t(J)$ is a set of means. The J-regularity of J implies that the $L^t(J)$-convergence is an extension of the J-convergence and therefore $L^t(J) \subset J$ since J is saturated.#

When L is a matrix operator (a_{nk}), a sufficient condition for L to be J-positive is that the sequence $\sum_{k=1}^{\infty} a^-_{nk}$, where $a^-_{nk} = a_{nk} \wedge 0$, be J-convergent to zero. In particular, when J=B, a sufficient condition for $L^t(B) \subset B$ is that the matrix (a_{nk}) is almost regular (see King [10, Definition 2.3]) and $\sum_{k=1}^{\infty} a^-_{nk}$ almost convergent to zero.

Similary, we obtain, in the case $J=\mathcal{L}$, that a sufficient condition for $L^t(\mathcal{L}) \subset \mathcal{L}$ is that the matrix defining the operator L be regular and almost positive (namely, that it satisfies the condition of the Silverman-Toeplitz's theorem and also $\lim_n \sum_k a^-_{nk} = 0$). From the results of S. Simons [14, § 4] it follows that in this case, such conditions are also necessary for $L^t(\mathcal{L}) \subset \mathcal{L}$.

Remarks.- It follows from Proposition 2 that the J-regular and J-positive operators are characterized in terms of ρ_J by the condition $\rho_J(Lx) \leqslant \rho_J(x)$ for every $x \in l^\infty$. As this condition is translated to $\overline{\lim}(Lx) \leqslant \overline{\lim}\, x$ for every $x \in l^\infty$ when $J=\mathcal{L}$, Proposition 2 could be thought of as an abstract version of the classic Knopp's core Theorem (see Cooke [1, p. 149]).

Let us denote the convex hull of the semigroup of the operators G by co(G), and by O(x) the orbit of x under the action of co(G), namely $O(x) = \{L(x) : L \in co(G)\}$. Finally, we will denote the seminorm on l^∞, $\sup\{|\mu(x)| : \mu \in J\}$ by $\|x\|_J$. When $J=\mathcal{L}$ is $\|x\|_{\mathcal{L}} = \overline{\lim}_n |x(n)|$. For this, it is sufficient to observe that $\mu(|x|) = |\mu(x)|$ when $\mu \in \mathcal{L}_o$ and for some $\mu\in\mathcal{L}_0$ is $\overline{\lim_n} |x(n)| = \mu(|x|)=|\mu(x)|$, therefore $\overline{\lim_n} |x(n)| \leq \|x\|_{\mathcal{L}}$. The other inequality is immediate.

Definition 3.- A semigroup G of operators on l^∞ is J-ergodic on the right if the following condition are satisfied:

(3.1) Every $L\in G$ is J-positive and J-regular.

(3.2) A net of operators on l^∞ exists $\{A_i : i\in I\}$ holding:

(a) $\lim_i \|A_i(Lx-x)\|_J=0$ for all $x\in l^\infty$ and all $L\in G$.

(b) $A_i(x)$ is $\| \|_J$-adherent to $O(x)$ for all $x\in l^\infty$ and all $i\in I$.

Then we will say that the net $(A_i)_{i\in I}$ is an ergodic system on the right related to the family J.

If it is also satisfied that

(c) $\lim_i \|L(A_i x)-A_i x\|_J=0$ for all $x\in l^\infty$ and all $L\in G$,

we will say that $(A_i)_{i\in I}$ is an ergodic system related to J and that G is ergodic.

If J=M then $\| \|_M=\| \|_\infty$ and the previous Definition is particularized in Eberlein [7, Definition 2.1]. If $J=\mathcal{L}$, Definition 3 is given by Duran in [6, Definition 3.2].

Theorem 4.- Let J be a saturated family of means whose fundamental sublinear functional is q and G a J-ergodic semigroup on the right on l^∞. Then $K=[G]\cap J\neq\phi$ and the fundamental sublinear functional ρ of the family K is is given by

(4.1) $\rho(x)=\lim_i q(A_i x)=\inf_i q(A_i x)$

where $(A_i)_{i\in I}$ is any ergodic system on the right on l^∞ related to J.

Proof.-

By Proposition 2, $L^t(J)\subset J$ for every $L\in G$. As J is convex, this condition is also verified for every $L\in co(G)$. As $q(x)=\sup\{\mu(x): \mu\in J\}$ it follows that $q(Lx)\leq q(x)$ for every $L\in co(G)$ and every $x\in l^\infty$.

Given $i\in I$, $x\in l^\infty$ and $\varepsilon>0$, by (3.2)(b) $L\in co(G)$ exists such that $q(A_i x-Lx)\leq \|A_i x-Lx\|_J<\varepsilon$. Then $q(A_i x)\leq q(Lx)+\varepsilon\leq q(x)+\varepsilon$, so $q(A_i x)\leq q(x)$ for every $i\in I$ and if we define $\rho(x)=\overline{\lim_i}\, q(A_i x)$, it is satisfied $\rho(x)\leq q(x)$ for every $x\in l^\infty$. It is immediately proved that ρ is a sublinear functional. As $\rho(e)\leq q(e)=1$ and $\rho(e)\geq -\rho(-e)\geq -q(-e)=1$ it follows that $\rho(e)=1$. So, by the Hahn-Banach Theorem $\{\mu\in(l^\infty)^* : \mu\leq\rho\}=P$ is a non empty set of means. As $\rho\leq q$ it follows that $P\subset J$.

On the other hand, given $x\in l^\infty$ and $L\in G$, by (3.2)(a) we have:

$$\rho(Lx-x)=\overline{\lim_i}\, q(A_i(Lx-x))\leq\overline{\lim_i}\|A_i(Lx-x)\|_J=0$$

from which it can be deduce that $P\subset[G]$ and therefore $P\subset[G]\cap J$.

Given $\mu\in[G]\cap J$, $x\in l^\infty$, $\varepsilon>0$ and $i\in I$ fixed, $L\in co(G)$ exists holding the condition $||A_i x-Lx||_J<\varepsilon$. As $L^t(\mu)=\mu$, we obtain $|\mu(A_i x)-\mu(x)|=|\mu(A_i x-Lx)|<\varepsilon$, so $\mu(x)=\mu(A_i x)\leqslant q(A_i x)$ and then $\mu(x)\leqslant\rho(x)$ for all $x\in l^\infty$, namely $\mu\in P$.

We have proved the equality $P=[G]\cap J=K$; thus ρ is the fundamental sublinear functional of $[G]\cap J$. We must still prove that $\lim_i q(A_i x)$ exists and takes the value $\inf_i q(A_i x)$. For this it is sufficient to define $\rho'(x)=\underline{\lim}_i\, q(A_i x)$ ($\leqslant\rho(x)$). As $\mu(x)=\mu(A_i x)$ for all $\mu\in K$, it follows that $\rho(x)=\sup\{\mu(A_i x):\mu\in K\}\leqslant\sup\{\mu(A_i x):\mu\in J\}=q(A_i x)$ and so $\rho(x)\leqslant\inf_i q(A_i x)\leqslant\rho'(x)$ and therefore (4.1) is proved.#

Corollary 5.- In the condition of the previous Theorem, if $x\in l^\infty$, they are equivalent:

(a) $K\text{-}\lim_n x_n=\alpha$.

(b) $\lim_i\|A_i(x-\alpha e)\|_J=0$.

If e is also a fixed point of the semigroup G, the condition (b) can be replaced equivalently by

(b') $\lim_i\|A_i x-\alpha e\|_J=0$.

Proof.-

Let $y=x-\alpha e$. From the inequalities $\|z\|_J\leqslant|q(z)|+|q(-z)|\leqslant 2\|z\|_J$, the condition (b) is equivalent to $\lim_i q(z_i)=0=\lim_i q(-z_i)$ where $z_i=A_i y$. From (4.1), (b) is equivalent to $\rho(y)=0=\rho(-y)$ and this relation equivalent to (a).

If $O(e)=\{e\}$, the condition (3.2)(b) implies that $\|A_i e-e\|_J=0$, so (b) and (b') are equivalent.#

EXAMPLES AND APPLICATIONS

1.- Two operators A,B on l^∞ are J-equivalent if Bx-Ax is J-convergent to zero for each $x\in l^\infty$. In particular, when A and B are matrix operators and $J=\mathcal{L}$, the usual notion of absolutely equivalent matrices is obtained.

If G is a semigroup of operators that satisfies (3.1) and such that $A\circ B$ and $B\circ A$ are J-equivalent for each A,B of G, then G is ergodic related to the family J. If we denote the restriction of H^t to J by $\hat{H}$, then $\{\hat{H}: H\in co(G)\}$ is a commutative semigroup of transformations from J to J. This fact allows us to direct co(G) in the following way:

$$A\geqslant B \text{ if } C \text{ exists such that } \hat{A}=\hat{C}\circ\hat{B}.$$

Therefore we obtain that $\{L: L\in co(G)\}$ (which is a net of operators) satisfies (3.2). This can be deduced from the following observation. Given $L\in G$, if $A_n=[I+L+\ldots+L^n]/(n+1)$ and if $A\geqslant A_n$ then

$\|LAx-Ax\|_J = \|A(Lx-x)\|_J \leqslant \|A_n(Lx-x)\|_J \leqslant 2\|x\|_J/(n+1)$.

In this case, the analytic expression of ρ will be

$\rho(x)=\inf\{q(Lx): L\in co(G)\} = \lim_L q(Lx)$.

If we knew another ergodic system simpler than the latter, we could obtain an easier analytic expression of the functional ρ.

2.- A particular case of 1.- is when $J=\mathscr{L}$ and G is the semigroup of operators defined by regular and almost positive matrices with the property that A°B and B°A are absolutely equivalent for each couple of elements A,B of G. This is the situation established by Das [3, p. 504] and Cooke [2, p. 411]. If G is the convex and commutative semigroup H composed of the positive and regular Hausdorff matrix, then $[G]\cap\mathscr{L}$ is the set of limits of Hausdorff whose fundamental sublinear functional is $\rho(x)=\inf\{\overline{\lim_n}(Ax)_n : A\in H\}$.

The limits of Banach-Hausdorff were introduced by Eberlein [8, p.662] as the elements of $[H]\cap B$. The equality $[H]\cap B=[H]\cap\mathscr{L}$ follows from the fact that H contains an element C (the Cesàro matrix) that satisfies $C^t(\mathscr{L})\subset B$. From this inclusion it follows that $[H]\cap\mathscr{L}=C^t([H]\cap\mathscr{L})\subset C^t(\mathscr{L})\subset B$.

3.- Another particular case of 1.- is when J=M and we consider a commutative semigroup G composed of positive operators L such that L(e)=e. The fundamental sublinear functional of [G] will be given by

$$\rho(x)=\inf\{\sup_n (Lx)(n): L\in co(G)\} = \lim_L \sup_n (Lx)(n) \qquad \text{(A)}$$

where co(G) is directed by:

$L_1 \geqslant L_2$ if $L_1=L_2{}^\circ C$ for some $C\in co(G)$.

When G is generated by a unique positive operator L such that L(e)=e, instead of the indicated ergodic system of 1.-, let us consider the sequence $A_m=(I+L+\ldots+L^m)/(m+1)$ which leads to an easier analytic expression of ρ:

$$\rho(x)=\lim_m \sup_n (A_m x)(n) = \inf_m \sup_n (A_m x)(n) \qquad \text{(B)}$$

From Corollary 5, the sequences $x\in l^\infty$ that are [G]-convergent to α are characterized by $\lim_m \|A_m x-\alpha e\|_\infty=0$.

4.- Let us suppose that the semigroup G satisfies the conditions of the example 3.-, $[G]\subset J$ and every $L\in G$ is J-regular. Then the hypothesis of Theorem 4 are satisfied for both families M and J and therefore, for the sublinear functional ρ of the family $J\cap[G]=[G]$ we obtain (A) and also

$$\rho(x)=\inf\{q(Lx): L\in co(G)\} = \lim_L q(Lx). \qquad \text{(A')}$$

(here, q is the sublinear functional associated to the family J).

If G is generated by a unique positive and J-regular operator

$\rho(x)= \lim_m q(A_m x)= \inf_m q(A_m x)$ (B')

This situation holds when $J=\mathcal{L}$ and L is the induced operator by an injective mapping $\sigma: \mathbb{N} \longrightarrow \mathbb{N}$ without finite orbits, namely $Lx=x\circ\sigma$ (for example L=S could be the shift operator). The condition that σ does not have periodic orbits implies that $[G]\subset\mathcal{L}$ and the condition that σ be injective implies that L is regular ($\mathcal{L}$-regular). Thus, the formulas (A),(B),(A') and (B') lead to different expressions of $\rho=\rho_{[G]}$.

As when $J=\mathcal{L}$ is $q=\overline{\lim}$, the expressions deduced from (B) and (B') are:

$\rho(x)= \inf_m \sup_n s(n,m)=\lim_m \sup_n s(n,m)= \inf_m \overline{\lim_n} s(n,m)= \lim_m \overline{\lim_n} s(n,m)$

where $s(n,m)= [x(n)+x(\sigma(n))+\ldots+x(\sigma^m(n))]/(m+1)$.

When $\sigma(n)=n+1$, we obtain different analytic expressions of the sublinear functional of Banach $w= \rho_B$ given by Jerison [9, p. 87], Simons [14, p. 642] and Raimi [12, p. 21]

5.- Finally, let J be a family of means different from M or $\mathcal{L}$. Let L be the matrix

$$a_{mn}=\begin{cases}[1+(-1)^m]/(m+1) & \text{if} \quad 0\leqslant n\leqslant m \\ 0 & \text{if} \quad n>m\end{cases}$$

The operator L is B-regular but not $\mathcal{L}$-regular (see [5, p.81]). The family J=B is saturated and the semigroup G generated by L is B-ergodic where $A_n=(I+L+\ldots+L^n)/(n+1)$ is an ergodic system related to the norm $\|\ \|_\infty$ (and therefore with respect to the seminorm $\|\ \|_B$).

We find then that the fundamental sublinear functional of the family of L-invariant limits of Banach, $K=[G]\cap B$ is

$\rho(x)= \lim_n w(A_n x)=\inf_n w(A_n x)$

where w is the sublinear functional of Banach.

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