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Almost convergent functions and their multipliers

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Synopsis

In this paper we give a characterisation of the multipliers of a space of almost convergent functions with respect to invariant means related to ergodic semigroups of operators. The characterisation extends several results of the literature.

1. Introduction and preliminary results

In [**3**, Theorem 3.1], Ching-Chou characterised the multipliers of the space of almost convergent sequences introduced by Lorentz [**11**] as the sequences which are convergent through the filter basis composed of the subsets of $\mathbb{N}$ whose characteristic function is almost convergent to one. Interesting extensions were made by Duran [**7**] to the setting of almost convergence with respect to semigroups of positive and regular matrices, and by Ching-Chou and Duran [**5**] to countable left cancellative and left amenable semigroups without finite left ideals. Also, in [**4**], Ching-Chou studied this problem for amenable groups having additional properties.

The purpose of this paper is to study a similar problem in the setting of the space $C_b(T)$ of the real-valued bounded continuous functions on T, where T is a paracompact and locally compact, or a σ-compact and locally compact, space. We consider almost convergent functions with respect to invariant means related to a semigroup of operators $G \subset \mathscr{L}(C_b(T))$. A result has been obtained by one of the authors in [**1**] considering invariant means with respect to the semigroup of operators generated by a transformation $h: T \to T$ holding an additional property.

Our main result is Theorem 4, where we give a characterisation of the space of the almost convergent functions as those functions which are convergent through a filter basis of subsets of T associated to the family of invariant means. This result contains as a particular case the previous results, and when it is applied to other situations we obtain results which seem to be new.

Let T be a Hausdorff locally compact and noncompact space, ν the filter basis composed of the subsets $T\backslash K$, where K runs over all the compact subsets of T and $c(T)$ (respectively $c_0(T)$), the subspace of $C_b(T)$ composed of the functions that converge (respectively converge to zero) through ν.

Let $M(T)$ be the space of the finitely additive measures defined on $B(T)$ that are bounded and $Z(T)$-regular, where $B(T)$ is the algebra of subsets of T generated by the lattice $Z(T)$ of zeros of continuous functions.

With the total variation norm, $\|\mu\| = |\mu|(T)$, $M(T)$ is isometric to the dual space of $(C_b(T), \| \|_\infty)$ ($\| \|_\infty$ denotes the sup. norm). The set of σ-additive measures defined on the σ-algebra of Baire subsets of T can be identified with a subspace $M\sigma(T)$ of $M(T)$. By $M_t(T)$ we denote the subspace of $M\sigma(T)$ composed of the t-additive measures. It is well known that the t-additive measures can be extended in a unique way to Radon measures defined over the σ-algebra of Borel sets. When T is locally compact and σ-compact it is known that $M_t(T) = M\sigma(T)$ (see [**15**]).

By $M(T)_1^+$ we denote the set $\{\mu \in M(T)\colon \mu(f) \geqq 0$ if $f \geqq 0$ and $\mu(T) = 1\}$, the set of means over T. The set $v^0 = c_0(T) \cap M(T)_1^+$ will be said to be the set of generalised v-limits since it is composed of means that extend the limit through v. It is easy to see that $v^0 = \{\mu \in M(T)_1^+\colon \mu(M) = 1$ if $M \in \tilde{v} \cap B(T)\}$, where $\tilde{v}$ denotes the filter generated by v.

Observation 1. It can be proved (see [**1**]) that $f \in c(T)$ with $\lim_v f = \alpha$ if and only if $\mu(f) = \alpha$ for every $\mu \in v^0$.

The convergence through v is a particular case of the following notion of convergence. If $\phi \neq J \subset M(T)_1^+$, we say that $f \in C_b(T)$ is J-convergent to α if $\mu(f) = \alpha$ for every $\mu \in J$. In such a case we write $J\text{-}\lim f = \alpha$. In this setting we stand by $J(T)$ (respectively $J_0(T)$) the closed subspace of $C_b(T)$ composed of the J-convergent (respectively J-convergent to zero) continuous functions.

Associated with each family J of means over T is the filter basis $J^0 = \{E \in B(T)\colon \mu(E) = 1$ for every $\mu \in J\}$. We can consider the set of means $(J^0)^0 = J^{00}$ associated to the filter basis J^0. It is obvious that $J^{00} \supset J$.

According to Observation 1, $\lim_{J^0} f = \alpha$ if and only if f is J^0 convergent to α. In this case we will say that f is strongly J-convergent to α. From $J \subset J^{00}$ it follows that if $f \in C_b(T)$ is strongly J-convergent to α, then f is J-convergent to α.

In the setting of sequences, two particular cases of strongly J-convergent functions are the strongly almost convergent and the strongly summable Cesàro sequences [**12**]. These are obtained, respectively, if we consider $J = [\sigma]$, where $\sigma(n) = n + 1$ for every $n \in \mathbb{N}$, or $J = C^*(\phi^0)$, where ϕ^0 is the set of invariant means associated with the Frèchet filter ϕ on $\mathbb{N}$ and C^* is the adjoint of the Cesàro operator $C\colon l^\infty \to l^\infty$ given by $C(x)(n) = (x(1) + \ldots + x(n))/n$.

We denote by $mJ(T)$ the vector subspace of $J(T)$ composed of its multipliers, that is, those functions $\psi \in J(T)$ such that $\psi f \in J(T)$ whenever $f \in J(T)$, and by $\bar{m}J(T)$ the subspace of $mJ(T)$ composed of those multipliers ψ such that $J - \lim(\psi f) = (J - \lim \psi)(J - \lim f)$ for every $f \in J(T)$. The subspace $\bar{m}J(T)$ will be called the set of *strict* multipliers. They are characterised by the following result in which for $c \in \mathbb{R}$, $\tilde{c}(t) = c$ for all $t \in T$ (see [**1**]).

Lemma 2. *Let $\psi \in C_b(T)$. The following statements are equivalent*:

(1) $\psi \in \bar{m}J(T)$.

(2) *There exists $c \in \mathbb{R}$ such that $J\text{-}\lim|\psi - \tilde{c}| = 0$.*

(3) *ψ is strongly convergent to c.*

We will be interested in families J of the form $J = [G]$ where $[G] = \{\mu \in M(T)\colon \mu(Lf) = \mu(f)$ for every $f \in C_b(T)$ and every $L \in G\}$ is the set of the G-invariant means with respect to the semigroup of operators generated by $G \subset \mathscr{L}(C_b(T))$.

2. Ergodic semigroups and almost convergent functions

If we consider a semigroup of operators $G \subset \mathscr{L}(C_b(T))$, an interesting question is to study when it is possible to ensure that $[G] \neq \phi$ and how to characterise the almost convergent functions with respect to $[G]$. This holds when G is an ergodic semigroup of operators on the right (this notion comes from **[8]**, taking away one of the required conditions).

A semigroup G is said to be ergodic on the right if a norm bounded net of operators $(U_i)_{i \in I}$ exists in $\mathscr{L}(C_b(T))$ such that $\lim_i \|U_i(Lf - f)\| = 0$ for every $L \in G$ and $U_i(f) \in \overline{0(f)}$ for every $i \in I$ and $f \in C_b(T)$, where $0(f) = \{L(f): f \in \text{co}(G)\}$ and co (G) stands for the convex hull of g. In such a case we will say that $(U_i)_{i \in I}$ is an ergodic system on the right.

We need the next result proved in **[8**, Theorem 3.1**]**.

THEOREM 3. *Let G be an ergodic semigroup on the right and $(U_i)_{i \in I}$ be an ergodic system on the right for G. Let $e \in C_b(T)$ verify $L(e) = e$ for every $L \in G$. For $f \in C_b(T)$, the following conditions are equivalent*:

(a) $e \in \overline{0(f)}$.
(b) $\lim_i U_i(f) = e$ *in norm.*
(c) $\lim_i U_i(f) = e$ *weakly.*

Moreover, given $g \in C_b(T)$, the following are equivalent:
(d) $\lim_i \|U_i(g)\| = 0$,
(e) $g \in B$ *where* $B = \{Lf - f : f \in C_b(T),\ L \in G\}^{\perp\perp}$.

The following result gives a characterisation of the almost convergent functions with respect to the family $[G]$.

THEOREM 4. *Let G be an ergodic semigroup on the right composed of positive operators such that $L(\tilde{1}) = \tilde{1}$ for every $L \in G$. Then $[G] \neq \phi$. Moreover, if $(U_i)_{i \in I}$ is an ergodic system on the right, given $f \in C_b(T)$ the following conditions are equivalent*:
(a) $[G]$-$\lim f = \alpha$,
(b) $f - \tilde{\alpha} \in B$,
(c) $\tilde{\alpha} \in \overline{0(f)}$,
(d) $\lim_i \|U_i(f) - \tilde{\alpha}\| = 0$,
(e) $\lim U_i(f) = \tilde{\alpha}$ *weakly.*

Proof. By applying Theorem 3, we show that (c), (d) and (e) are equivalent. Also (b) and (d) are equivalent ($U_i(\tilde{\alpha}) = \tilde{\alpha}$ since $U_i(\tilde{\alpha}) \in \overline{0(\tilde{\alpha})} = \{\tilde{\alpha}\}$).

If $f \geqq 0$, the set $0(f)$ is composed of positive functions since every $L \in G$ is positive. Therefore $U_i(f) \geqq 0$ for every $i \in I$. Let $\mathscr{U}$ be an ultrafilter containing the filter basis composed of the sections in I and let us denote by $\mathscr{U} - \lim_i a_i$ the limit through $\mathscr{U}$ of the bounded net $(a_i)_{i \in I}$.

Given a mean $\lambda \in M(T)_1^+$ and $f \in C_b(T)$, the net $(\lambda(U_i(f)))_{i \in I}$ is bounded and it is possible to define $\mu(f) = \mathscr{U} - \lim_i \lambda(U_i(f))$. Then it is clear that $\mu(f) \geqq 0$ if

$f \geqq 0$ and $\mu(\tilde{1}) = \tilde{1}$. Therefore μ is a mean. If $L \in G$ and $f \in C_b(T)$, we have $\lim_i \lambda(U_i(Lf - f)) = 0$ and then $\mu \in [G]$.

(a) $\Rightarrow$ (e). It is sufficient to see that if $\lambda \in M(T)_1^+$ then $\lambda(\alpha) = \lim_i \lambda(U_i f)$. To see this it is enough to prove that for every ultrafilter $\mathcal{U}$ that contains the filter basis composed of the sections in I we have $\alpha = \mathcal{U} - \lim_i \lambda(U_i f)$. But this is immediate since $\mathcal{U} - \lim_i \lambda(U_i f) = \mu(f)$ where $\mu \in [G]$ is the element associated to $\mathcal{U}$ and λ.

(c) $\Rightarrow$ (a). If $g \in 0(f)$, then for every $\mu \in [G]$ we have $\mu(g) = \mu(f)$. Therefore, if $\tilde{\alpha} \in \overline{0(f)}$, then $\alpha = \mu(\tilde{\alpha}) = \mu(f)$ for all $\mu \in [G]$, and f is $[G]$-convergent to α. □

In what follows, $G \subset \mathcal{L}(C_b(T))$ will be composed of positive operators holding $L(\tilde{1}) = \tilde{1}$ and $L^*(M_t(T)) \subset M_t(T)$ for every $L \in G$ (we will say that L is of class t). Besides, it is interesting to know when every G-invariant mean $\mu \in [G]$ extends the limit through ν, that is, when $\mu(f) = \alpha$ whenever $\lim f = \alpha$. It is easy to see that a necessary and sufficient condition for this is $[G] \subset \nu^0$.

On the other hand, the condition for $L \in G$ to transform every convergent function through ν into a function converging to the same value (L is ν-regular) is equivalent to $L^*(\nu^0) \subset \nu^0$ (we note that if $L \geqq 0$ and $L(\tilde{1}) = \tilde{1}$, then $L^*(\nu^0) \subset M(T)_1^+$).

If a semigroup G holds the condition $[G] \subset \nu^0$ and $L^*(\nu^0) \subset \nu^0$ for every $L \in G$, we will say that $[G]$ is ν-adequate. In all examples given later, this condition is held.

For some families $J \subset M(T)_1^+$, a sequence of measures (Π_n) in $M_t(T)_1^+$ exists such that if $f \in C_b(T)$ is J-convergent to α, then $\alpha = \lim_n \Pi_n(f)$. We will say then that J has the property (Π).

If the conditions of Theorem 4 hold, when a countable ergodic system on the right (U_n) exists and all the operators U_n are of class t, it is easy to see that such a sequence (Π_n) exists for the family $[G]$. A particular case of this occurs when G is the bounded abelian semigroup generated by a positive operator $L \in \mathcal{L}(C_b(T))$ of class t with $L(\tilde{1}) = \tilde{1}$. The corresponding ergodic system is (U_n) where $U_n = (I + L + \ldots + L^n)/(n+1)$.

The semigroups G of operators in which we are interested are those which are ν-adequate and have property (Π). The following theorem applies to this kind of semigroup.

THEOREM 5. *Let T be a locally compact and paracompact or a locally compact and σ-compact space and $G \subset \mathcal{L}(C_b(T))$ be an ergodic semigroup on the right such that every $L \in G$ is positive of class t and verifies $L(\tilde{1}) = \tilde{1}$. We will suppose that G is ν-adequate and $[G]$ holds property (Π). Then, given $\psi \in C_b(T)$, the following conditions are equivalent*:

(1) $\psi \in m[G](T)$,
(2) *ψ is convergent through the filter basis $[G]^0$ (to $\alpha \in \mathbb{R}$),*
(3) *$\alpha \in \mathbb{R}$ exists such that $|\psi - \tilde{\alpha}|$ is $[G]$-convergent to zero,*
(4) *$\lim_i U_i(|\psi - \tilde{\alpha}|) = 0$ for some $\alpha \in \mathbb{R}$ (weakly or in norm), where $(U_i)_{i \in I}$ is any ergodic system for the semigroup G.*

Proof. (2)$\Leftrightarrow$(3) by Lemma 2.
(3)$\Leftrightarrow$(4) by Theorem 4.
(2)$\Rightarrow$(1) by Lemma 2.
(1)$\Rightarrow$(2): It is sufficient to prove that for every $[G]$-convergent function g, $M_\psi(g) = \psi g$ is $[G]$-convergent to $([G] - \lim \psi)\,([G] - \lim g)$. Then $\psi \in \bar{m}[G]$, and Lemma 2 and Observation 1 complete the proof.

Given $L \in G$, we define the operator $A = M_\psi(L - I)$. Since $\psi \in m[G]$ and $(L - I)f$ is $[G]$-convergent to zero for every $f \in C_b(T)$, then Af is $[G]$-convergent and therefore we can define $\mu \in C_b(T)^*$ by $\mu(f) = [G] - \lim Af$.

It is easy to see that A is an operator of class $t(A^*(M_t(T)) \subset M_t(T))$. Since $[G]$ has property (Π), there exists a sequence (Π_n) in $M_t(T)$ such that:

$\mu(f) = [G] - \lim Af = \lim_n \Pi_n(Af) = \lim_n A^*(\Pi_n)f = \lim_n \mu_n(f)$ for every $f \in C_b(T)$, where $\mu_n \in M_t(T)$ (since A is of class t).

As T is a paracompact space, we can apply Conway's result [**2**] to see that $\mu \in M_t(T)$. If T is locally compact and σ-compact, we can apply the analogous result of Prokhorof [**15**] to see that $\mu \in M_\sigma(T) = M_t(T)$.

On the other hand, if $h \in c_0(T)$ and since L is ν-regular, $\psi(Lh - h)$ also belongs to $c_0(T)$. As $[G] \subset \nu^0$, it follows that $\mu(h) = 0$ and we have $\mu = 0$, since the Radon measure μ is zero on $c_0(T)$. Then Af is $[G]$-convergent to zero for every $f \in C_b(T)$.

Since $A = M_\psi(L - I)$, M_ψ transforms functions in B into functions which are $[G]$-convergent to zero. If g is $[G]$-convergent to b, then by Theorem 4, $g - \tilde{b}$ is in B, from which we obtain

$$\begin{aligned}[G] - \lim M_\psi(g) &= [G] - \lim M_\psi(\tilde{b}) = b([G] - \lim \psi)\\ &= ([G] - \lim g)([G] - \lim \psi). \qquad \square\end{aligned}$$

3. Applications

In the following examples, it is left to the reader to test that all the conditions established in Theorem 5 hold. It is possible to characterise the multipliers in each case in the way given by Theorem 5.

EXAMPLE 1. The space T is paracompact and locally compact, and $h: T \to T$ is a continuous proper map (the inverse image of every compact is compact) such that for every compact $K \subset T$, an $n \in \mathbb{N}$ exists such that $h^n(K) \cap K = \phi$. The semigroup G is generated by the operator $L_h(L_h(f) = f \circ h)$ (see [**1**]). This example includes as a particular case that considered in [**3**].

EXAMPLE 2. The semigroup S is a countable left cancellative and left amenable semigroup without finite left ideals endowed with the discrete topology. The semigroup G is of left translations and $G = \{L_t : t \in S,\ L_t(f)(s) = f(ts)\}$, and ν is the basis filter of the cofinite subsets of S. To test that all the conditions hold we can use [**10**], [**9**] and [**5**].

EXAMPLE 3. The group T is an amenable locally compact non compact and σ-compact topological group and $G = \{L_t : t \in T\}$ is the group of left translations. (See [**13**] and [**4**].) In this case, $\Pi_n(f)$ is given by $(1/|K_n|) \int_{K_n} f(s)\, ds$, where (K_n) is the F-sequence given by Ching-Chou [**4**].

EXAMPLE 4. The semigroup $G \subset \mathcal{L}(C_b(\mathbb{R}))$ is the bounded commutative ergodic semigroup generated by the operator $(U_s f)(x) = (\frac{1}{2s}) \int_{-s}^{+s} f(x+t)\,dt$, $(s > 0)$. (See [**14**].) In this example the almost convergent functions are precisely those functions with mean value, that is, those for which the limit $m(f) = \lim_s U_s(f)(x)$ exists uniformly in $x \in \mathbb{R}$.

EXAMPLE 5. The semigroup $G \subset \mathcal{L}(C_b(\mathbb{R}^n))$ could also be the semigroup generated by $\{U_r : r > 0\} \cup \mathcal{M}$, where $U_r(f) = (1/|B_r|) \int_{B_r} f(x+t)\,dt$ (B_r denotes the closed ball of radius $r > 0$) and $\mathcal{M}$ is the group of operators $L_h \in \mathcal{L}(C_b(\mathbb{R}^n))$ associated with the rigid motions $h: \mathbb{R}^n \to \mathbb{R}^n$.

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