BAIRE MEASURABILITY OF SEPARATELY CONTINUOUS FUNCTIONS

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1. Introduction and terminology

THE main object of this note is to improve some of the well known results concerning the measurability of separately continuous functions (s.c.f.) defined on the product of two completely regular Hausdorff spaces X and Y (see **[8]**, **[9]**, and **[15]**). If X or Y is a metrizable space, the fact that every s.c.f. $f\colon X \times Y \to \mathbb{R}$ is the pointwise limit of a sequence of jointly continuous functions (we shall write then $f \in B_1(X \times Y)$) was proved in **[15]**. Consequently, if the pointwise compact subsets of $C(X)$ are metrizable, every s.c.f. defined on $X \times Y$ is in $B_1(X \times Y)$, whenever Y is a compact space. For instance, this situation holds when X is the support of some Borel measure and has a dense σ-compact subset **[15]**. In this paper we shall extend the class of spaces X and Y for which these results hold. Moreover, we establish that if X has the discrete countable chain condition (D.C.C.C.), the reciprocal assertion is also verified. We will prove, for a completely regular space X, the equivalence of the following conditions:

(a) All the pointwise compact subsets of $C(X)$ are metrizable.
(b) X has the D.C.C.C. and every s.c.f. $f\colon X \times Y \to \mathbb{R}$ is in $B_1(X \times Y)$, whenever Y is a compact space.

Moreover, we will see that condition (b) is equivalent to the one that results replacing the D.C.C.C. by the C.C.C. (countable chain condition). Let us recall that a topological space X is said to have the D.C.C.C. (resp. C.C.C.) if every discrete (resp. pairwise disjoint) family $\mathcal{U}$ of non empty open sets is at most countable (the family $\mathcal{U}$ is discrete if each point of X has a neighbourhood meeting at most one member of $\mathcal{U}$). Spaces with the D.C.C.C. property are called pseudo-$\aleph_1$-compact by some authors. A space has the D.C.C.C. if and only if every continuous metric image is separable. Obviously C.C.C. implies D.C.C.C. If X is a Lindelöf space or $C(X)$ is an angelic space for the pointwise topology (for instance, if X has a dense σ-compact subset), then X has the D.C.C.C. (see **[20]**).

In order to simplify the enunciation of our results it will be useful to introduce the following terminology: we will say that the topological

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space X is a Moran space (resp. a weak Moran space) if every s.c.f. f defined on $X \times Y$ is in $B_1(X \times Y)$ (resp. is Baire measurable), whenever Y is any compact space. The reason for this terminology is given by Moran's results in **[9]**. These results are contained in the following theorem due to Moran and Rosenthal. All of it was shown by Moran **[9]** except for the implication (5) implies (1)–(4) which was established by Rosenthal **[14]**.

1. THEOREM. *If X is a compact Hausdorff space, then the following conditions are equivalent:* (1) *every pointwise compact subset of $C(X)$ is pointwise metrizable* (*resp. pointwise separable, norm separable*); (2) *every weakly compact subset of $C(X)$ is weakly metrizable* (*resp. weakly separable, norm separable*); (3) *if Y is a compact Hausdorff space, then every separately continuous $f\colon X \times Y \to \mathbb{R}$ is in $B_1(X \times Y)$;* (4) *if Y is a compact Hausdorff space, then every separately continuous $f\colon X \times Y \to \mathbb{R}$ is Baire measurable;* (5) *X has the countable chain condition.*

In addition to the extension of Moran's work we also give some new results about the measurability of separate continuous functions. We obtain them as a consequence of the linking between this problem and some Namioka's type theorems about the existence of a "wide" subset of points of jointly continuity (**[2]**, **[4]**, **[5]**, **[6]**, **[10]**, **[16]**, **[18]** and **[19]**). A comprehensive survey about separate continuous functions is contained in **[13]**.

2. Baire measurability of S.C.F. and pointwise compactness

If $f\colon X \times Y \to \mathbb{R}$ is a s.c.f., we write f_x and f^y for the continuous functions given by $f_x(y) = f^y(x) = f(x, y)$, and we denote X_f and Y_f the subsets of $C(Y)$ and $C(X)$ defined by $X_f = \{f_x\colon x \in X\}$ and $Y_f = \{f^y\colon y \in Y\}$, respectively. We will always suppose, unless explicitly mentioned, that X_f and Y_f are equipped with the topology of pointwise convergence on Y and X, respectively, which will be denoted by t_p.

When Y is a compact space, a theorem of Troallic (see **[19]**) establishes that every pointwise separable subset H of $C(Y)$ is norm separable. Of course, if Y is a compact space, then Y_f is also t_p-compact. A proof of Troallic's result, can be obtained applying the next theorem to the s.c.f. $f\colon H \times Y \to \mathbb{R}$ given by $f(h, y) = h(y)$.

2. THEOREM. *Let $f\colon X \times Y \to \mathbb{R}$ be a* s.c.f. *where Y is a compact space. Let us consider the following conditions:* (a) *Y_f is metrizable;* (b) *X_f is norm or t_p-separable;* (c) *f belongs to $B_1(X \times Y)$; and* (d) *f is Baire measurable. Then* (a)$\Rightarrow$(b)$\Rightarrow$(c). *Further,* (d)$\Rightarrow$(a) *whenever X has the* D.C.C.C.

Proof. (a)$\Rightarrow$(b). If (a) is verified, the Banach space $C(Y_f)$ (sup. norm) is separable. We can define on X an equivalence relation in the following way: $x_1 \sim x_2$ if and only if $f(x_1, y) = f(x_2, y)$ for every $y \in Y$. The set of equivalence classes $X^{\sim}$ is a metric space with the metric ρ defined by

$$\rho(\tilde{x}_1, \tilde{x}_2) = \sup\{|f(x_1, y) - f(x_2, y)|: y \in Y\}.$$

Since $(X^{\sim}, \rho)$ is isometric with a subset of $C(Y_f)$, we have that $(X^{\sim}, \rho)$ is separable. We can now consider the continuous map $\varphi: (X^{\sim}, \rho) \to (C(Y), \text{norm})$, defined by $\varphi(\tilde{x}) = f_x$, and we have that $\varphi(X^{\sim}) = X_f$ is norm separable.

(b)$\Rightarrow$(a). If $\{x_n: n \in \mathbb{N}\}$ is a countable subset of X such that $\{f_{x_n}: n \in \mathbb{N}\}$ is t_p-dense in X_f, the metric $d(h_1, h_2) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |h_1(x_n) - h_2(x_n)|\}$ describes the topology of the compact space (Y_f, t_p).

(a)$\Rightarrow$(c). If Y_f is metrizable, the s.c.f. $\tilde{f}: X \times Y_f \to \mathbb{R}$, defined by $\tilde{f}(x, h) = h(x)$ is in $B_1(X \times Y_f)$ **[15]**. Since $f = \tilde{f} \circ \alpha$ where α is the continuous function $\alpha(x, y) = (x, f^y)$, we have that $f \in B_1(X \times Y)$.

(c)$\Rightarrow$(d) is obvious. The proof will be complete if we show that (d)$\Rightarrow$(b) when X satisfies the D.C.C.C. In this proof we will use the fact that all the continuous metric images of a space X with the D.C.C.C. are separable. So, if $g: X \times Y \to \mathbb{R}$ is continuous, the associated mapping $G: X \to C(Y)$, given by $G(x) = g_x$, is norm continuous, and the continuous metric image $G(X) = X_g$ is separable. Let $\mathscr{C}$ be the class of all the functions $g: X \times Y \to \mathbb{R}$ such that X_g is norm separable. Since $\mathscr{C}$ contains all the continuous functions, it will be enough to show that $\mathscr{C}$ is closed with respect to pointwise limits of sequences of functions belonging to it. Let f be the pointwise limit of a sequence f_n in $\mathscr{C}$. Then, a closed norm separable subspace E of $C(Y)$ exists such that $X_{f_n} \subset E$ for each $n \in \mathbb{N}$. The Lebesgue dominated convergence theorem implies that f_x is the weak limit of $(f_n)_x$ when f_n is uniformly bounded, and so $X_f \subset E$. If f_n is not a uniformly bounded sequence, given $i \in \mathbb{N}$, we can consider the uniformly bounded sequence $f^i_n = \max\{-i, \min\{f_n, i\}\}$. Since $f^i_n \in \mathscr{C}$ we conclude that f^i belongs to $\mathscr{C}$ by the former case. Then $X_f = \bigcup_{i=1}^{m} X_{f^i}$ is norm separable.

3. THEOREM. *For a completely regular Hausdorff space X the following conditions are equivalent:* (a) *All the pointwise compact subsets of $C(X)$ are metrizable;* (b) *X is a Moran (or a weak Moran) space with the* D.C.C.C.; *and* (c) *X is a Moran (or a weak Moran) space with the* C.C.C.

Proof. (c)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(a). It follows from the application of Theorem 2 to the s.c.f.

$f: X \times H \to \mathbb{R}$, where $H \subset C(X)$ is pointwise compact and $f(x, h) = h(x)$.

(a)$\Rightarrow$(c) It was proved in **[9]** that for a compact space X the condition (a) implies the C.C.C. Since the proof also works for a completely regular space X, Theorem 2 gives us that (a) implies (c).

4. COROLLARY. *Let X be a space with the* C.C.C., *and Y a compact space. If f: $X \times Y \to \mathbb{R}$ is a* s.c.f. *that verifies:* (N) "*A dense G_δ subset X_0 of X exists such that f is jointly continuous at each point of $X_0 \times Y$*", *then f belongs to $B_1(X \times Y)$.*

Proof. It will be sufficient to show the assertion supposing that $f(X \times Y) \subset [-1, +1]$. In this case, from the proof of the main theorem in **[5]**, it follows that X_f is separable and Theorem 2 can be applied.

Let us recall that a topological space X is said to be a Namioka space if for every compact space Y all the s.c.f: $X \times Y \to \mathbb{R}$ verify the condition (N) of Corollary 4. The class of Namioka spaces contains all the Baire spaces which are separable or metrizable, and all the strongly countably complete regular spaces. Therefore, the Čech-complete spaces are Namioka spaces. Basic references about Namioka spaces are **[4]**, **[10]**, **[13]** and **[16]**.

5. PROPOSITION. *If X is a completely regular Hausdorff space, the following conditions are equivalent:* (a) *X is a Namioka space with the* C.C.C.; *and* (b) *X is a Baire Moran* (*or weak Moran*) *space with the* D.C.C.C.

Proof. (a)$\Rightarrow$(b) It is a consequence of Corollary 4, since all the completely regular Namioka spaces are Baire spaces **[16]**.

(b)$\Rightarrow$(a) If (b) holds and f: $X \times Y \to \mathbb{R}$ is a s.c.f. where Y is compact, Theorem 2 gives us that Y_f satisfies the second axiom of countability. Now, by a result of J. Calbrix and J. P. Troallic **[2]**, the associated separate continuous function $\tilde{f}$: $X \times Y_f \to \mathbb{R}$, defined by $\tilde{f}(x, h) = h(x)$ verifies the condition (N) of Corollary 4. So, f verifies (N) too and X is a Namioka space. From Theorem 3, X verifies the C.C.C.

Some simplification in the statement of some of the above results is obtained if we consider spaces X such that $C(X)$ is angelic for the pointwise topology, because these spaces verify the D.C.C.C. **[20]**. (A topological space T is angelic if the closure of each relatively countably compact subset A of T is compact and precisely consists of the limits of sequences from A). On the other hand, if $(C(X), t_p)$ is angelic and X is the support of a Baire measure, then X is a Moran space. In fact, every pointwise compact $H \subset C(X)$ is sequentially compact, and from the metrizability theorem in **[7]** we have that H is metrizable.

J. Orihuela has proved in **[11]** that $C(X)$ is pointwise angelic if X is in the class of web-compact spaces introduced by him in the following way: A topological space X is web-compact if there is a non void subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a family $\{A_\sigma\colon \sigma \in \Sigma\}$ of subsets of X verifying the following conditions: (i) $\bigcup\{A_\sigma\colon \sigma \in \Sigma\}$ is dense in X; (ii) given $\sigma = (\sigma(j))$ in Σ, if $C_{\sigma|n}$ denotes the union of $\{A_\tau\colon \tau \in \Sigma\colon \tau(j) = \sigma(j), 1 \leq j \leq n\}$, and x_n is a sequence in X such that $x_n \in C_{\sigma|n}$ for every n in $\mathbb{N}$, then this sequence has an adherent point in X.

If X has a dense countably determined subset, in particular if X is K-analytic (see **[17]** for definitions), then X is web-compact, and the class of web-compact spaces is larger than the previous one because it contains non Lindeföf spaces **[11]**.

Let us recall that a compact space K is called Gul'ko compact if the Banach space $C(K)$ is weakly countably determined (see **[1]** and **[17]** for definitions). It is proved in **[3]** that if X is a web-compact space and K is a pointwise compact subset of $C(X)$, then K is Gul'ko compact.

On the other hand, G. Debs has shown in **[6]** that if X is a Baire space and K is Gul'ko compact (even if K is Corson compact), then every s.c.f. $f\colon X \times K \to \mathbb{R}$ verifies the condition (N) in Corollary 4.

6. THEOREM. *Let Y be a countably compact space and X a web-compact space. If $f\colon X \times Y \to \mathbb{R}$ is a* s.c.f., *then f belongs to $B_1(X \times Y)$ in the following two cases:* (1) *if X is a Baire space with the* C.C.C.; (2) *if Y verifies the* C.C.C.

Proof. (1) Since $(C(X), t_P)$ is angelic **[11]**, Y_f is a pointwise compact subset of $C(X)$, and then Y_f is Gul'ko compact **[3]**. From the Debs' Theorem **[6]**, the associated function $\tilde{f}\colon X \times Y_f \to \mathbb{R}$ verifies the condition (N) in Corollary 4. Then f also verifies (N), so f is in $B_1(X \times Y)$.

(2) Since Y_f is a continuous image of Y we have that Y_f has the C.C.C., and a theorem of S. Argyros and S. Negrepontis **[1]** implies that the Gul'ko compact space Y_f is metrizable. Then we can use Theorem 2.

Remark. According to the proof of Theorem 6, all the Baire web-compact spaces are Namioka spaces.

Theorem 3 shows that, inside a wide class of topological spaces, the Moran spaces are exactly the topological spaces X verifying the condition that the t_p-compact subsets of $C_p(X)$ are metrizable. This condition does not give a general characterization of Moran spaces as the example of a metrizable and non-separable space shows. Indeed, X is a Moran space but the condition (a) of Theorem 3 is not verified because X does not have the C.C.C. However, every Moran space has the property that the t_p-compact subsets of $C(X)$ have essential metrizability conditions defined through the separable measures of Dudley (see **[21]**). A Baire measure μ defined on the Baire σ-algebra of X is said to be a separable measure ($\mu \in M_\infty(X)$) if for each continuous pseudometric d on X, there is a d-separable, d-closed subset $X_0 \subset X$ with $|\mu|(X) = |\mu|(X_0)$. In the absence of real-valued measurable cardinals, every countably additive Baire measure is a separable measure.

7. THEOREM. *Let X be a Moran space and H a pointwise compact subset of $C(X)$. For each separable Baire probability μ in $M_\infty(X)$ there exists a Baire subset X_0 of X such that $\mu(X_0) = 1$ and $H_{|X_0}$ is pointwise metrizable.*

Proof. Let f: $X \times H \to R$ be the s.c.f. defined by $f(x, h) = h(x)$, and f_n: $X \times H \to \mathbb{R}$ a sequence of jointly continuous functions converging to f in the pointwise topology. For every n, let F_n: $X \to C(H)$ be the function associated with f_n in the usual way. Since F_n is continuous in norm, we can define on the Baire σ-algebra of ($C(H)$-norm), that coincides with the Borel σ-algebra, the image measure μF_n^{-1} that is also a separable Dudley measure. Then we have a closed and separable subspace $E_n \subset C(H)$ such that $\mu F_n^{-1}(E_n) = 1$. Let E be the separable subspace generated by $\bigcup \{E_n$: $n \in \mathbb{N}\}$ and $X_0 = \bigcap \{F_n^{-1}(E_n)$: $n \in \mathbb{N}\}$ which is a Baire subset of X with $\mu(X_0) = 1$. Since $F_n(X_0) \subset E$ for every n in $\mathbb{N}$, proceeding as we have done in the proof of (d)$\Rightarrow$(b) in Theorem 2, we can obtain that $F(X_0)$ is separable where we are denoting by F: $X \to C(H)$ the mapping $F(x) = f_x$. We can apply now Theorem 2 to the restriction of f on $X_0 \times H$ and we conclude that $H_{|X_0}$ is pointwise metrizable.

Remark. The space $M^*(X)$ of Baire measures μ verifying the property established in Theorem 7 was considered in [**12**]. It was shown that its members can be characterized by the property of making Bochner integrable every weakly continuous and bounded function F: $X \to E$, whenever E is any Banach space ([**12**], Theorem 12). Then, if X is a Moran space, E a Banach space, and F: $X \to E$ is weakly continuous and bounded, then F is Bochner integrable with respect to each separable measure.

As an application of Theorem 7 another sufficient condition is shown for the Bochner measurability of a weakly continuous function. Let us remark that the hypothesis on E in the next proposition is verified when E is the dual of a separable Banach space.

8. PROPOSITION. *Let E be a Banach space such that the unit ball $B(E^*)$ verifies the* C.C.C. *for the weak*-topology. If X is web-compact and F: $X \to E$ is weakly continuous, then $F(X)$ is a separable subset of E.*

Proof. Let us consider the s.c.f. f: $X \times Y \to \mathbb{R}$ where $Y = (B(E^*)$, weak*) and $f(t, x') = \langle x', F(t) \rangle$. $Y_f = \{x' \circ F$: $x' \in B(E^*)\}$ is metrizable bearing in mind the proof of (b) in Theorem 6 and so $F(X) = X_f$ is norm separable.

An open problem around these ideas should be to obtain a useful topological characterization of Moran spaces. At present, the only Moran spaces we know are the metrizable spaces or the topological spaces that verify the condition (a) of Theorem 3. If a Baire space X belongs to one of the former classes it is a Namioka space (because of Proposition 5 and **[16]**). It seems to be natural now to ask the following question: Is it true that every Baire and Moran space is a Namioka space?

In connection with this question it is natural to examine Talagrand's example of a Baire space T which is not a Namioka space **[18]**. This space is not a weak Moran space and does not have the D.C.C.C. Since T is a P-space (i.e., every countable intersection of open sets is open), the first assertion follows from the following observation. If X is a weak Moran P-space, then X is a Namioka space. Indeed, if Y is a compact space and $g: X \times Y \to \mathbb{R}$ is a jointly continuous function then g has the following property (*). For each point x of X, there is a neighborhood V_x of x such that g is constant on $V_x \times Y$. This is a consequence of the P-space property. Another application of the P-space property shows that the pointwise limit f of a sequence of functions f_n having the property (*) also has this property. Then every Baire measurable function $f: X \times Y \to \mathbb{R}$ verifies (*) and it will be continuous on $X \times Y$.

In order to show that in Talagrand space T the D.C.C.C. fails, it is enough to consider I equal to $[0, 1]$ in the description of T. For each irrational t in $[0, 1]$, let A_t be a fixed sequence of rationals converging to t and let $B_t = Q - A_t$, where Q is the set of rationals in $[0, 1]$. Let $U_t = \{h: h \mid A_t = 1, h \mid B_t = 0\}$. Each U_t is a clopen set in T. If g is any member of T, then $\{h: h|_Q = g|_Q\}$ is a neighborhood of g in T which meets at most one U_t. Hence $\{U_t: t \text{ irrational}\}$ is an uncountable discrete family of non-empty clopen subsets of T.

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