SOME NAMIOKA LIKE THEOREMS

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INTRODUCTION

In **[F]** R.E.Feiock proves the following result: Let Z be a space which is compact and metrizable, let T be an arbitrary topological space, and let Y be second countable. Suppose that D is an dense subset of Y, and φ:TxY ⟶ Z is everywhere continuous in y∈Y and continuous in t on DxY. Then there is a set Ω which is residual in T such that φ is jointly continuous on ΩxY. In [K1] Kenderov extends above result to the case of a separable metric space Z. For a metrizable compact space Y and Z = ℝ, this result can be reformulated in the following way: If f:T ⟶ C(Y) is continuous for $t_p(D)$ (the topology of pointwise convergence of a dense subset D of Y) then there is a residual subset Ω of T such that f is norm continuous at each point of Ω. From this particular case follows the following previous result in [AO]: Let T be a complete metric space, X a Banach space and and B a norming subset of the unit dual ball B_X*. Suppose that f:T ⟶ X is σ(X,B) continuous and f(T) is norm-separable. Then f is norm continuous at each point of a residual set Ω ⊆ T.

In [St1] Ch.Stegall gives the following reformulation of Namioka's theorem: Suppose T is a Cech-Complete space and f:T ⟶ X is a function into the Banach space X which is σ(X,Ext) continuous, where Ext is the set of extreme points of the unit ball B_X*. Then f is norm continuous at each point of a dense G_δ-subset of T.

In this paper we give some more results about the existence of point of continuity in the norm for a map f:T ⟶ X wich is assumed to be σ(X,B) continuous, where B is a norming subset of B_X*.

Throughout the paper we consider the more general framework of a map f:T ⟶ C(K) which is continuous for the topology $t_p(D)$ of pointwise convergence on D, where D is a fixed dense subset of the compact space K. Our main result extends and give a unified proof of previous results of Debs **[D1]** and Deville **[De]** relatives to the topology of pointwise convergence on K.

VALDIVIA COMPACT SPACES

For any infinite set I, let $\Sigma(I)$ be denote the subset of the cube $[0,1]^I$ consisting of functions $x:I\longrightarrow[0,1]$ such that their support $S(x) = \{i\in I: x(i)>0\}$ is at most countable. If K is (homeomorfic to) a closed subset of some $[0,1]^I$ and $K\cap\Sigma(I)$ is dense in K then K is said to be a Valdivia compact. If K is a Valdivia compact subset of some $[0,1]^I$ we assumethat we have fixed a dense subset D of K such that $D \subseteq K\cap\Sigma(I)$.

We start fixing some notation. If M is a subset of I then R_M will be the continuous map from $[0,1]^I$ into $[0,1]^I$ defined by $R_M(x)(i) = x(i)$ if $i\in M$, $R_M(x)(i) = 0$ if $i\notin M$. If $R_M(K)\subseteq K$ we shall say that M is a good subset of I (relative to K); in this case the restriction $r_M = R_M|_K$ is a continuous retraction from the compact space K onto the metrizable compact subset $K_M = R_M(K)$. The retraction r_M define a continuous linear projection Π_M on C(K) by the formula $\Pi_M(f) = f\circ r_M$. The image of this projection is the subspace of C(K) formed by the functions f of C(K) such that $f(x) = f(r_M(x))$ for every $x\in K$ and it can be be identified to the Banach space $C(K_M)$ in a natural way. If C is a subset of K let $Supp(C) = \bigcup\{Supp(x):x\in C\}$.

1.-**Lemma:** *If C_0 be a countable subset of D, there exists a countable subset C of D such that $C_0\subseteq C$, $M = Supp(C)$ is a good subset of I and $\bar{C} = K_M$.*

Proof: We start with the countable set $M_0 = Supp(C_0)$. Suppose that we have obtained a finite increasing sequence $C_0\subseteq C_1\subseteq C_2 \ldots C_n$ of countable subsets of D, and a finite increasing sequence $M_0\subseteq M_1\subseteq M_2\ldots M_n$ of countable subsets of I such that $M_k = Supp(C_k)$ for k=1,..n. In order to define C_{n+1} we consider the metrizable compact subset $R_{M_n}(K)$ of $[0,1]^I$ and the dense subset $R_{M_n}(D)$ and obtain a countable subset D_n of D such that $R_{M_n}(D_n)$ is dense in $R_{M_n}(K)$. Now we define $C_{n+1} = C_n \cup D_n$ and $M_{n+1} = Supp(C_{n+1})$ and this completes the inductive definition of the sequences C_n, M_n. The union C of the increasing sequence C_n is a countable subset of D such that $M = Sop(C)$ is a good subset of I. Indeed, it is clear that M is the union of the increasing sequence M_n and $R_M(x) = \lim_n R_{M_n}(x)$ for each $x\in K$. If $x\in K$ the sequence $R_{M_n}(x)$ is in the closure of $R_{M_n}(D_n) \subseteq R_M(C) = C$ and we can assure that $R_M(x) \in \bar{C} \subseteq K$ so M is a good set and $K_M \subseteq \bar{C}$. Since $C\subseteq K_M$ the equality

$\bar{C} = K_M$ follows.

Now we denote by $\mathcal{Z}$ the family formed by all the compact subset Z of the Valdivia compact K which are formed taking the closure of some countable subset C of D such that Supp(C) is a good subset of I:

$$\mathcal{Z} = \{\bar{C} : C \subseteq D,\ C \text{ is countable and Supp(C) is a good subset of } I\}$$

Using lemma 1 it is easy to check that the family $\mathcal{Z}$ has the following properties:

a) $\mathcal{Z}$ is directed by inclusion, $K = \overline{\bigcup \mathcal{Z}}$, and $D \cap Z$ is dense in Z for each Z in $\mathcal{Z}$.

b) Each $Z \in \mathcal{Z}$ is a metrizable compact set and there exists a continuous retraction $r_Z : K \longrightarrow Z$.

c) If $Z_1 \subseteq Z_2 \subseteq \ldots \subseteq Z_n \ldots$ is an increasing sequence in $\mathcal{Z}$ then $Z = \overline{\bigcup_{n=1}^{\infty} Z_n}$ belongs to $\mathcal{Z}$ and $r_Z(x) = \lim_n r_{Z_n}(x)$ for every $x \in K$.

Now we point out other class of compact spaces such that given a dense subset D of a compact K in the class it is posible to obtain a family $\mathcal{Z}$ of compact subsets of K with the above properties.

Let us consider the class of all compact intervals $[0,\Gamma]$ where Γ is an ordinal. This class is not contained in the one of Valdivia compact because if ω_2 is the first ordinal of cardinality strictly larger than ω_1 (the first uncountable ordinal) then $[0,\omega_2]$ is not a Valdivia compact (See [*]). If D is a dense subset of $[0,\Gamma]$ we can take for $\mathcal{Z}$ the class of compact subsets of $[0,\Gamma]$ which are the closure of countable subsets of D. Since the closure of a countable subset C of $[0,\Gamma]$ is countable, every set Z in $\mathcal{Z}$ is metrizable.

We can consider the continuous retraction $R_Z : [0,\Gamma] \longrightarrow Z$ defined by $R_Z(\alpha) = \mathrm{Min}\,(Z \cap [\alpha,\Gamma])$ if $Z \cap [\alpha,\Gamma] \neq \phi$, $R_Z(\alpha) = \mathrm{Max}\, Z$ if $Z \subseteq [0,\alpha)$, and it is easy to check that a),b) and c) holds.

THE MAIN RESULT

2.-Lemma: *Let K be a compact space, D a dense subset of K and f a continuous map from the Baire space W into the space $(C(K), t_p(D))$ such that for every non empty open subset V of W we have $\|\ \|$-diameter$f(V) > \varepsilon > 0$. Suppose that there exists a family $\mathcal{Z}$ of compact subsets of K holding condition a),*

Then for every finite subset $\mathcal{F}$ of C(K) and each non empty open subset U of W, it is possible to find an element Z of $\mathcal{Z}$ and a non empty open subset V of U such that $\|\Pi_Z(f(x)-\varphi)\| > \varepsilon/3$ for each x in V and each φ in $\mathcal{F}$.

Proof: For $\varphi \in C(K)$ and $\varepsilon > 0$ let $B(\varphi,\varepsilon) = \{\psi \in C(K): \|\psi - \phi\| \leq \varepsilon\}$. Since $B(\varphi,\varepsilon)$ is closed for the topology $t_p(D)$ and f is $t_p(D)$-continuous we have that $C(\varphi,\varepsilon) = f^{-1}(B(\varphi,\varepsilon))$ is a closed subset of W. From the hypothesis about f the closed sets $C(\varphi,\varepsilon/3)$ have empty interior. The set $W - \bigcup\{C(\varphi,\varepsilon/3): \varphi \in \mathcal{F}\}$ is non empty (because W is a Baire space and $\mathcal{F}$ is finite) an we can choose a point w in this set. It is clear that $\|f(w) - \varphi\| > \varepsilon/3$ for each $\varphi \in \mathcal{F}$. By property a) of $\mathcal{Z}$ we can obtain an element Z of $\mathcal{Z}$ and a finite subset $\{t_\varphi : \varphi \in \mathcal{F}\}$ of $Z \cap D$ such that $|f(w)(t_\varphi) - \varphi(t_\varphi)| > \varepsilon/3$ for each $\varphi \in \mathcal{F}$. Now, the open set $V = \{x \in W: |f(x)(t_\varphi) - \varphi(t_\varphi)| > \varepsilon/3\}$ is non empty and it is clear that for every $x \in V$ and each $\varphi \in \mathcal{F}$ we have $\|(f(x) - \varphi) \circ r_Z\| > \varepsilon/3$.

In order to prove the next theorem we need a remarkable result, due to Krom ([K, Theorem 1]) that characterizes Baire spaces in terms of the topological game of Banach-Mazur: A Hausdorff space T is Baire iff it is β-defavourable for the game of Banach-Mazur on T. This game between players α and β can be described as follow: Player β is the first to move and at his *n*th move he chooses a non empty open set V_n in T. Player α answers by choosing an open non empty set U_n contained in V_n, then β chooses V_{n+1} lying in the opponents's previously chosen open set U_n etc... . The rule is that α wins whenever he can choose his U_n sets so that $\bigcap_{n=0}^{\infty} U_n \neq \phi$. The space T is called β-defavourable if β does not have any winning strategy in the game of Banach-Mazur over T.

The proof of the following theorem is indeed an adaptation of the proof of the analogous result of G.Debs [D] relative to a Corson compact K and the pointwise topology on C(K)

3.-Theorem: *Let K be a compact space and D a dense subset of K. Suppose that there are a family $\mathcal{Z}$ of compact subsets of K holding conditions a), b) and c). If T is a Baire space and $f: T \longrightarrow C(K)$ is $t_p(D)$-continuous, there exists a $\mathcal{G}_\delta$ dense subset Ω of T such that f is norm continuous at every point of Ω.*

Proof: Using the usual argument the whole proof of the theorem can be reduced to derive a contradiction from the following assumption: There exists $\varepsilon > 0$ and a non empty open set $W \subseteq T$ such that $\| \ \|$-diam $f(V) \geq \varepsilon$ for each non empty open subset V of W.

Let us start the Banach-Mazur game on W, between the players α and β.

In order to give a description of a strategy for the player β we start fixing a countable dense subset $\{\psi_{Z,n}:n\in\mathbb{N}\}$ of each C(Z). In the first move β chooses $V_0 = W$ and α answer by choosing some non empty open subset $U_0 \subseteq V_0$. Now β fixes some $Z_0 \in \mathcal{Z}$, considers the finite set $\mathcal{F}_0 = \{\psi_{Z_0,1}\}$ and using lemma **2** obtains a non empty open subset $V_1 \subseteq U_0$ and $Z_1 \in \mathcal{Z}$, such that $Z_0 \subseteq Z_1$ and $\|\Pi_{Z_1}(f(x)-\varphi)\| > \varepsilon/3$ for each $x\in V_1$ and each $\varphi\in\mathcal{F}_0$

If α answers choosing $U_1 \subseteq V_1$ then β considers the finite set

$$\mathcal{F}_1 = \{ \psi_{Z_0,1} , \psi_{Z_0,2} , \psi_{Z_1,1} , \psi_{Z_1,2} \}$$

and by lemma **2** he obtains a non empty open set $V_2 \subseteq U_1$ and a new set $Z_2 \in \mathcal{Z}$, such that $Z_1 \subseteq Z_2$ and $\|\Pi_{Z_2}(f(x)-\varphi)\| > \varepsilon/3$ for each $x\in V_2$ and each $\varphi\in\mathcal{F}_1$.

The game continues according the strategy of the player β and he obtains an increasing sequence $\mathcal{F}_n$ of finite subsets of C(K), a increasing sequence Z_n in $\mathcal{Z}$ and a decreasing sequence of non empty open sets $V_n \subseteq W$ such that:

i) $\mathcal{F}_n = \{\psi_{Z_0,1},\dots\psi_{Z_0,n+1}, \psi_{Z_1,1},\dots \psi_{Z_1,n+1},\dots \psi_{Z_n,1},\dots \psi_{Z_n,n+1}\}$

ii) $\|\Pi_{Z_n}(f(x)-\varphi)\| > \varepsilon/3$ for each $x\in V_{n+1}$ and each $\varphi\in\mathcal{F}_n$

Since W is a Baire space, if we assume that β plays according this strategy, the player α can choose his moves in such a way that he wins and we can obtain a point $x \in \bigcap_{n=1}^{\infty} V_n$. From the property c) of the family $\mathcal{Z}$ the set $Z = \overline{\bigcup_{n=1}^{\infty} Z_n}$ belongs to $\mathcal{Z}$ and $\bigcup_{n=1}^{\infty} C(Z_n)$ is a dense subalgebra of C(Z) (we are considering each $C(Z_n)$ as a complemented subspace of C(Z)). Since $\mathcal{F} = \bigcup_{n=1}^{\infty}\mathcal{F}_n$ is dense in $\bigcup_{n=1}^{\infty} C(Z_n)$ we obtain that $\mathcal{F}$ is dense in C(Z) and this is in contradiction with the following fact: For every $\varphi \in \mathcal{F}$ there is some $n\in\mathbb{N}$ such that $\varphi\in\mathcal{F}_{n-1}$, so $\|\Pi_Z(f(x))-\varphi\| = \|\Pi_Z(f(x)-\varphi)\| > \|\Pi_{Z_n}(f(x)-\varphi)\| > \varepsilon/3$.

4.-Theorem: *Let K be a Valdivia compact subset of $[0,1]^I$, D dense subset of K formed by elements of countable support, T a topological space and $f:T \longrightarrow C(K)$ a $t_p(D)$-continuous map. If T is a Baire space then there exists a $\mathcal{G}_\delta$ dense subset Ω of T such that f is norm continuous at every point point of Ω.*

5.Corollary: *Let X be a Banach space such that $(B_{X^*},weak^*)$ is a Corson compact, B a norming subset of B_{X^*} and T a Baire space. If $f:T \longrightarrow X$ is a $\sigma(X,B)$-continuous map then there exists a $\mathcal{G}_\delta$ dense subset Ω of T such that f is norm continuous at every point point of Ω.*

6.-Theorem: *Let Γ an ordinal, D a dense subset of the compact [0,Γ] and f:T ⟶ C(K) a t_p(D)-continuous map. If T is a Baire space then there exists a $\mathcal{G}_\delta$ dense subset Ω of T such that f is norm continuous at every point point of Ω.*

7.-Theorem: *Let T be a Baire space with a dense countably determined subspace, K a compact space and D a dense subset of K. If f:T ⟶ C(K) is t_p(D)-continuous, there exists a $\mathcal{G}_\delta$ dense subset Ω of T such that f is norm continuous at every point point of Ω.*

Proof: For each x in K let φ_x be the real function on T defined by $\varphi_x(b) = f(b)(x)$. Then $\hat{K} = \{\varphi_x : x \in K\}$ is a pointwise compact subset of C(T) and by [*] we know that $\hat{K}$ is Gul'ko compact and hence Corson compact [*]. Now we can consider the map $\hat{f}:T \longrightarrow C(\hat{K})$ defined by: $\hat{f}(b)(\varphi_x) = f(b)(x)$ which is $t_p(\hat{D})$-continuous. By theorem 5 there exists a $\mathcal{G}_\delta$ dense subset Ω of T such that $\hat{f}$ is norm continuous at every point point of Ω and it is clear that f has the same property.

8.-Corollary: *Let T be a Baire space with a dense countably determined subspace, X a Banach space B a norming subset of B_{X^*} and f:T ⟶ X a σ(X,B)-continuous map. Then there exists a $\mathcal{G}_\delta$ dense subset Ω of T such that f is norm continuous at every point point of Ω.*

9.-Theorem: *Let T be a Baire Cech-analytic space, K a compact space such that (C(K),t(K)) is Lindelof and D a dense subset of K. If f:T ⟶ C(K) is t_p(D)-continuous, there exists a $\mathcal{G}_\delta$ dense subset Ω of T such that f is norm continuous at every point of Ω.*

10.-Theorem: *Let T be a Baire Cech-analytic space, X a Banach space such that (B_{X^*},weak*) is sequentially compact and B a boundary of B_{X^*}. If f:T ⟶ X is σ(X,B)-continuous then there exists a $\mathcal{G}_\delta$ dense subset Ω of T such that f is norm continuous at every point of Ω.*

Remark: If we assume in above results that T is a hereditarily Baire metric space then we conclude that f is fragmentable and so we can apply a result from [*] to obtain that, in this cases, f is the norm pointwise limit of a sequence of norm continuous functions f_n:T ⟶ C(K).

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Lemma: *If $C \subseteq [0,\Gamma]$ is countable then $\bar{C}$ is countable*

Proof: Let $C' = \{x_\alpha : 0 \le \alpha < \beta\}$ be the set of all acumulation point of C. We can assume that $x_\alpha < x_\gamma$ iff $\alpha < \gamma$, so for each $\alpha \in [0,\beta]$ we have $\mathrm{Min}\{x \in C' : x > x_\alpha\} = x_{\alpha+1}$. Hence, there exists $c_\alpha \in C$ such that $x_\alpha < c_\alpha < x_{\alpha+1}$.

The map $x_\alpha \longrightarrow c_\alpha$ is a biyection between $[0,\beta)$ and a subset of C, so $[0,\beta)$ is countable.