# FUNCTIONS OF THE FIRST CLASS

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## Introduction

The significance of some recent results about first class selectors of some upper semicontinuous multi-valued maps (**[JR1]**, **[JR2]**, **[JR3]**, **[HJT]**, **[Han3]**, **[St]**, and **[JOPV]**) has impelled the study of the non separable theory of first class functions (**[Rog]**, **[Fos]**, **[Han4]**, **[St]** and **[Ves]**). It is our aim in this talk to give a report of results concerning the characterization of first Borel class and first Baire class functions $f:X \longrightarrow E$, from a topological space X into a metric space E, pointing out the more general situation where they hold.

Every function in the first Baire class is in the first Borel class and for a Banach space E and a complete metric space X both classes are the same class that can be characterized, in the classical way, by the point of continuity property (**[St]** and **[Han4]**).

If E is a Banach space and the metric space X is not assumed to be complete, the first Baire class functions coincide with the $\sigma$-discrete first Borel class functions (**[Han2]**, **[Han3]**, **[Han4]**, **[HJT]**, and **[Rog]**).

However, in **[Fos]** it is shown that this result does not hold for an arbitrary metric space E, and it is proved the coincidence of both classes when X is a metric space and E is an arcwise connected and locally arcwise connected metric space. Hansell **[Han4]** showed the same result for a collectionwise normal topological space X and a closed convex subset E of a Banach space. A unification of these results has been given in **[Ves]** showing that the first Baire class functions coincide with the $\sigma$-discrete first Borel class functions if X is collectionwise normal and the metric space E is arcwise connected and locally arcwise connected.

On the other hand, when the Baire assumptions are removed from the space X, it is a well known fact that the point of continuity characterization of first class functions is not feasible. Nevertheless, in the case of a Banach space E and a perfectly paracompact space X, in **[JOPV]** a similar characterization has been obtained replacing the point of continuity property by a condition of small oscillations similar to the one considered by Lebesgue (**[Le1]**, **[Le2]** and **[Le3]**).

For a detailed study of classical results concerning first class functions from a metric space X into a separable metric space E, a complete account of the standard theory can be found in **[Ku]**.

**Definitions and preliminary results.**

We shall denote by X a Hausdorff topological space and by E a metric space whit metric $\rho$. The family of all closed (resp. open) subsets of X will be denoted by $\mathcal{F}(X)$ (resp. $\mathcal{G}(X)$). We shall denote by $\mathcal{Z}(X)$ (resp. $\mathcal{U}(X)$) the subfamily of $\mathcal{F}(X)$ (resp. $\mathcal{G}(X)$) formed by the zeros (resp. cozeros) of continuous real functions on X. If $\mathcal{H}$ is a family of subsets of X then $\mathcal{H}_\sigma$, (resp. $\mathcal{H}_\delta$) will be the family of countable unions (resp. intersections) of sets in $\mathcal{H}$. A topological space X is said to be a <u>perfect</u> space if $\mathcal{F}(X) \subseteq \mathcal{G}_\delta(X)$. However, for every topological space X we have $\mathcal{Z}(X) \subseteq \mathcal{U}_\delta(X)$.

**1 Definitions:** *We shall denote by $F_\sigma(X,E)$ (resp. $Z_\sigma(X,E)$) the set of all functions $f:X \longrightarrow E$ such that $f^{-1}(G) \in \mathcal{F}_\sigma(X)$ (resp. $f^{-1}(G) \in \mathcal{Z}_\sigma(X)$) for each open subset G of E and by $B_1(X,E)$ the set formed by the functions $f:X \longrightarrow E$ which are the pointwise limit of some sequence of continuous functions $f_n:X \longrightarrow E$. If f belongs to $F_\sigma(X,E)$ (resp. $B_1(X,E)$) then f is said to be of the first Borel class (resp. first Baire class).*

*PC(X,E) will be denote the set of functions $f:X \longrightarrow E$ having the point of continuity property (i.e. the restriction of f to each closed subset of X has a point of continuity).*

**2 Proposition:** *Each function of the first Baire class is of the first Borel class: $B_1(X,E) \subseteq F_\sigma(X,E)$.*

This is an easy and well known result (**[Ku]**, 31.VIII, Th.1), but its converse may fail if the space E is disconnected between two of its points (**[Ku]**, remark on p.391). It has been pointed out in **[La]** that the equality $B_1(X,E) = F_\sigma(X,E)$ may also fail, if $E = \mathbb{R}$ and X is a completely regular space X. Hence, to require that every function of the first Borel class is the limit of a sequence of continuous functions is a strong condition that impose some restrictions on the domain space X and on the range space E. Some of the results collected in this paper indicates under what general conditions the converse is true.

In the characterization of the first Baire class function as the first

Borel class function, even in the case of real valued functions, usually has been assumed that X is a perfect space. If X is a normal space, the elimination of this restrictive assumption has been pointed out in **[LMZ]** and **[La]** for real valued functions, and in **[Han4]** and **[Ves]** in the case of functions valued in some metric spaces. Hansell attains the result showing that in this situation each first Borel function has a $\sigma$-discrete base consisting of closed $\mathcal{G}_\delta$-sets. Vesely **[Ves]** obtains the same conclusion using the following easy but not well known result:

**3 Proposition:** *If X is a normal space then* $F_\sigma(X,E) = Z_\sigma(X,E)$.

The proof follows in an easy way from the following fact: If X is a normal space, $A \in \mathcal{F}_\sigma(X)$, $B \in \mathcal{G}_\delta(X)$, and $A \subseteq B$, then there exists $H \in \mathcal{U}_\delta(X)$ such that $A \subseteq H \subseteq B$ (**[Ves]**, Lemmas 1.7, 1.8).

**4 Proposition:** *If X is a metric space and* $f:X \longrightarrow E$ *has the point of continuity property then f is of the first Borel class:* $PC(X,E) \subseteq F_\sigma(X,E)$.

See **[Ku]**, 31.X, Th.2 and remark 1 on p.395. The converse is not true by remark 3 on p.396.

One of the results in **[JOPV]** give us an improvement of Prop.4 assuming that f is $\sigma$-fragmented by closed sets (a condition weaker than the point of continuity property; see definition 8). On the other hand, the converse of the improvement of Prop.4 holds under rather general hypothesis (Th. 10).

**Resume of the main results.**

We start collecting some of the definitions to be used for the extension of the classical results (for functions with separable range) to the general case of functions with non separable range.

A family $\mathcal{H}$ of subsets of X is said to be discrete if each point of X has a neighborhood meeting at most one member of $\mathcal{H}$. If there is families $\{Z_H : H \in \mathcal{H}\} \subseteq \mathcal{Z}(X)$, $\{V_H : H \in \mathcal{H}\} \subseteq \mathcal{U}(X)$ such that $H \in Z_H \subseteq V_H$ for each $H \in \mathcal{H}$ and $\{V_H : H \in \mathcal{H}\}$ is discrete then we shall say that $\mathcal{H}$ is strongly discrete.

If X is a normal space, the family $\mathcal{H}$ is strongly discrete iff there is a discrete family of open sets $\{G_H : H \in \mathcal{H}\} \subseteq \mathcal{G}(X)$ such that $\bar{H} \in G_H$ for each $H \in \mathcal{H}$ (this is the notion of strongly discrete family that has been considered in **[Ves]**). The family $\mathcal{H}$ is called $\sigma$-discrete when $\mathcal{H}$ is a countable union of discrete families.

A family $\{H_i : i \in I\}$ of subsets of X is said to be $\sigma$-discretely decomposable (abbreviated $\sigma$-d.d.) if for every $i \in I$, $H_i = \bigcup \{H_{i,n} : n \in \mathbb{N}\}$ where each family $\{H_{i,n}:i \in I\}$ is discrete.

A family of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is said to be a base for $\mathcal{H}$ if every $H \in \mathcal{H}$ is a union of elements of $\mathcal{B}$. $\mathcal{B}$ is said to be a base for $f:X \longrightarrow E$ if $\mathcal{B}$ is a base for the family $\{f^{-1}(G):G \in \mathcal{G}(E)\}$. The map f is called <u>$\sigma$-discrete</u> if there exists a $\sigma$-discrete base $\mathcal{B}$ for f. Basic reference for $\sigma$-discrete maps are **[Han1]**,**[Han2]**,and **[Han4]** where they take part in the extension to the non-separable case of the usual properties of Borel measurable functions taking values in a separable metric space. If in above definitions we replace discrete families by strongly discrete families we obtain the corresponding notions of strongly $\sigma$-discrete family, strongly $\sigma$-d.d. family, strongly $\sigma$-discrete function, etc. Note that in collectionwise normal spaces these notions coincide with the previous ones. (We recall that a Hausdorff space X is said to be <u>collectionwise normal</u> if for every discrete family $\{F_\alpha:\alpha \in A\}$ of closed subsets of X there exists a discrete family $\{G_\alpha:\alpha \in A\}$ of open subsets of X such that $F_\alpha \subseteq G_\alpha$ for every $\alpha \in A$).

The class of all $\sigma$-discrete (resp. strongly $\sigma$-discrete) functions $f:X \longrightarrow E$, will be denoted $\Sigma(X,E)$ (resp. $\Sigma^*(X,E)$). It is obvious that every map with values in a separable metric space is strongly $\sigma$-discrete and it is easy to check that each continuous function is strongly $\sigma$-discrete.

Under suitable hypothesis about X every function of the first Borel class is $\sigma$-discrete and then some of the next results are true with the reference to $\sigma$-discrete functions omitted. In **[Han1]** Hansell shows that if X is an absolutely analytic metric space (i.e. X is a Souslin $\mathcal{F}$-set in its completion) then every Borel measurable function from X into a metric space E is $\sigma$-discrete. W.G.Fleissner **[Fl]** has shown that is relatively consistent with the axioms of set theory to assume that every Borel map between metric spaces is $\sigma$-discrete.

We continue with the terminology introduced in **[JOPV]**:

5 **Definition:** *We shall say that a partition $\{H_i:i \in I\}$ of X is a good partition (resp. a good partition in the strong sense) if it is a $\sigma$-d.d. family formed by $\mathcal{F}_\sigma$-subsets of X (resp. a strongly $\sigma$-d.d. family formed by zero sets). A map $f:X \rightarrow E$ will be said to be $\tau$-constant (resp. $\tau^*$-constant) if there is a good partition (resp. a good partition in the strong sense) $\{H_i:i \in I\}$ of X such that every restriction $f|H_i$ is constant.*

*We shall say that f is piecewise continuous if X can be expressed as the union of an increasing sequence of closed sets $X_n$, such that all the restrictions $f|X_n$ are continuous.*

6 **Remarks:** Every countable partition of X formed by $\mathcal{F}_\sigma$-sets (resp. $\mathcal{Z}_\sigma$-sets) is a good partition (resp. a good partition in the strong sense) and we shall say that a map f is $\sigma$-constant (resp. $\sigma^*$-constant) if there is a countable partition $\{H_i:i \in I\}$ of X formed by $\mathcal{F}_\sigma$-sets (resp. $\mathcal{Z}_\sigma$-sets) such that every restriction $f|H_i$ is constant. It follows from Prop.3 that if X is normal then every $\sigma$-constant function is $\sigma^*$-constant.

Note that if X is collectionwise normal every $\tau$-constant function is $\tau^*$-constant. If f is $\tau$-constant (resp. $\tau^*$-constant) then $f^{-1}(T)$ belongs to $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$ (resp. $\mathcal{Z}_\sigma \cap \mathcal{U}_\delta$) for every subset T of E, so f is in the first Borel class when we consider E endowed whit its discrete topology.

If the metric space E has the <u>extension property</u> for X (i.e. each continuous function defined on a closed subset of X into E, has a continuous extension to X) and $f:X \longrightarrow E$ is piecewise continuous then there exists a sequence of continuous functions $f_n:X \longrightarrow E$ such that for each $x \in X$ the sequence $f_n(x)$ converges to $f(x)$ in the discrete topology ($f_n(x) = f(x)$ if $n \geq n(x)$) so f is of the first Borel class.

Next proposition, concerning the structure of $\tau$-constant and $\tau^*$-constant functions, is the key for understand the role performed by these functions. It is implicit in **[Ves]** and **[Han4]**, (see also**[JOPV]**).

7. **Proposition:** *Let $f:X \longrightarrow E$ be a $\tau$-constant (resp. $\tau^*$-constant) function and $\{H_i:i \in I\}$ a good partition (resp. a good partition in the strong sense) of X such that every restriction $f|H_i$ is constant. If X is a perfect (resp. arbitrary) space then there is a family of closed sets (resp. zero sets) $\{F_{i,m}:i \in I, m \in \mathbb{N}\}$ such that:*

*a) For each $m \in \mathbb{N}$ the family $\{F_{i,m}:i \in I\}$ is discrete (resp, strongly discrete);*

*b) For each $i \in I$ the sequence $\{F_{i,m}:m \in \mathbb{N}\}$ is increasing and*

$$H_i = \bigcup\{F_{i,m}:m \in \mathbb{N}\}$$

*Its follows that $C_m = \bigcup\{F_{i,m}:i \in I\}$ is an increasing sequence of closed sets (resp. zero sets) such that every restriction $f|C_m$ is continuous, so f is piecewise continuous.*

*If E is arcwise connected and f is a $\tau^*$-constant function then there exists a sequence of continuous functions $f_n:X \longrightarrow E$, such that for each $x \in X$ the sequence $f_n(x)$ converges to $f(x)$ in the discrete topology ($f_n(x) = f(x)$ if $n \geq n(x)$), so f is in in the first Baire class.*

**8 Definition:** *Let $f$ be a map from the topological space $X$ into the metric space $E$ and $\varepsilon$ a positive number. We shall say that a subset $Y$ of $X$ has the property $(f,\varepsilon)$ if for each non empty subset $C$ of $Y$ there exists an open subset $U$ of $X$ such that $U \cap C \neq \phi$ and $\rho$-diam $f(U \cap C) \leq \varepsilon$.*

*The function $f$ is said to be fragmented on $Y$ if $Y$ has the property $(f,\varepsilon)$ for every $\varepsilon > 0$.*

*The function $f$ is said to be $\sigma$-fragmented by closed sets if the following property holds:*

*(D'): For every $\varepsilon > 0$ there is a sequence of closed subsets $X_n$ of $X$ such that $X = \bigcup \{X_n : n \in \mathbb{N}\}$ and every $X_n$ has the property $(f,\varepsilon)$.*

If in condition (D') the sets $X_n$ are required to be elements of $\mathcal{Z}(X)$ the map f will be said to be $\sigma$-fragmented by zero sets.

Before stating next theorem, that resume the main results about the non separable theory of first class functions, we recall some of the toplogical notions involved in it. A Hausdorff space X is said to be paracompact (resp. subparacompact **[Bu1]**) if every open cover of X has a locally finite open refinement (resp. a $\sigma$-finite closed refinement). A space X is paracompact if and only if it is collectionwise normal and subparacompact (see **[Bu1]** or **[Bu2]**). A topological space X is said to have countable tightness if for every subset $M \subseteq X$ and every $x \in \bar{M}$ there exists a countable subset $C \subseteq M$ such that $x \in \bar{C}$.

**9 Theorem:** *Let us consider the following properties of $f: X \longrightarrow E$:*

*(A) $f$ is of the first Baire class.*

*(A') $f$ is the uniform limit of a sequence in $B_1(X,E)$.*

*(B) $f$ is of the first Borel class.*

*(B') $f$ is $\sigma$-discrete of the first Borel class.*

*(C) $f$ has the point of continuity property.*

*(D') $f$ is $\sigma$-fragmented by closed sets.*

*(E') $f$ is the uniform limit of a sequence of $\tau$-constant functions.*

*Then we have:*

*i) The implications (A') $\Rightarrow$ (B'); (E') $\Rightarrow$ (B') $\Rightarrow$ (D'); (C) $\Rightarrow$ (D') are true for any metric space $E$ and topological space $X$.*

*ii) If $X$ is either a perfect space or a collectionwise normal space (B') $\Leftrightarrow$ (E').*

*iii) If X is a perfectly subparacompact space (B') ⇔ (D') ⇔ (E').*

*iv) If X is hereditarily Baire (C) ⇔ (D'). Moreover, if X has countable tightness (B) ⇒ (C). Hence, if X is an hereditarily Baire metric space then*

$$(B) \Leftrightarrow (B') \Leftrightarrow (C) \Leftrightarrow (D') \Leftrightarrow (E').$$

*v) If X is a perfect space and the metric space E has the extension property for X then (A') ⇔ (B').*

*vi) If X is a collectionwise normal space and the metric space E is arcwise connected (A') ⇔ (B'). Moreover, if E is locally arcwise connected, then (A) ⇔ (B').*

*vii) If X is a perfectly paracompact space and the metric space E is arcwise connected and locally arcwise connected, then*

$$(A) \Leftrightarrow (A') \Leftrightarrow (B') \Leftrightarrow (D') \Leftrightarrow (E')$$

*viii) If X is perfectly subparacompact hereditarily Baire and has countable tightness then then every function of the first Borel class $f:X \longrightarrow E$ is $\sigma$-discrete so (B) ⇔ (B').*

*ix) If X is a a paracompact Cech-complete space (or with more generality, an analytic space in the sense of Frolik-Holicky* [FH]*) then every function of the first Borel class $f:X \longrightarrow E$ is $\sigma$-discrete so (B) ⇔ (B').*

Now we point out the references where the proofs an be found:

For i) see Prop.1.10 of **[Ves]** (having in mind lemma 1.1 of **[Han4]** and Th.2 on p.386 of **[Ku]**), and Th.3.7 of **[JOPV]**. For ii) see Th.3 of **[Han2]** in the case of a perfect space; the case of a collectionwise normal space is implicit in **[Ves]** and **[Han4]**. The proof of Th.5 in **[JOPV]** give us iii).

The first assertion of iv) is an easy consequence of definitions and the other assertions can be found in **[Han4]**. v) follows from proposition 7 and vi) is implicit in **[Ves]**. vii) and viii) are consequences of above results (recall that a paracompact space is subparacompact and collectionwise normal), and ix) is a generalization of Hansell's theorem; although it is not explicitly stated in the references, it is an easy consequence of the main result in **[FH]**.

**10 Remark:** Let us consider the following properties of $f:X \longrightarrow E$:

(B'') $f \in \mathcal{Z}_\sigma(X,E) \cap \Sigma^*(X,E)$.

(D'') $f$ is $\sigma$-fragmented by zero sets.

(E'') $f$ is the uniform limit of a sequence of $\tau^*$-constant functions.

Note that (B'') ⇒ (B'), and (E'') ⇒ (E'), an if X is collectionwise normal then (B'') ⇔ B'), and (E'') ⇔ (E').

An adaptation of the proofs of some of the assertions of theorem 9 allows us to obtain that for any topological space X the following holds:

$$(A') \Rightarrow (B''); \qquad (E'') \Leftrightarrow (B'') \Rightarrow (D'');$$

If the metric space E is arcwise connected (A') ⇔ (B''). Moreover, if E is locally arcwise connected, then (A) ⇔ (A') ⇔ (B'').

Note that if X is a collectionwise normal space then $Z_\sigma(X,E) \cap \Sigma^*(X,E) = F_\sigma(X,E) \cap \Sigma(X,E)$, so in this case every $\sigma$-discrete function of the first Borel class is $\sigma$-fragmented by zero sets.

**11 Remark:** Note that for a separable space E we have (B) ⇔ (B'). In this case we can consider the following properties of f

(D) For each $\varepsilon > 0$ there is a countable closed covering $\{X_n : n \in \mathbb{N}\}$ of X such that $\rho$-diam$(f(X_n)) \leq \varepsilon$, for each $n \in \mathbb{N}$.

(E) f is the uniform limit of a sequence of $\sigma$-constant functions.

From the classical results that can be found in **[Ku]** and some of the previous results collected in this paper we have:

(E) ⇒ (B) ⇔ (D);

If X is either a perfect space or a normal space (B) ⇔ (E);

If X is hereditarily Baire (B) ⇒ (C);

If X is a normal space and the separable metric space E is arcwise connected and locally arcwise connected then (A) ⇔ (B) ⇔ (D) ⇔ (E);

However, an adaptation of the proofs allows us to obtain that for any topological space X and separable metric space E the following properties are equivalent:

$(B^*)$ $f \in Z_\sigma(X,E)$.

$(D^*)$ For each $\varepsilon > 0$ there is a countable covering $\{X_n : n \in \mathbb{N}\}$ of X, formed by zero sets, such that $\rho$-diam$(f(X_n)) \leq \varepsilon$, for each $n \in \mathbb{N}$.

$(E^*)$ f is the uniform limit of a sequence of $\sigma^*$-constant functions.

Moreover, if the separable metric space E is arcwise connected and locally arcwise connected then (A) ⇔ $(B^*)$, i.e. $B_1(X,E) = Z_\sigma(X,E)$.

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