JOURNAL OF FUNCTIONAL ANALYSIS 117, 243–273 (1993)

σ-Fragmentability of Multivalued Maps and Selection Theorems*

J. E. JAYNE

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, England

AND

J. ORIHUELA, A. J. PALLARÉS, AND G. VERA

Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Espinardo, Murcia, Spain

Communicated by A. Connes

Received April 28, 1991; revised April 14, 1992

We answer the question as to when a weak-star upper semi-continuous map F with arbitrary non-empty values from a metric space T to the dual X^* of an Asplund Banach space X has a selector of the first Baire class to the norm.

INTRODUCTION

A map F from a topological space T to the power set of a topological space E is said to be upper semi-continuous (u.s.c.) if the set $\{t \in T: F(t) \cap H \neq \varnothing\}$ is closed in T, whenever H is a closed subset of E. Jayne and Rogers [25] studied the structure of such maps in considerable depth, when T and E are both metric spaces. A map f from T into E is said to be a selector for F if $f(t) \in F(t)$ for all t in T. Jayne and Rogers proved that if F has non-empty arbitrary values, then F always has a selector f of the second Borel class; further, if $F(t)$ is complete for every t in T, they proved that F has a selector f of the first Borel class. Hansell showed in [18] that the proof in [25] can be modified to yield a selector of the first Borel class when $F(t)$ is only supposed to be non-empty for every t in T. Srivatsa in [38] obtained the same result, independently and at about the same time as Hansell, by a different approach.

* The authors have been partially supported by D.G.C.Y.T. PS88-0083 and by NATO Grant CRG 900231.

0022-1236/93 $5.00

Some of the more interesting u.s.c. set-valued maps are defined on a subset T of a Banach space and take their values in a Banach space X with its weak topology, or in a dual Banach space X^* with its weak-star topology. Jayne and Rogers proved in [26] that, if F is a weak (or weak-star) u.s.c. set-valued map from a metric space T to a Banach space X with the Point of Continuity Property (or a dual X^* of an Asplund Banach space X) and F takes only non-empty weakly compact (or weak-star compact) values, then F has a selector f of the first Baire class *to the norm*; i.e., f is the norm pointwise limit of a sequence of norm continuous functions from T into X (or X^*). The existence of these selectors has been successfully applied to several questions in the theory of Banach spaces [10, 12, 34], and adds a new facet to the theories of maximal monotone maps, of subdifferential maps, of attainment maps, and of metric projections [26]. Hansell *et al.* proved in [20] that it is also possible to obtain the former result when X is an arbitrary Banach space and T is a complete metric space. Srivatsa later (in 1985, [38]) showed that, if F is a weak u.s.c. set-valued map from a metric space T to a Banach space X, and F takes only non-empty values, then F has a selector f of the first Baire class to norm.

During 1989 the first named author visited Murcia University and gave seminar talks explaining the ideas underlying the former results, together with the unpublished work of Srivatsa. We then began to develop a unified approach to the known selection theorems and to examine the extent to which Srivatsa's result holds for dual Banach spaces with their weak-star topology. Namely, if F is a weak-star u.s.c. set-valued map from a metric space T into the dual X^* of an Asplund Banach space X and $F(t)$ is assumed to be non-empty for every t in T, is it possible to get a selector f for F of the first Baire class to the norm? Our main aim in this paper is to give a rather complete solution to this problem. Note that if X is not an Asplund space, then there exist weak-star u.s.c. maps with weak-star compact convex values without selectors in the first Baire class [28, Theorem 10] (Theorem 26 in this paper gives another proof).

Our main results are:

THEOREM 21. *Let (M, d) be a metric space and X^* a weakly compactly generated dual Banach space. Then every weak-star upper semi-continuous set-valued map $F: M \to 2^{X^*}$ with arbitrary non-empty values, has a selector f of the first Baire class to the norm.*

THEOREM 23. *Let (M, d) be a metric space and X^* the dual space of an Asplund space X such that the unit ball B_{X^*} in X^* is angelic for the weak-star topology (i.e., if y is a point in the weak-star closure of $A \subset B_{X^*}$, then there is sequence in A weak-star convergent to y). Then every weak-star* *upper semi-continuous set valued map $F: M \to 2^{X^*}$ with arbitrary non-empty values has a selector f of the first Baire class to the norm.*

We also observe the following counterexample: Let ω_1 be the first uncountable ordinal, and $[0, \omega_1]$ the ordinal interval with the order topology. $X = \mathscr{C}[0, \omega_1]$ is an Asplund space, and $F: X \to 2^{X^*}$ defined by

$$F(f) = \{\delta_\alpha : \alpha \in [0, \omega_1), \ \|f\| = |f(\alpha)|\}$$

is a weak-star u.s.c. map without a selector in the first Baire class to the norm (Example 28 below). This example was included in a previous paper [33] and can be considered as a particular case of the next theorem.

THEOREM 31. *Let X be an Asplund space. Then the following are equivalent:*

(i) *X has the property C of Corson;*

(ii) *if M is a metric space and F is a weak-star upper semi-continuous multivalued map from M into X^* for which $F(m)$ is a non-empty convex weak-star countably compact subset of X^* for each $m \in M$, then F has a selector of the first Baire class to the norm.*

Note that if X is a representable Banach space, then property C of Corson in X is equivalent to the weak-star angelicity of the unit dual ball B_{X^*} [15].

In order to organize some common ideas underlying the previous work on selections, we study, in the first part of this paper, which kind of multivalued maps F, from a metric space into a topological space (E, τ) with a lower semi-continuous metric ρ defined on E, can be uniformly approximated by functions in the first Baire class to the metric ρ. A version for set-valued maps of the notion of σ-fragmentability considered by Jayne *et al.* [22, 23, 24] plays a central role here. Using this notion, and basing our treatment in part on Srivatsa's proof of this theorem for weak-u.s.c. maps, we give selection theorems (Theorems 12 and 13) that include the known cases (Theorems 16 and 19), and, introducing new ideas, we obtain some new applications (Theorems 21 and 23) for weak-star u.s.c. maps. Finally, using the Choquet boundary set studied in [21], we are able to prove Theorem 23 stated above. This boundary set may be empty, but in the proof of Theorem 23 we can construct a selector in such a way that the points where the boundary set is empty can be fixed in advance. In order to obtain a selector with the suitable properties, another application of our selection results is required.

Some Definitions and Notation

Throughout the paper we denote by T a Hausdorff topological space.

We start by recalling some definitions and facts that have been used in previous selection work [20, 25, 28]. The following definition of a discretely σ-decomposable family of sets has been successfully used in this context.

> *An indexed family $\{H_i : i \in I\}$ of subsets of T is said to be discretely σ-decomposable* (d.σ.d) *if for each $i \in I$ we have $H_i = \bigcup \{H_{i,n} : n \in \mathbb{N}\}$, where the family $\{H_{i,n} : i \in I\}$ is discrete for every positive integer n (note that I will usually be an uncountable index set).*

If in the above definition we assume that the sets H_i are $\mathscr{F}_\sigma$-sets (countable unions of closed sets), then all the sets $H_{i,n}$ can be assumed to be closed sets. Indeed, given a sequence of closed sets $\{F_{i,m} : m \in \mathbb{N}\}$ such that $H_i = \bigcup \{F_{i,m} : m \in \mathbb{N}\}$, then

$$H_i = \bigcup \{\overline{H_{i,n}} \cap F_{i,m} : n, m \in \mathbb{N}\}$$

and the family $\{\overline{H_{i,n}} \cap F_{i,m} : i \in I\}$ is discrete for each pair of positive integers n, m.

Definition. We shall say that $\{H_i : i \in I\}$ is a good partition of T if it is a d.σ.d family of $\mathscr{F}_\sigma$-subsets of T such that $H_i \cap H_j = \varnothing$ if $i \neq j$ and $T = \bigcup \{H_i : i \in I\}$.

Remark 1. If $\{H_i : i \in I\}$ is a good partition of T, and for each $i \in I$ $\{M_\alpha : \alpha \in A_i\}$ is a good partition relative to the subspace H_i, where the index sets A_i are assumed to be pairwise disjoint, and $A = \bigcup \{A_i : i \in I\}$, then it is easy to check that $\{M_\alpha : \alpha \in A\}$ is a good partition of T (see [20, Lemma 5]).

If $\{H_i : i \in I\}$ and $\{M_j : j \in J\}$ are two good partitions of T, then the family $\{H_i \cap M_j : (i, j) \in I \times J\}$ is also a good partition of T. Just observe that $\{H_i \cap M_j : j \in J\}$ is a good partition of each set H_i.

Let $\{T_n : n \in \mathbb{N}\}$ be a countable cover of T by closed sets and suppose that each open subset of T is an $\mathscr{F}_\sigma$ set. If $H_1 = T_1$ and $H_n = T_n \setminus \bigcup \{T_m : 1 \leqslant m < n\}$ for $n > 1$, then $\{H_n : n \in \mathbb{N}\}$ is a good partition of T. Note that every countable family of sets is σ-discrete, so it is d.σ.d.

> *We shall say that a topological space T is perfectly paracompact if T is a paracompact space such that each open subset is an $\mathscr{F}_\sigma$-set.*

The following result is one of the tools for the proof of selection theorems and has been used in [20, Lemma 4].

PROPOSITION 2. *Let T be a perfectly paracompact space and $\{G_\gamma : \gamma < \Gamma\}$ a transfinite sequence of open sets covering T. If $M_\gamma = G_\gamma \setminus \bigcup \{G_\xi : \xi < \gamma\}$, then $\{M_\gamma : \gamma < \Gamma\}$ is a good partition of T.*

Throughout, (E, ρ) denotes a metric space with the distance ρ. In particular, E can be a normed space and ρ the metric associated to the norm.

Very often, we are concerned with maps $f: T \to E$ that are constant on the sets of a good partition; i.e., there exists a good partition $\{H_i : i \in I\}$ of T such that every restriction $f|_{H_i}$ is constant. We call them *piecewise constant* functions (note that piecewise is used in a somewhat non-standard way).

We recall that a function f from the topological space T into the metric space E is said to be in the *first Baire class* if there exists a sequence of continuous functions $f_n : T \to E$ such that for each point $t \in T$ the sequence $f_n(t)$ converges to $f(t)$. We denote by $B_1(T, E)$ the space of all functions $f: T \to E$ in the first Baire class.

The following proposition relates the piecewise constant functions to the functions in the first Baire class.

PROPOSITION 3. *Let T be a topological space such that each open set is an $\mathscr{F}_\sigma$-set. If $f: T \to E$ is constant on the sets of a good partition (i.e., f is piecewise constant), then there exists an increasing sequence D_n of closed subsets of T that covers T and such that each restriction $f|_{D_n}$ is continuous.*

Moreover, when E is a convex subset of a Banach space and T is a perfectly paracompact space, then f is in the first Baire class, and there exists a sequence of continuous functions $f_n : T \to E$ such that

$$\forall t \in T\ \exists n(t) \in \mathbb{N} \qquad \text{such that} \quad f_n(t) = f(t) \text{ if } n \geq n(t). \tag{1}$$

Proof. Let $\{H_i : i \in I\}$ be a good partition of T such that each restriction $f|_{H_i}$ is constant. Each H_i can be expressed as a countable union $H_i = \bigcup_{n \in \mathbb{N}} H_{i,n}$, where each $\{H_{i,n} : i \in I\}$ is a discrete familie of closed sets. Then $T_n = \bigcup \{H_{i,n} : i \in I\}$ is a sequence of closed sets such that each restriction $f|_{T_n}$ is continuous.

If we consider the sequence of $\mathscr{F}_\sigma$-sets: $C_1 = T_1$, $C_n = T_n \setminus \bigcup \{T_i : 1 \leq i < n\}$ $(n > 1)$, we can obtain increasing sequences of closed sets $\{C_{n,m} : m \in \mathbb{N}\}$ such that $C_n = \bigcup \{C_{n,m} : m \in \mathbb{N}\}$. Thus $D_m = C_{1,m} \cup C_{2,m} \cup \cdots \cup C_{m,m}$ is an increasing sequence of closed sets such that $T = \bigcup \{D_m : m \in \mathbb{N}\}$ and all restrictions $f|_{D_m}$ are continuous.

When E is a convex subset of a Banach space and T is a paracompact space, then E has the extension property with respect to T (i.e., every continuous function $f: C \to E$ defined on a closed subset C of T has a continuous extension to T) [1]. We can consider $f_m: T \to E$ as a continuous extension of $f|_{D_m}$ and so (1) holds. ∎

σ-Fragmented Maps

Lebesgue [30] proved that a real function $f: I \to \mathbb{R}$, defined on an interval I of the real line, is in the first Baire class if, and only if, for each $\varepsilon > 0$, I can be expressed as the union of a sequence (I_n) of closed subsets such that $\operatorname{diam}(f(I_n)) < \varepsilon$.

Throughout, we use the following notion that is based on the condition used by Lebesgue and on the notion of a σ-fragmented space given in [22].

Given a class $\mathscr{H}$ of subsets of T, we say that a function $f: T \to E$ is *σ-fragmented by sets of $\mathscr{H}$*, if for each $\varepsilon > 0$ there is a sequence T_n in $\mathscr{H}$ such that $T = \bigcup \{T_n : n \in \mathbb{N}\}$ and each T_n has the property

> (P_ε): For each non-empty subset C of T_n there exists an open subset V of T such that $V \cap C \neq \varnothing$ and $\rho\text{-diam}(f(V \cap C)) < \varepsilon$, where "$\rho$-diam" denotes the diameter of the set for the distance ρ.

For a multivalued map $F: T \to 2^E$, i.e., a map with values in the power set $2^E = \{A : A \subset E\}$, we extend the above definition, replacing (P_ε) by

> (P_ε^*): For each non-empty subset C of T_n there exists an open subset V of T and a subset D of E with $\rho\text{-diam}(D) < \varepsilon$ such that $V \cap C \neq \varnothing$ and $F(t) \cap D \neq \varnothing$ for each $t \in V \cap C$.

If $\mathscr{H}$ is the family of all subsets (resp., closed subsets) of T, we shall say that the function f, or the multivalued map F, is *σ-fragmented* (resp., *σ-fragmented by closed sets*).

Note that if a multivalued map F from T into E has a selector f, i.e., a function $f: T \to E$ such that $f(t) \in F(t)$ for every $t \in T$, which is σ-fragmented by closed sets, then F is also σ-fragmented by closed sets. A particular case is when the selector is in the first Baire class, as the following proposition shows.

Proposition 4. *Let T be a Hausdorff topological space and E a metric space. Every map in $B_1(T, E)$ is σ-fragmented by closed sets. If E is a* *separable metric space, then for every* $\varepsilon > 0$ *there exists a sequence of closed sets* I_n *such that* $E = \bigcup_n I_n$ *and* $\rho\text{-diam}\, f(I_n) < \varepsilon$.

Proof. We assume that f is the pointwise limit of a sequence of continuous functions $f_n: T \to E$. Given $\varepsilon > 0$, the sequence of closed sets

$$T_n = \bigcap_{j \geq n} \left\{ t \in T : \rho(f_n(t), f_j(t)) \leq \frac{\varepsilon}{3} \right\}$$

covers T. Since $\rho(f_n(t), f(t)) \leq \varepsilon/3$ for every $t \in T_n$, and f_n is continuous, each closed set T_n has property (P_ε).

If E is separable, then each one of the above sets T_n can be expressed as a countable union of closed sets $H_{n,k}$ such that $\rho\text{-diam}\, f(H_{n,k}) < \varepsilon$. Just take $H_{n,k} = T_n \cap f_n^{-1}(B_k)$, where B_k is a countable cover of E by closed balls with $\rho\text{-diam} < \varepsilon/3$. ∎

The following theorem gives a characterization of σ-fragmentability in terms of uniform approximations by piecewise constant functions.

THEOREM 5. *Let F be a multivalued map from the perfectly paracompact space T into the subsets of a metric space (E, ρ). The following are equivalent*:

(a) *F is σ-fragmented by closed sets*;

(b) *for each* $\varepsilon > 0$ *there exists a function constant on the sets of a good partition* $f_\varepsilon: T \to E$ *such that, for every* $t \in T$, $\rho\text{-dist}(f_\varepsilon(t), F(t)) < \varepsilon$.

Proof. (a) ⇒ (b). Given $\varepsilon > 0$, let $\{T_n : n \in \mathbb{N}\}$ be a countable cover of T such that each T_n verifies (P_ε^*), and let $\{Y_n : n \in \mathbb{N}\}$ be the good partition of T defined by $Y_1 = T_1$, and for $n > 1$, $Y_n = T_n \backslash \bigcup \{T_i : 1 \leq i < n\}$ (Remark 1). We prove that, for each $n \in \mathbb{N}$, there exists a piecewise constant map $f_n: Y_n \to E$ such that $\rho\text{-dist}(f_n(t), F(t)) < \varepsilon$ for every $t \in Y_n$. By Remark 1 the map $f_\varepsilon: T \to E$ defined by $f_\varepsilon|_{Y_n} = f_n$, $n \in \mathbb{N}$, is a piecewise constant function satisfying (b).

Each Y_n with the induced topology is paracompact, because it is an $\mathscr{F}_\sigma$-subset of T [9, p. 383].

By property (P_ε^*) applied to $C = Y$ $(\subset T_n)$ there exists a non-empty (relatively) open subset G_0 of Y and a set $D_0 \subset E$, with $\rho\text{-diam}(D_0) < \varepsilon$, such that $F(t) \cap D_0 \neq \varnothing$ for every $t \in G_0$. Now we proceed by transfinite induction. If the closed set $Y \backslash G_0$ is non-empty, we apply (P_ε^*) another time and obtain a (relatively) open subset G_1 of Y and a set $D_1 \subset E$ such that $M_1 = G_1 \cap (Y \backslash G_0) \neq \varnothing$, $\rho\text{-diam}(D_1) < \varepsilon$, and $F(t) \cap D_1 \neq \varnothing$ for each $t \in M_1$.

Let γ be an ordinal number such that for every ordinal $\mu < \gamma$ a relatively open subset G_μ of Y and a subset D_μ of E have been defined so that $\rho\text{-diam}(D_\mu) < \varepsilon$ and $F(t) \cap D_\mu \neq \emptyset$ for each $t \in M_\mu$, where $M_\mu = G_\mu \cap (Y \setminus \bigcup \{G_\xi : \xi < \mu\})$.

The construction continues by applying (P_ε^*) to $Y \setminus \bigcup \{G_\xi : \xi < \gamma\}$ until we have some ordinal Γ such that $Y = \bigcup \{G_\gamma : \gamma < \Gamma\}$.

By Proposition 2, the family $\{M_\gamma : \gamma \in \Gamma\}$ is a good partition of Y. Choosing a fixed point y_γ in each D_γ we can define a piecewise constant function f_n on Y $(= Y_n)$ by $f_n(t) = y_\gamma$ if $t \in M_\gamma$. It is clear that $\rho\text{-dist}(f_n(t), F(t)) < \varepsilon$ for each $t \in Y_n$ as required.

(b) $\Rightarrow$ (a). Given $\varepsilon > 0$, let f_ε be a piecewise constant function such that

$$\rho\text{-dist}(f_\varepsilon(t), F(t)) < \frac{\varepsilon}{3} \qquad \text{for each} \quad t \in T.$$

By Proposition 3, there is a countable cover $\{T_n : n \in \mathbb{N}\}$ of T such that for each $n \in \mathbb{N}$ the set T_n is closed and $f_\varepsilon|_{T_n}$ is continuous. If $\emptyset \neq C \subset T_n$, then there is an open subset V of T such that $V \cap C \neq \emptyset$ and $\rho\text{-diam}(f_\varepsilon(V \cap C)) < \varepsilon/3$.

Taking $D = \{y \in E : \rho\text{-dist}(y, f_\varepsilon(V \cap C)) < \varepsilon/3\}$, we have $\rho\text{-diam}(D) < \varepsilon$ and $F(t) \cap D \neq \emptyset$ if $t \in V \cap C$. ▮

Remark 6. Theorem 5 and Remark 1 show that a multivalued map $F : T \to 2^E$, defined on a perfectly paracompact space T, is σ-fragmented by closed sets if, and only if, the restrictions $F|_{H_i}$ to the pieces of a good partition $\{H_i : i \in I\}$ of T are σ-fragmented by closed sets.

In order to relate the notion of a function σ-fragmented by closed sets with those that have been used in previous papers [16–18, 20] we recall the notion of a σ-discrete function in the first Borel class.

A function f from the topological space T into the metric space E is said to be in the *first Borel class* if $f^{-1}(V)$ is an $\mathscr{F}_\sigma$ set for each open subset $V \subset E$.

A collection $\mathscr{B}$ of subsets of T is said to be a base for $f : T \to E$ if for each open set $G \subset E$ the set $f^{-1}(G)$ is the union of sets in $\mathscr{B}$. The function f is called σ-discrete if there exists a sequence of discrete families of sets $(\mathscr{B}_n)$ in T such that $\mathscr{B} = \bigcup \{\mathscr{B}_n : n \in \mathbb{N}\}$ is a base for f. The class of σ-discrete functions contains the continuous functions, all the functions with a separable range, and each Borel measurable function whose domain is a Borel subset of a complete metric space (see [16]).

COROLLARY 7. *Let T be a perfectly paracompact space, E a metric space, and $f : T \to E$. Then the following are equivalent*:

(i) *f is σ-fragmented by closed sets;*

(ii) *f is a uniform limit of a sequence of functions which are constant on the sets of good partitions (i.e., of piecewise constant functions);*

(iii) *f is σ-discrete and of the first Borel class.*

Moreover, if E is a convex subset of a Banach space, they are equivalent to:

(iv) *f is in the first Baire class $B_1(T, E)$.*

Proof. (i) $\Leftrightarrow$ (ii) is Theorem 5.

(ii) $\Rightarrow$ (iii). It follows from Proposition 3 that every piecewise constant function is in the first Borel class. It is clear that each piecewise constant map is σ-discrete. Since the uniform limit of a sequence of σ-discrete functions is σ-discrete [19, Theorem 3.3], and the uniform limit of a sequence of functions in the first Borel class is also in the first Borel class [29, p. 386], we have (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). Given $\varepsilon > 0$ we consider the open cover of E consisting of all $(\varepsilon/2)$-balls. By reason of Stone's Theorem [9, p. 349] we can obtain a σ-discrete open cover $\{B_\alpha : \alpha \in A\}$ of E where ρ-diam $B_\alpha \leqslant \varepsilon$ for every $\alpha \in A$. Let $\{A_n : n \in \mathbb{N}\}$ be a countable partition of A such that every family $\{B_\alpha : \alpha \in A_n\}$ is discrete, and let $\mathscr{B} = \bigcup \{\mathscr{B}_n : n \in \mathbb{N}\}$ be a base of f where each $\mathscr{B}_n$ is a discrete family. For a fixed natural number k we denote by $\mathscr{B}_n^k$ the subfamily of $\mathscr{B}_n$ composed of sets $U \in \mathscr{B}_n$ such that there exists an element $\alpha \in A_k$ with $f(U) \subset B_\alpha$. We observe that given $U \in \mathscr{B}_n^k$, the element $\alpha \in A_k$ with $f(U) \subset B_\alpha$ is unique; we denote it by $\alpha(k, U)$. Since f is in the first Borel class, all the sets in the discrete family $\mathscr{D}_n^k = \{\bar{U} \cap f^{-1}(B_{\alpha(k, U)}) : U \in \mathscr{B}_n^k\}$ are $\mathscr{F}_\sigma$-set, so the union $D_n^k = \bigcup \{C : C \in \mathscr{D}_n^k\}$ is a $\mathscr{F}_\sigma$-set in T. One readily sees that $T = \bigcup \{D_n^k : n, k \in \mathbb{N}\}$. Given a non-empty subset $C \subset D_n^k$ there exists $U \in \mathscr{B}_n^k$ such that $C \cap \bar{U} \cap f^{-1}(B_{\alpha(k, U)}) \neq \varnothing$. If we fix a point a in this non-empty set, we can obtain an open neighbourhood G of a such that U is the only member of $\mathscr{B}_n^k$ which intersects G. Now we have that $C \cap G \subset C \cap \bar{U} \cap f^{-1}(B_{\alpha(k, U)})$ and so ρ-diam$(f(C \cap G)) < \varepsilon$. We have shown that f is σ-fragmentable by the countable family of $\mathscr{F}_\sigma$-sets $\{D_n^k : n, k \in \mathbb{N}\}$, and it follows that f is σ-fragmented by closed sets.

If E is a Banach space, the uniform limit of a sequence in $B_1(T, E)$ belongs to $B_1(T, E)$ (see [18, p. 391]). By Proposition 3, $B_1(T, E)$ contains the piecewise constant functions, and so (ii) $\Rightarrow$ (iv). And Proposition 4 gives the implication (vi) $\Rightarrow$ (i). ∎

In the case of a metric space T, Hansell [18, Lemma 7] has shown that (iv) and (iii) are equivalent above.

The equivalence between (iii) and (iv) is not true if E is allowed to be an arbitrary metric space. Indeed, recently Fosgerau [11] has shown that if T is a metric space and E is an arcwise connected and locally connected metric space, then (iii) and (iv) are equivalent; and, if T contains a copy of the interval $[0, 1]$, E is a complete metric space, and (iii)$\Leftrightarrow$(iv), then E must be arcwise connected and locally connected.

Now we are going to give sufficient conditions for the σ-fragmentability by closed sets of multivalued maps.

We shall denote by τ a Hausdorff topology on a metric space (E, ρ) such that ρ is τ-lower semi-continuous. A standard example is obtained when $E=X$ is a Banach space (resp., $E=X^*$ is a dual Banach space) with the metric associated to the norm of E and τ is the weak (resp., weak-star) topology on E. Another example is obtained when $\tau=\sigma(X, \text{ext}(B_{X^*}))$ is the extremal topology on the Banach space X (the coarsest topology on X such that each extreme point of the unit ball of the dual space X^* is continuous). It is well known that if $X=C(K)$ is the space of the continuous function defined on the compact space K, then the extremal topology coincides with the topology of the pointwise convergence.

A multivalued map $F: T \to 2^E$ is said to be *τ-upper semi-continuous* (τ-u.s.c.) if $\{t \in T: F(t) \cap C \neq \varnothing\}$ is a closed subset of T for each τ-closed subset C of E. If $T=(M, d)$ is a metric space, then the τ-upper semi-continuity of F has the following characterization: if t_n is a sequence in T convergent to a point $t \in T$ and $x_n \in F(t_n)$, then the τ-closure of $\{x_n\}$ intersects $F(t)$. In particular, if $x_n \notin F(t)$ for every $n \in \mathbb{N}$, then the sequence x_n has a τ-cluster point $x \in F(t)$.

We call a multivalued map *piecewise upper semi-continuous* (piecewise u.s.c.) if its restrictions to the pieces of a good partition are upper semi-continuous.

The following two results (Lemma 8 and Proposition 9) are based upon Srivatsa's proof of his selection theorem [38].

LEMMA 8. *Let τ be a Hausdorff topology on the metric space (E, ρ) such that ρ is τ-lower semi-continuous, T is a perfectly paracompact space, and $F: T \to 2^E$ is a piecewise τ-upper semi-continuous map. Assume that for every $\varepsilon > 0$ there exists a sequence of piecewise constant functions $g_n: T \to E$ such that the sequence of subsets $B_n = \{t \in T: \rho\text{-dist}(g_n(t), F(t)) \leqslant \varepsilon\}$ covers T. Then F is σ-fragmented by closed sets.*

Proof. Because of Remark 6 we can suppose that F is u.s.c. Let $\varepsilon > 0$. By Proposition 3, for each $n \in \mathbb{N}$ there exists a sequence of closed subsets $\{D_{n,m}: m \in \mathbb{N}\}$ such that $T = \bigcup \{D_{n,m}: m \in \mathbb{N}\}$ and every restriction $g_n|_{D_{n,m}}$ is continuous. We check that the subsets

$$B_{n,m} = \{t \in D_{n,m}: \rho\text{-dist}(g_n(t), F(t)) \leqslant \varepsilon\}$$

are closed. If $s \in D_{n,m} \backslash B_{n,m}$, then there exists $\alpha > \varepsilon$ such that the closed ball $B(g_n(s), \alpha)$ does not intersect $F(s)$. By the continuity of g_n there exists a neighbourhood U of s such that $B(g_n(t), \varepsilon) \subset B(g_n(s), \alpha)$ for every $t \in U \cap D_{n,m}$. By the τ-upper semi-continuity of F there exists a neighbourhood $V \subset U$ of s such that

$$F(t) \cap B(g_n(t), \varepsilon) \subset F(t) \cap B(g_n(s), \alpha) = \varnothing$$

for every $t \in V \cap D_{n,m}$. This proves that $B_{n,m}$ is a closed subset of $D_{n,m}$, and so a closed subset of T. By hypothesis $T = \bigcup \{B_{n,m} : n, m \in \mathbb{N}\}$.

Let C be a non-empty subset of some $B_{n,m}$. If we fix a point $a \in C$, there is an open neighbourhood V of a such that ρ-dist$(g_n(t), g_n(a)) < \varepsilon$ for each $t \in V \cap C$. If $D = B(g_n(a), 2\varepsilon)$, then ρ-diam$(D) \leqslant 4\varepsilon$ and $F(t) \cap D \neq \varnothing$ for every $t \in V \cap C$. This completes the proof. ∎

The lemma above is useful in checking that a given multivalued map is σ-fragmented by closed sets.

PROPOSITION 9. *Let (M, d) and (E, ρ) be metric spaces and $F: M \to 2^E$ a multivalued map with arbitrary non-empty values which is assumed to be piecewise upper semi-continuous. Then F is σ-fragmented by closed sets.*

If E is a Banach space and F is assumed to be piecewise upper semi-continuous for the weak topology, then F is σ-fragmented by closed sets.

Proof. We can suppose that F is u.s.c. (Remark 6). Let $\{G_\gamma : \gamma < \Gamma(n)\}$ be an open cover of M such that d-diam$(G_\gamma) < 1/n$ for each $\gamma < \Gamma(n)$. By Proposition 2 the family $\{M_\gamma : \gamma < \Gamma\}$ defined by

$$M_\gamma = G_\gamma \Big\backslash \bigcup \{G_\xi : \xi < \gamma\}$$

is a good partition of M. If $M_\gamma \neq \varnothing$, we fix a point $t_\gamma \in M_\gamma$ and a point $y_\gamma \in F(t_\gamma)$. Then we define a piecewise constant function $f_n: M \to X$ by $f_n(t) = y_\gamma$ if $t \in M_\gamma$.

Let $\varepsilon > 0$ and assume that $t \in M$. By the construction of f_n, there exists a point $t_n \in M$ such that $d(t_n, t) < 1/n$ and $f_n(t) \in F(t_n)$. The upper semi-continuity of F gives the existence of a point $a \in F(t)$ in the ρ-closure of $\{f_n(t): n \in \mathbb{N}\}$. The subsets

$$B_n = \{t \in M : \rho\text{-dist}(g_n(t), F(t)) < \varepsilon\}$$

cover M. And Lemma 8 shows the σ-fragmentability of F.

With the above notation, if $E = X$ is a Banach space and F is upper semi-continuous for the weak topology, then for each $t \in M$ there exists a point $a \in F(t)$ in the weak closure of $\{f_n(t): n \in \mathbb{N}\}$, and thus there exists a sequence of rational convex combinations of $\{f_n(t)\}$ that converges to a in the norm of X.

If we denote by $\{g_n\}$ the sequence of all the rational convex combinations of functions from the sequence $\{f_n\}$, then the above argument shows that the subsets

$$B_n = \{t \in M : \| \,\|\text{-dist}(g_n(t), F(t)) < \varepsilon\}$$

cover M. And Lemma 8 shows again the σ-fragmentability of F. ∎

PROPOSITION 10. *Let M be a metric space, X a convex subset of a Banach space and $F: M \to 2^X$ a multivalued map with non-empty values which is assumed to be piecewise u.s.c. for the topology $\sigma(X, \text{ext}(B_{X^*}))$ of pointwise convergence on the extreme points of the dual unit ball B_{X^*}. Then F is σ-fragmented by closed sets.*

Proof. As in the above cases, we can assume that F is u.s.c. for the extremal topology.

We show that if F takes its values in a bounded subset $B \subset X$, then F is u.s.c. for the weak topology. If not, we could fine a sequence $t_n \in M$ convergent to a point t and a sequence $y_n \in F(t_n) \setminus F(t)$ such that $\{y_n\}$ does not have weak cluster points in $F(t)$. But by the $\sigma(X, \text{ext}(B_{X^*}))$-u.s.c. of F we have that each subsequence of $\{y_n\}$ has $\sigma(X, \text{ext}(B_{X^*}))$ cluster points in $F(t)$. This implies that $\{y_n : n \in \mathbb{N}\}$ is a relatively $\sigma(X, \text{ext}(B_{X^*}))$ countably compact set and so is relatively weakly compact [3]. Then (y_n) has a weak convergent subsequence whose limit must be in $F(t)$, and that contradicts the choice of (y_n).

If F does not take its values in a bounded set, we take the sequence of closed set $T_n = \{t \in M : F(t) \cap B(0, n) \neq \varnothing\}$, which covers M. The sequence $H_1 = T_1$, $H_n = T_n \setminus T_{n-1}$ (if $n \geqslant 2$) is a good partition of M, and $F'(t) = F(t) \cap B(0, n)$ if $t \in H_n$ defines a piecewise $\sigma(X, \text{ext}(B_{X^*}))$-u.s.c. map. This map is piecewise weak u.s.c. by the above paragraph. The last proposition and Remark 6 show that F' is σ-fragmented by closed sets, and consequently so is F. ∎

A τ-upper semi-continuous map $F: T \to 2^E$ with τ-compact values (denoted τ-usco) is said to be minimal, if given another τ-usco map

$$F_0: T \to 2^E \qquad \text{such that} \quad F_0(t) \subset F(t),$$

for every $t \in T$, and $F_0(t) \neq \varnothing$ if $F(t) \neq \varnothing$, then $F = F_0$. If $F: T \to 2^E$ is τ-usco with non-empty values, the axiom of choice implies that there exists a minimal τ-usco $F_0: T \to 2^E$ with non-empty values such that $F_0(t) \subset F(t)$ for every $t \in T$.

Let E be a weak-star compact subset of a dual Banach space X^*. E is said to be *fragmented by the norm of* X^* if for every $\varepsilon > 0$ and every non-empty subset $D \subset E$ there exists a weak-star open set W such that $W \cap D \neq \emptyset$ and $\| \|$-diam$(W \cap D) < \varepsilon$. In [32] Namioka and Phelps proved that X is an Asplund space if, and only if, the unit ball of X^* is fragmented by the norm; they also proved that X^* has the Radon–Nikodým property when X is an Asplund space. In [31] it is proved that the weak-star compact subsets of dual Banach spaces fragmented by the norm can be embedded into dual spaces of Asplund spaces.

We say that a multivalued map F from T into E is *fragmented* if for each $\varepsilon > 0$ the property (P_ε^*) holds with $T_n = T$. This notion is different from the one used in [20, p. 219] for multivalued usco maps, where such a map was called "fragmented" by the metric ρ of E if the following property holds:

(FP) For each $\varepsilon > 0$ and each open subset U of T with $F(U) \neq \emptyset$ there is an open subset V of U such that $F(V) \neq \emptyset$ and ρ-diam$(F(V)) < \varepsilon$.

We remark that if F is a fragmented usco minimal map, then F has the property (FP) (just apply (P_ε^*) to the set $C = U$ and note that every fragmented map has non-empty values). The same conclusion can be obtained if F is σ-fragmented by closed sets and T is a Baire space. Working as in [20, Lemma 6], we get the following proposition.

PROPOSITION 11. *Let E be a weak-star compact subset of a dual Banach space X^* such that E is fragmented by the norm (e.g., E is a subset of the dual X^* of an Asplund Banach space X) and T is a Hausdorff topological space. If $F: T \to 2^E$ is a weak-star (piecewise) upper semi-continuous map with non-empty weak-star compact values, then F is fragmented (σ-fragmented by closed sets).*

Proof. We assume that F is weak-star usco. We begin with E weak-star compact and fragmented by the norm. Let $\varepsilon > 0$, and C be a non-empty subset of T. We consider a minimal map $F_0: C \to 2^E$ with $F_0(t) \subset F(t)$ for all $t \in T$.

Since $F_0(C) \subset E$ and E is fragmented by the norm, there exists a weak-star open set $W \subset X^*$ such that $D = W \cap F_0(C)$ is non-empty and

$$\| \|\text{-diam}(D) < \varepsilon.$$

By the minimality of F_0 there is an open subset V of T such that $C \cap V \neq \emptyset$ and $F_0(C \cap V) \subset W \cap F_0(C) = D$. This implies that F is fragmented.

In the case $E = X^*$, the dual of an Asplund space, the closed balls $B_n = B(0, n) = \{x \in X^*: \|x^*\| \leqslant n\}$ are weak-star compact sets fragmented by the norm.

Let $T_n = \{t \in T: F(t) \cap B_n \neq \varnothing\}$. The sets T_n are closed subsets of T by the upper semi-continuity of F, and they cover T.

We consider the weak-star usco maps $F_n: T_n \to 2^{B_n}$ defined by $F_n(t) = F(t) \cap B_n$. The first part of the proof applied to these maps shows that F verifies the property (P_ε^*) with these T_n, and thus F is σ-fragmented. ∎

Selection Theorems for Multivalued Maps

A selector f of a multivalued map $F: T \to 2^E$ is a single valued map $f: T \to E$ such that $f(t) \in F(t)$ for each $t \in T$. If F has a σ-discrete first Borel class selector f, then f is σ-fragmented by closed sets, and it follows from the definition that F is σ-fragmented by closed sets.

On the other hand, looking at the proofs of the known selection theorems, and in particular, at that of Srivatsa's theorem, we see that a kind of hereditary σ-fragmentability of F (by closed sets) is enough to obtain this kind of selector. More precisely, the condition is the following: For every closed ball $B \subset E$ and every $M \subset T$ which is an $\mathscr{F}_\sigma$ and $\mathscr{G}_\delta$-subset, such that $F(t) \cap B \neq \varnothing$ for all $t \in M$, the multivalued map $F': M \to 2^E$ defined by $F'(t) = F(t) \cap B$ is σ-fragmented by closed sets.

In order to simplify our exposition, we introduce the following concept.

> Let $F: T \to 2^E$ be a multivalued map with non-empty values. We say that $F': T \to 2^E$ is a *reduction* of F if there exists a good partition $\{H_i: i \in I\}$ of T, and indexed families $\{x_i: i \in I\} \subset E$, $\{r_i \geqslant 0: i \in I\} \subset \mathbb{R}$ such that
>
> $$F'(t) = F(t) \cap B_i \neq \varnothing, \qquad \text{for each} \quad t \in H_i,$$
>
> where $B_i = \{x \in E: \rho(x, x_i) \leqslant r_i\}$. We say that F_n is an *nth order reduction* of F if, writing $F = F_0$, there exist multivalued maps $F_1, F_2, ..., F_{n-1}$ such that F_{i+1} is a reduction of F_i for $i = 0, 1, 2, ..., n-1$.

We consider multivalued maps whose reductions are σ-fragmented by closed sets, and show that they are the maps with the above hereditary σ-fragmentability property.

This condition holds for u.s.c. maps in the context of Propositions 9–11, because the reduction of u.s.c. maps are piecewise u.s.c.

The next Theorems 12 and 13 closely follow the ideas from Srivatsa's unpublished work. Nevertheless, in order to obtain a unified approach to his result and the Jayne–Rogers Theorem (Theorem 16), we use as a main condition the hereditary σ-fragmentability of the multivalued map, instead of working with additional properties for the spaces T and E. We see in the next section how to apply these theorems to obtain new selection results.

THEOREM 12. *Let T be a perfectly paracompact space, let (E, ρ) be a metric space, and let $F: T \to 2^E$ be a multivalued map with non-empty complete values, such that every finite order reduction of F is σ-fragmented by closed sets. Then F has a σ-discrete first Borel class selector f.*

Proof. We proceed by induction to construct a sequence of piecewise constant functions that uniformly converges to a selector of F.

Let $F_0 = F$. Since F_0 is σ-fragmented by closed sets, given $\varepsilon = 1/2$, Theorem 5 gives us a piecewise constant function $f_1: T \to E$ such that $\rho\text{-dist}(f_1(t), F_0(t)) < 1/2$ for each $t \in T$.

Let $F_1(t) = F_0(t) \cap B(f_1(t), 1/2)$, where $B(f_1(t), 1/2)$ denotes the closed ball in E with centre $f_1(t)$ and radius $\frac{1}{2}$. Since f_1 is piecewise constant, F_1 is a reduction of F_0.

Suppose we have constructed piecewise constant functions

$$f_1, f_2, ..., f_n$$

and multivalued maps $F_1, F_2, ..., F_n$ such that F_k is the reduction of F_{k-1} given by

$$F_k(t) = F_{k-1}(t) \cap B(f_k(t), 1/2^k) \qquad \text{for every} \quad t \in T \text{ and } k = 1, 2, ..., n.$$

Since the intersection of the good partitions associated with $f_1, f_2, ..., f_n$ defines a good partition by Remark 1, F_n is a reduction of F.

By hypothesis F_n is σ-fragmented by closed sets, and using Theorem 5 again, with $\varepsilon = 1/2^{n+1}$, we obtain a piecewise constant function f_{n+1} such that

$$\rho\text{-dist}(f_{n+1}(t), F_n(t)) < 1/2^{n+1}.$$

To continue with the induction we define $F_{n+1}(t) = F_n(t) \cap B(f_{n+1}(t), 1/2^{n+1})$ for every $t \in T$.

By the construction we have

1. $\rho\text{-dist}(f_n(t), F(t)) < 1/2^n$, and
2. $\rho(f_n(t), f_{n+1}(t)) < 1/2^{n-1}$ for every $t \in T$ and $n \in \mathbb{N}$.

By Property 1, for every $t \in T$, we can choose a sequence $x_n \in F(t)$ such that $\rho(f_n(t), x_n) < 1/2^n$. By Property 2 the sequence (x_n) is a Cauchy sequence, and the completeness of $F(t)$ gives us a limit $f(t) \in F(t)$.

Property 2 also gives the uniform convergence of f_n to f, and we have found a selector f of F, which is σ-fragmented by closed sets. ∎

If the values of the map are not known to be complete, there is a similar result when the domain is a metric space.

THEOREM 13. *Let (E, ρ) be a metric space, τ a Hausdorff topology on E such that ρ is τ-lower semi-continuous, and let (M, d) be a metric space. If $F: M \to 2^E$ is a τ-upper semi-continuous multivalued map with non-empty values such that every finite order reduction of F is σ-fragmented by closed sets, then F has a σ-discrete first Borel class selector f.*

Proof. We proceed by induction to construct a sequence of piecewise constant functions $\{\tilde{f}_n\}$ which is uniformly convergent to a function f, and such that for each $t \in M$ we have the two exclusive possibilities:

(a) There exists some $k \in \mathbb{N}$ such that $\tilde{f}_n(t) = \tilde{f}_k(t) \in F(t)$ for all $n \geq k$, and thus $f(t) \in F(t)$, or

(b) $\tilde{f}_n(t) \notin F(t)$ for all $n \in \mathbb{N}$, but there is a sequence $t_n \in M$ converging to t such that $\tilde{f}_n(t) \in F(t_n)$, and by the τ-upper semi-continuity of F the ρ-limit $f(t)$, which is the only τ-cluster point of $\tilde{f}_n(t)$, is in $F(t)$.

The function f is the selector of F for which we are looking.

To construct $\tilde{f}_1$ we start with $\varepsilon = \frac{1}{2}$ and the piecewise constant function $f_1: M \to E$ such that $\rho\text{-dist}(f_1(t), F(t)) < \frac{1}{2}$ for each $t \in M$, which is given by Theorem 5.

Let $\{H_i : i \in I\}$ be the good partition associated to f_1. For each H_i we take a good partition $\{K_j : j \in A_i\}$ associated to a cover of H_i by open sets with diameter less than $\varepsilon = \frac{1}{2}$ (Proposition 2), and we denote by $\{K_j : j \in \bigcup \{A_i : i \in I\}\}$ the good partition of M given by Remark 1. Now, we choose a point $t_j \in K_j$ and a point $y_j \in F(t_j) \cap B(f_1(t_j), \frac{1}{2})$, and we define a new piecewise constant function

$$\tilde{f}_1(t) = y_j \qquad \text{if} \quad t \in K_j.$$

By the upper semi-continuity of F the sets $Y_j = \{t \in K_j : \tilde{f}_1(t) \in F(t)\}$ are closed subsets relative to the K_j's, so K_j has a good partition given by Y_j and $N_j = K_j \setminus Y_j$. Taking these good partitions together we have that the following multivalued map is a reduction of F:

$$F_1(t) = \{\tilde{f}_1(t)\} \qquad \text{if} \quad \tilde{f}_1(t) \in F(t) \quad (\text{i.e., if } t \in Y_j \text{ for some } j),$$

and

$$F_1(t) = F(t) \cap B(\tilde{f}_1(t), 1) \qquad \text{otherwise.}$$

Suppose that we have defined piecewise constant functions $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_n$ and reductions $F_1, F_2, ..., F_n$ of F in such a way that for each $k = 2, 3, ..., n$ we have:

(1) $F_k(t) = \{\tilde{f}_k(t)\}$ if $\tilde{f}_k(t) \in F(t)$, and $F_k(t) = F_{k-1}(t) \cap B(\tilde{f}_k(t), 1/2^{k-1})$ otherwise,

(2) for each $t \in M$ there exists $t_k \in M$ such that $d(t, t_k) < 1/2^k$ and $\tilde{f}_k(t) \in F(t_k)$,

(3) $\rho(\tilde{f}_k(t), \tilde{f}_{k-1}(t)) < 1/2^{k-3}$ for every $t \in M$, and

(4) $\tilde{f}_k(t) = \tilde{f}_j(t)$ if $k > j$ and $\tilde{f}_j(t) \in F(t)$.

Take $\varepsilon = 1/2^{n+1}$. By the σ-fragmentability of F_n we can find a piecewise constant function f_{n+1} such that ρ-dist$(f_{n+1}(t), F_n(t)) < 1/2^{n+1}$ for every $t \in T$. Now, we define a good partition $\{H_i : i \in I_n\}$ of M by taking the pieces of the good partition associated to F_n where $F_n(t) = \{\tilde{f}_n(t)\}$, and intersecting the other pieces with the good partition associated to f_{n+1}. We take a good partition $\{K_j : j \in A_i\}$ of each H_i associated with a cover of open sets with diameter less than $\varepsilon = 1/2^{n+1}$ (Proposition 2), and we take $\{K_j : j \in \bigcup \{A_i : i \in I_n\}\}$ to be the good partition of M given by Remark 1. We fix a point $t_j \in K_j$ and a point $y_j \in F_n(t_j) \cap B(f_{n+1}(t_j), 1/2^{n+1})$ and we define the piecewise constant function $\tilde{f}_{n+1}$ in the following way:

$$\tilde{f}_{n+1}(t) = \tilde{f}_n(t) \qquad \text{if} \quad F_n(t) = \{\tilde{f}_n(t)\},$$

and

$$\tilde{f}_{n+1}(t) = y_j \qquad \text{if} \quad t \in K_j \qquad \text{and} \qquad F_n(t) \neq \{\tilde{f}_n(t)\}.$$

With the same arguments as in the definition of F_1 we have that the multivalued map defined by

$$F_{n+1}(t) = \{\tilde{f}_{n+1}(t)\} \qquad \text{if} \quad \tilde{f}_{n+1}(t) \in F(t),$$

and

$$F_{n+1}(t) = F_n(t) \cap B(\tilde{f}_{n+1}(t), 1/2^n), \qquad \text{otherwise}$$

is a reduction of F. It is clear that $\tilde{f}_{n+1}$ and F_{n+1} verify Conditions 1–4.

The sequence of piecewise constant functions $\{\tilde{f}_n\}$ defined above is uniformly convergent by Condition 3 and, by the τ-upper semi-continuity of F, verifies the Conditions a and b at the beginning, and so we have finished the construction of the selector for F. ∎

Note that if τ is the topology associated to the metric ρ, then Proposition 9 and this last theorem give the following corollary.

COROLLARY 14 (Jayne–Rogers [25], Hansell [18], Srivatsa [38]). *Let* (M, d) *and* (E, ρ) *be metric spaces. If* $F \to 2^E$ *is upper semi-continuous with non-empty values, then* F *has a* σ*-discrete first Borel class selector* f.

Remark 15. Suppose that (E, ρ) and τ are as in Theorem 13 and that they satisfy the following property: Every τ-u.s.c. multivalued map from any domain in arbitrary metric spaces and values in E has a selector, which lies in the ρ-pointwise closure of some sequence of piecewise constant functions (for instance, is in some Baire class). Then each of these multivalued maps has a selector which is σ-fragmented by closed sets (e.g., in the first Baire class to the norm if E is a Banach space and τ is the weak or the weak-star topology). Indeed, the reductions of u.s.c. maps are σ-fragmented by closed sets by Remark 6 and Lemma 8, so the hypothesis of the last theorem holds.

As a consequence of these general selections theorems, we obtain the following results (Theorems 16 and 19).

The first one is a theorem by Jayne and Rogers [28], extended to perfectly paracompact domain space as in [20].

THEOREM 16 (Jayne–Rogers). *Let* T *be a perfectly paracompact space and* E *a convex subset of the dual* X^* *of an Asplund Banach space* X. *If* $F: T \to 2^E$ *is a weak-star upper semi-continuous map with weak-star compact non-empty values, then* F *has a selector* f *in the first Baire class* $B_1(T, E)$.

Proof. It is a consequence of Proposition 11 and Theorem 12. ∎

Remark 17. If in the last theorem we take $E \subset X^*$ to be a weak-star compact subset of a dual Banach space such that E is fragmented by the norm instead of X^* being the dual of an Asplund space, then we can find a selector f of F that is σ-fragmented by closed sets. This function f is in the first Baire class $B_1(T, D)$, where D is the weak-star closed convex hull of E in X^*. This is because the weak-star closed convex hull of E is also fragmented by the norm metric [31, Theorem 2.5].

This last theorem was proved in [20] as a consequence of the main theorem [20, Theorem 1′]. The following lemma, together with Theorem 12, gives a proof of their main result under the additional hypothesis of lower semi-continuity for the metric involved.

LEMMA 18. *Let* T *be a perfectly paracompact space,* (E, ρ) *a metric space, and* τ *a topology on* E *such that* ρ *is* τ*-lower semi-continuous. If all the minimal* τ*-usco maps from* T *into* E *have property* (FP) (*are "fragmented" by the metric* ρ, *in terms of* [20]), *then every reduction of a given* τ*-usco multivalued map with non-empty values* F *from* T *into* E *is* σ*-fragmented by closed sets.*

Proof. Let F' be a reduction of F and M one of the pieces of the good partition of T associated to F', so that $F'(t) = F(t) \cap B$ for each $t \in M$, where B is a closed ball in E. We shall show that $F'|_M$ is fragmented, and so F' will be σ-fragmented by closed sets (Remark 6).

Let $\bar{C}$ be a non-empty subset of M and $\bar{C}$ its closure in T. The multivalued map G from $\bar{C}$ into E defined by $G(t) = F(t) \cap B$ ($\neq \varnothing$ by definition) for all $t \in \bar{C}$ is τ-usco, because B is τ-closed and F is u.s.c.

We consider a minimal τ-usco map G_0 contained in G, and we extend it to a minimal usco map F_0 from T into E defining $F_0(t) = \varnothing$ if $t \notin \bar{C}$. By hypothesis F_0 has the property (FP).

Let $\varepsilon > 0$. If U is an open subset of T such that $U \cap C \neq \varnothing$, then $F_0(U) \neq \varnothing$, and we can find an open subset V of U such that $F_0(V) \neq \varnothing$ (so $V \cap \bar{C} \neq \varnothing$) and ρ-diam $F_0(V) < \varepsilon$. We have that $V \cap C \neq \varnothing$ and $F'(t) \cap F_0(V) \supset F_0(t) \neq \varnothing$ for each $t \in C \cap V$. ∎

Next is the theorem by Srivatsa [38]. We formulate it for the extremal topology.

THEOREM 19 (Srivatsa). *Let M be a metric space, X a Banach space, and F from M into X a multivalued map which is upper semi-continuous for the weak topology, or for the topology $\sigma(X, \mathrm{ext}(B_{X^*}))$, with non-empty values. Then F has a selector f in the first Baire class $B_1(M, X)$.*

Proof. Just apply Theorem 13 and Proposition 9 or Proposition 10. ∎

A particular case is the following corollary.

COROLLARY 20. *Let M be a metric space, K a compact Hausdorff topological space, and $F\colon M \to 2^{C(K)}$ an upper semi-continuous map for the topology of the pointwise convergence, with non-empty values. Then F has a selector f in the first Baire class $B_1(M, C(K))$.*

SELECTORS OF U.S.C. MAPS WITH VALUES IN DUAL BANACH SPACES

In this section we obtain a first Baire class selector for arbitrary weak-star u.s.c. maps with non-empty values in dual spaces X^* of Asplund spaces X in the following cases:

(I) X^* is weakly compactly generated (WCG); i.e., there is a weakly compact subset whose linear span is dense in X^*;

(II) the unit ball B_{X^*} of X^* is angelic for the weak-star topology; i.e., the weak-star closure of every subset $A \subset B_{X^*}$ is the set of the weak-star limits of sequences from A.

These two cases are distinguished by the following examples:

The dual of the space X given by Johnson and Lindenstrauss (see [8, Example 5.6]) is WCG, so X is Asplund [32, Corollary 7], but its unit ball is not angelic for the weak-star topology. This is an example of one space which satisfies Case I but not II.

The space considered by Pol in [35] is a space of continuous functions $C(K)$ on an uncountable scattered compact space K. This $C(K)$ is Asplund, weakly Lindelöf, but not WCG, and its dual $l^1(K)$ is not Lindelöf for the weak topology (and so not WCG), because the set $\{\delta_x : x \in K\} \subset l^1(K)$ is an uncountable discrete weak-closed subset. On the other hand, Valdivia [40] has proved that the unit ball of this $l^1(K) = C(K)^*$ is angelic for the weak-star topology, so this dual space satisfies Case II, but not I.

THEOREM 21. *Let (M, d) be a metric space and X^* a WCG dual Banach space. Then every weak-star upper semi-continuous map $F: M \to 2^{X^*}$ with arbitrary non-empty values has a selector f in the first Baire class $B_1(M, X^*)$.*

Proof. We shall use the $\mathscr{K}$-analytic structure of the WCG dual Banach spaces given by a family of sets indexed by the set of the finite sequences of natural numbers to prove that every (piecewise) weak-star upper semi-continuous set-valued map is σ-fragmented by closed sets. The proof of this is obtained by constructing a sequence of piecewise constant function that verifies the hypothesis of Lemma 8. Once this has been established, Theorem 13 will provide a selector which is σ-fragmented by closed sets, and so is in $B_1(M, X^*)$, for each weak-star u.s.c. map with non-empty values.

We denote by $\mathbb{N}^{[\mathbb{N}]}$ the set of all finite sequences $(m_1, m_2, ..., m_k)$ of natural numbers, and by $\mathbb{N}^{\mathbb{N}}$ the set of all infinite sequences of natural numbers.

If $\sigma = (m_1, ..., m_n, ...) \in \mathbb{N}^{\mathbb{N}}$ we write $\sigma \mid k = (m_1, m_2, ..., m_k)$.

If X^* is a WCG dual Banach space, then we construct a family

$$\{K_{(m_1, m_2, ..., m_k)}: (m_1, m_2, ..., m_k) \in \mathbb{N}^{[\mathbb{N}]}\}$$

of weak-star compact subsets of X^* such that

(1) $X^* = \bigcup \{K_{(m)}: m \in \mathbb{N}\}$;

(2) for each $(m_1, m_2, ..., m_k) \in \mathbb{N}^{[\mathbb{N}]}$,

$$K_{(m_1, m_2, ..., m_k)} = \bigcup \{K_{(m_1, m_2, ..., m_k, m)}: m \in \mathbb{N}\};$$

(3) if $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $y_k \in K_{\sigma \mid k}$ for each $k \in \mathbb{N}$, then the sequence (y_k) has a subsequence which is weak convergent to a point of $K_\sigma = \bigcap \{K_{\sigma \mid k}: k \in \mathbb{N}\}$.

If X^* is generated by an absolutely convex weak compact subset K and we write $B(0, \varepsilon) = \{y \in X^*: \|y\| \leqslant \varepsilon\}$ and $mK = \{my: y \in K\}$, then we take $K_{(m)} = mK + B(0, \frac{1}{2})$ for all $m \in \mathbb{N}$, and

$$K_{(m_1, \dots, m_k, m)} = K_{(m_1, \dots, m_k)} \cap (mK + B(0, 1/2^{k+1}))$$

for all $m \in \mathbb{N}$ and $k \geqslant 1$.

Denote by $B^*(0, 1/2^k)$ the closed $(1/2^k)$-ball with centre at 0 in X^{***}. Then the sets $mK + B^*(0, 1/2^k)$ are weak-star compact in X^{***} for each pair of positive integers m, k, because they are sums of weak-star compact sets. In order to prove (3), observe that for each $\sigma = (m_1, m_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ and each sequence $y_k \in K_{\sigma \mid k}$ the weak-star cluster points in X^{***} of the sequence lie in

$$K_\sigma = \bigcap \{m_k K + B^*(0, 1/2^k): k \in \mathbb{N}\} \subset X^*,$$

where $B^*(0, 1/2^k)$ denotes the weak-star closure of $B(0, 1/2^k)$ in X^{***}. Then we have that the set $\{y_k: k \in \mathbb{N}\}$ is relatively compact in X^* for the weak topology, and by the Eberlein-Smulian Theorem the sequence has a subsequence which is weak convergent to a point that must be in K_σ.

Using the weak-star upper semi-continuity of F, for each $m \in \mathbb{N}$ we consider the countable cover of M given by the closed sets $T_{(m)} = \{t \in M: F(t) \cap K_{(m)} \neq \emptyset\}$. We take consecutive differences of the elements of the sequence of closed sets $T_{(m)}$ and we obtain a good partition $\{H_{(m)}: m \in \mathbb{N}\}$ of M such that $H_{(m)} \subset T_{(m)}$.

Suppose that for each $k = 1, \dots, n$ we have obtained a good partition

$$\{H_{(m_1, m_2, \dots, m_k)}: (m_1, m_2, \dots, m_k) \in \mathbb{N}^k\}$$

of M that verifies

(a) $H_{(m_1, m_2, \dots, m_{k-1})} = \bigcup \{H_{(m_1, \dots, m_{k-1}, m)}: m \in \mathbb{N}\}$,

if $k > 1$ and $(m_1, m_2, \dots, m_{k-1}) \in \mathbb{N}^{k-1}$, and

(b) $H_{(m_1, m_2, \dots, m_k)} \subset \{t \in M: F(t) \cap K_{(m_1, m_2, \dots, m_k)} \neq \emptyset\}$,

for all $(m_1, m_2, \dots, m_k) \in \mathbb{N}^k$.

For each $(m_1, m_2, \dots, m_n) \in \mathbb{N}^n$ we consider the countable cover of $H_{(m_1, m_2, \dots, m_n)}$ given by its closed subsets

$$T_{(m_1, m_2, \dots, m_n, m)} = \{t \in H_{(m_1, m_2, \dots, m_n)}: F(t) \cap K_{(m_1, m_2, \dots, m_n, m)} \neq \emptyset\}.$$

Then its consecutive differences give a good partition $\{H_{(m_1, m_2, \dots, m_n, m)}: m \in \mathbb{N}\}$ of $H_{(m_1, m_2, \dots, m_n)}$ such that $H_{(m_1, m_2, \dots, m_n, m)} \subset T_{(m_1, m_2, \dots, m_n, m)}$ for each $m \in \mathbb{N}$. By Remark 1 the family $\{H_{(m_1, \dots, m_n, m_{n+1})}: (m_1, m_2, \dots, m_{n+1}) \in \mathbb{N}^{n+1}\}$ is a good partition that satisfies Conditions a and b.

This induction procedure gives us a sequence of good partitions verifying a and b.

For each $n \in \mathbb{N}$ and each $(m_1, m_2, ..., m_n) \in \mathbb{N}^n$, Proposition 2 gives a good partition of $H_{(m_1, m_2, ..., m_{n+1})}$ associated with a cover by open sets with d-diameter less than $1/2^n$ that we denote by $\{M_j : j \in J(m_1, ..., m_{n+1})\}$. For each $j \in J(m_1, m_2, ..., m_n)$ such that $M_j \neq \varnothing$, we fix $t_j \in M_j$ and $y_j \in F(t_j) \cap K_{(m_1, m_2, ..., m_n)}$.

Now we define the piecewise constant function f_n by $f_n(t) = y_j$, if $t \in M_j$ for some $(m_1, m_2, ..., m_n) \in \mathbb{N}^n$ and some $j \in J(m_1, m_2, ..., m_n)$.

By the construction of the sequence f_n, for each $t \in M$ there exist a sequence $\sigma \in \mathbb{N}^{\mathbb{N}}$ and a sequence (s_n) in M such that $d(t, s_n) < 1/2^n$ and $f_n(t) \in F(s_n) \cap K_{\sigma | n}$ for all $n \in \mathbb{N}$.

Thus for each $t \in M$ we have two possibilities:

(i) $f_n(t) \in F(t)$ for some $n \in \mathbb{N}$ and so $\| \|$-dist$(f_n(t), F(t)) = 0$, or

(ii) $f_n(t) \notin F(t)$ for all $n \in \mathbb{N}$.

In case (ii) we have a sequence $\sigma \in \mathbb{N}^{\mathbb{N}}$ and a sequence (s_n) in M that converges to t such that $f_n(t) \in (F(s_n) \backslash F(t)) \cap K_{\sigma | n}$ for all $n \in \mathbb{N}$. Property 3 of the $K_{(m_1, m_2, ..., m_k)}$ says that there exists a weak convergent subsequence $f_{n_p}(t)$. The upper semi-continuity of F says that this subsequence has a weak-star cluster point in $F(t)$, so the weak limit of $f_{n_p}(t)$ is a point of $F(t)$ and there exists a sequence of rational convex combinations of f_{n_p} that converges in norm to a point of $F(t)$.

If we denote by $\{g_n : n \in \mathbb{N}\}$ the sequence of the rational convex combinations of elements of the sequence f_n, we obtain the sequence of piecewise constant functions satisfying the hypothesis of Lemma 8. ∎

For the proof of the selection theorem in the angelic case we use the fragmentability of the usco maps together with Lemma 8 and the following boundary theorem of Hansell, Jayne, Labuda, and Rogers [21, Theorem 1].

THEOREM 22 (Hansell, Jayne, Labuda, Rogers). *Let (M, d) be a metric space, $t \in M$, and K an angelic compact space; i.e., the closure of every subset $A \subset K$ is the set of limits of sequences from A. If U_n denotes a neighbourhood basis of t, and the multivalued map F from M into K is u.s.c., then*

$$\bigcap_{n \geq 1} \overline{F(U_n) \backslash F(t)} \subset F(t).$$

THEOREM 23. *Let (M, d) be a metric space and X^* the dual space of an Asplund space, whose unit ball B_{X^*} is angelic for the weak-star topology. Then every weak-star upper semi-continuous map $F: M \to 2^{X^*}$ with arbitrary non-empty values has a selector f in the first Baire class $B_1(M, X^*)$.*

Proof. If we can prove that F is σ-fragmented by closed sets, then every reduction of F which is piecewise u.s.c. with non-empty values has the same property, and then Theorem 13 gives a selector which is σ-fragmented by closed sets, and consequently in $B_1(M, X^*)$.

We can suppose that F takes its values in the unit ball B_{X^*}. Otherwise we could work with the pieces M_n of the good partition of M associated to the sequence of closed subsets $T_n = \{t \in M\colon F(t) \cap B(0, n) \neq \emptyset\}$ that covers M, and with the reduction of F defined by $F(t) \cap B(0, n)$ if $t \in M_n$.

Let $\bar{F}\colon M \to 2^{X^*}$ be the multivalued map defined by $\bar{F}(t) = \overline{F(t)}$, where the closure is taken in the weak-star topology. Then $\bar{F}$ is a weak-star usco map. Indeed, let $t \in M$ and $U \subset B_{X^*}$ be a weak-star open set in the induced topology such that $\bar{F}(t) \subset U$. By the compactness of $\bar{F}(t)$ there is a weak-star open subset V of B_{X^*} such that

$$F(t) \subset \bar{F}(t) \subset V \subset \bar{V} \subset U.$$

As F is u.s.c. there is a neighbourhood N of t such that $F(s) \subset V$ for every $s \in N$. Clearly, for each $s \in N$ we have $\bar{F}(s) \subset \bar{V} \subset U$. This proves that $\bar{F}$ is an usco map.

Now, we use the fact that F and $\bar{F}$ are u.s.c. together with the angelicity of B_{X^*} to find a sequence of piecewise constant functions $\tilde{f}_n$ such that for every $t \in M$ the limit $\tilde{f}(t) = \lim_{n \to +\infty} \tilde{f}_n(t)$ exists in norm, and $\tilde{f}(t) \in F(t)$. With this sequence, the hypothesis of Lemma 8 holds and so F is σ-fragmented by closed sets.

We construct the sequence $\tilde{f}_n$ by induction.

We put $\tilde{f}_0(t) = 0$ and $F_0(t) = \bar{F}(t)$ for each $t \in M$.

Suppose we have defined piecewise constant functions

$$\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{n-1}$$

and piecewise usco maps $F_0, F_1, \ldots, F_{n-1}$ which are reductions of $\bar{F}$.

By Proposition 11 we know that F_{n-1} is σ-fragmented by closed sets. Thus, if $\varepsilon = 1/2^n$, there exists a piecewise constant function f such that $F_{n-1}(t) \cap B(f(t), 1/2^n) \neq \emptyset$ for each $t \in M$ (Theorem 5).

Let $\{H_i\colon i \in I\}$ be a good partition of M such that f is constant and F_{n-1} is usco on each H_i.

By Proposition 2 we take a good partition of each H_i associated with a cover by open sets with d-diameter less than $1/2^n$, and we denote by $\{M_j\colon j \in J\}$ the new good partition of M.

For each $j \in J$ we fix a point $t_j \in M_j$, a point

$$y_j \in F_{n-1}(t_j) \cap B(f(t), 1/2^n) \subset \bar{F}(t_j),$$

and a sequence $\{y_{m,j}\colon m \in \mathbb{N}\} \subset F(t_j)$ which is weak-star convergent to y_j (this is possible because B_{X^*} is angelic with the weak-star topology).

By the upper semi-continuity of F, the following subsets are closed in the induced topology of M_j,

$$N_j = \{t \in M_j : F(t) \cap \{y_j, y_{1,j,\ldots}, y_{m,j,\ldots}\} \neq \emptyset\}$$
$$H_{0,j} = \{t \in M_j : y_j \in F(t)\}$$
$$H_{m,j} = \{t \in M_j : y_{m,j} \in F(t)\} \qquad \text{for } m \in \mathbb{N}.$$

If $G_j = M_j \backslash N_j$, then $\{N_j, G_j\}$ is a good partition of M_j because G_j is a $\mathscr{F}_\sigma$ subset of M_j. On the other hand, the sets $M_{0,j} = H_{0,j}$ and $M_{m,j} = H_{m,j} \backslash H_{m-1,j}$, for $m \in \mathbb{N}$, define a good partition of N_j. Then these sets together with G_j define a good partition of M_j.

Now we define $\tilde{f}_n(t)$ for each $t \in M_j$ as follows:

$$\tilde{f}_n(t) = y_j \qquad \text{if} \quad t \in G_j \cup M_{0,j},$$

and

$$\tilde{f}_n(t) = y_{m,j} \qquad \text{if} \quad t \in M_{m,j}.$$

We note that if $\tilde{f}_n(t) \notin F(t)$, then $t \in G_j$. We write $s_n(t) = t_j$, the above fixed point in M_j, so $\tilde{f}_n(t) = y_j \in \overline{F(s_n(t)) F(t)}$ where the closure is taken in the weak-star topology, and $d(t, s_n(t)) < 1/2^n$, because d-diam$(M_j) < 1/2^n$.

We define the reduction F_n of $\bar{F}$ by

$$F_n(t) = \{\tilde{f}_n(t)\} \qquad \text{if} \quad \tilde{f}_n(t) \in F(t),$$

and

$$F_n(t) = F_{n-1}(t) \cap B(\tilde{f}_n(t), 1/2^{n-1}) \qquad \text{if} \quad \tilde{f}_n(t) \notin F(t).$$

In the last case $F_n(t) \neq \emptyset$, because $B(f(t), 1/2^n) \subset B(\tilde{f}_n(t), 1/2^{n-1})$ and

$$B(f(t), 1/2^n) \cap F_{n-1}(t) \neq \emptyset.$$

The sequence $\tilde{f}_n$ that we have constructed has the property that every $t \in M$ satisfies one of the following two possibilities:

1. There exists some $m \in \mathbb{N}$ such that $\tilde{f}_m(t) \in F(t)$;
2. for all $n \in \mathbb{N}$, $\tilde{f}_n(t) \notin F(t)$.

In Case 1 $\tilde{f}_n(t) = \tilde{f}_m(t)$ for all $n \geqslant m$, so $\tilde{f}(t) = \tilde{f}_m(t) \in F(t)$.

In Case 2 there exists a sequence $(s_n(t))$ in M such that $d(t, s_n(t)) < 1/2^n$ and

$$\tilde{f}_n(t) \in \overline{F(s_n(t)) \backslash F(t)}.$$

On the other hand, by the construction of $\tilde{f}_n$,

$$\tilde{f}_n(t) \in B(\tilde{f}_{n-1}(t), 1/2^{n-2})$$

for all $n \geq 2$, so the sequence $\tilde{f}_n(t)$ is a norm Cauchy sequence in X^* and so it is norm convergent to some $\tilde{f}(t)$.

Write $U_n = \{s \in M : d(s, t) < 1/2^n\}$. Then we have in this case that

$$\tilde{f}(t) \in \bigcap_{n \geq 1} \overline{F(U_n) \backslash F(t)}.$$

But by the Boundary Theorem 22 stated above, this last intersection is a subset of $F(t)$, because F is weak-star upper semi-continuous and B_{X^*} is weak-star angelic. Then we have that $\tilde{f}(t) \in F(t)$.

In both cases we have shown that the sequence $\tilde{f}_n(t)$ is norm convergent to a point $\tilde{f}(t) \in F(t)$, and the proof is complete. ∎

Remark 24. Note that in the last proof we cannot assure that the selector $\tilde{f}$ is in the first Baire class, because the construction of the sequence $\tilde{f}_n$ does not give uniform convergence. But the sequence $\tilde{f}_n$ verifies the hypothesis of Lemma 8, so F is σ-fragmented by closed sets, and this fact, together with Theorem 13, allows us to find a selector in the first Baire class. Also observe that this is one of the cases where Remark 15 applies.

Counter Examples

Let X be a Banach space and K a weak-star compact subset of X^*. A subset $B \subset K$ is called a *boundary* of K if for every $x \in X$ there exists $b \in B$ such that

$$\langle x, b \rangle = \sup\{\langle x, y \rangle : y \in K\}.$$

For instance, if K is convex, the set B of all extreme points of K is a boundary of K by the usual proof of the Krein–Milman Theorem. We shall be interested here in the following multivalued map:

$$F_B : X \to 2^B$$

defined by $F_B(x) = \{b \in B : \langle x, b \rangle = \sup\{\langle x, y \rangle : y \in K\}\} \neq \varnothing$. We need more properties on B to be sure that F_B is norm to weak-star upper semi-continuous. For instance, this happens when B is weak-star countably compact.

PROPOSITION 25. *If X is a Banach space, K is a weak-star compact subset of X^* and B is a weak-star countably compact boundary of K, then the multivalued map F_B is norm to weak-star upper semi-continuous.*

Proof. If $x \in X$ and x_n is a sequence in X which converges to x, then for every positive integer n we take $y_n \in F_B(x_n)$. The sequence (y_n) has a weak-star cluster point y in B. We prove that $y \in F_B(x)$, and so F_B is weak-star upper semi-continuous. We suppose that $y \notin F_B(x)$. Then there is $y_0 \in K$ and $\delta > 0$ such that

$$\langle x, y \rangle \leqslant \langle x, y_0 \rangle - \delta.$$

Choose a positve integer n such that $|\langle x_n - x, k \rangle| < \delta/2$ for every $k \in K$ and

$$\langle x, y_n \rangle \leqslant \langle x, y_0 \rangle - \delta.$$

Then we have

$$\langle x_n, y_n \rangle < \langle x, y_n \rangle + \delta/2 \leqslant \langle x, y_0 \rangle - \delta/2 < \langle x_n, y_0 \rangle,$$

which is a contradiction, since $y_n \in F_B(x_n)$. ∎

With the same notation as above, we let $F_K: X \to 2^K$ denote the norm to weak-star usco map given by

$$F_K(x) = \{k \in K: \langle x, k \rangle = \sup\{\langle x, y \rangle: y \in K\}\}.$$

The next result characterizes when F_K has a selector in the first Baire class to the norm $B_1(X, X^*)$.

THEOREM 26. *If X is a Banach space and K is a weak-star compact subset of X^*, then F_K has a selector in the first Baire class $B_1(X, X^*)$ if, and only if, K is fragmented by the norm. Moreover, if $f: X \to K$ is such a selector of F_K, then we have*

$$\overline{co(K)}^{\text{weak-star}} = \overline{co(f(X))}^{\|\ \|}.$$

Proof. When K is fragmented by the norm, Remark 17 to the Jayne–Rogers Selection Theorem gives a selector in the first Baire class for F_K. Conversely, if $f: X \to K$ is a first Baire class selector for F_K and S is a separable subspace of X, we denote by i_S the canonical embedding from S into X and by K_S the weak-star compact subset of S^* equal to $i_S^*(K)$. Let $B = i_S^*(f(S))$. Since f is in the first Baire class, $f(S)$ is norm separable in X^* and B is a norm separable boundary of K_S in S^*. It is clear that B is also a boundary of $\overline{co(K_S)}^{\text{weak-star}}$. Now a theorem of Godefroy [13, Theorem I.2] applies to give $\overline{co(K_S)}^{\text{weak-star}} = \overline{co(B)}^{\|\ \|_{S^*}}$, from which it follows that K_S is $\| \|_{S^*}$-separable. So we have proved that for any separable subspace S of X, K_S is $\| \|_{S^*}$-separable. A theorem of Namioka [31, Theorem 3.4] says that K is norm fragmented in X^*. Moreover, we have seen that for any separable subspace $S \subset X$ we have the identity

$$\overline{co(K_S)}^{\text{weak-star}} = \overline{co(i_S^*(f(S)))}^{\| \|_{S^*}}.$$

It is our objective now to prove the same equality when $S = X$. The separable reduction argument used by Fabian and Godefroy [10] dealing with $K = B_{X^*}$, the unit ball of X^*, can be translated word by word to finish the proof. We shall do it for completeness. Let $f_k: X \to \overline{co(f(X))}^{\| \|}$ be a sequence of norm-to-norm continuous functions that converges pointwise in norm to the selector f (see Remark 17). Consider $g \in \overline{co(K)}^{\text{weak-star}}$ and let $\varepsilon > 0$. Arguing by induction, for each positive integer n, we obtain a closed subspace Y_n of X, a countable dense subset D_n of Y_n, and a countable subset $F_n = \{v_{n,j} : j = 1, 2, ...\}$ of B_X, the unit ball of X, such that

(i) $D_n \subset D_{n+1}$,

(ii) $Y_{n+1} \supset Y_n \cup F_n$,

(iii) if $C_n = \{h_{n,j} : j = 1, 2, ...\}$ is the countable set of all rational linear combinations of vectors in $\bigcup \{f_k(D_n) : k = 1, 2, ...\}$, then we have the inequality

$$\langle g - h_{n,j}, v_{n,j} \rangle \geqslant \|g - h_{n,j}\| - \frac{1}{n}$$

for each $j \in \mathbb{N}$.

The induction process is straightforward when we begin with any separable subspace $Y_1 \subset X$, $Y_1 \neq \{0\}$. Let S be the separable subspace of X obtained as the norm closure of $\bigcup \{Y_n : n = 1, 2, ...\}$. Since $\overline{co(K_S)}^{\text{weak-star}} = \overline{co(i_S^*(f(S)))}^{\| \|_{S^*}}$, there exists a rational convex combination $h = \sum_{i=1}^{p} b_i f(s_i)$, where $s_i \in S$, $b_i \in \mathbb{Q}^+$, and $\sum_{i=1}^{p} b_i = 1$, such that $\|i_S^*(g - h)\| < \varepsilon/4$. Because of the continuity of the functions f_k, we can find an integer k and a finite subset $\{t_1, t_2, ..., t_p\}$ in some D_n such that $\|h - \sum_{i=1}^{p} b_i f_k(t_i)\| < \varepsilon/4$. We have obtained an element $h_{n,j} = \sum_{i=1}^{p} b_i f_k(t_i)$ of C_n such that $\|h - h_{n,j}\| < \varepsilon/4$. Observe that we can also assume $n \geqslant 4/\varepsilon$, because of Condition i. Then we have

$$\begin{aligned}\|g - h\| &\leqslant \|g - h_{n_j}\| + \varepsilon/4 \leqslant 1/n + \langle g - h_{n_j}, v_{n_j} \rangle + \varepsilon/4 \\ &\leqslant \varepsilon/2 + \|i_S^*(g - h_{n_j})\| \leqslant \varepsilon/2 + \|i_S^*(g - h)\| + \|i_S^*(h - h_{n_j})\| \\ &\leqslant \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon.\end{aligned}$$

This shows that $g \in \overline{co(f(X))}^{\| \|}$ and the proof is complete. ∎

If X is an Asplund space and K is a weak-star compact subset of X^*, the multivalued map F_K has a selector f in the first Baire class and we have proved that $\overline{co(K)}^{\text{weak-star}} = \overline{co(f(X))}^{\|\ \|}$. Now suppose we have a boundary B of K. If $f(X) \subset B$, then we must have $\overline{co(K)}^{\text{weak-star}} = \overline{co(B)}^{\|\ \|}$ and we arrive at the following:

COROLLARY 27. *Let X be an Asplund space and K a weak-star compact subset of X^* with a weak-star countably compact boundary B such that*

$$\overline{co(K)}^{\text{weak-star}} \neq \overline{co(B)}^{\|\ \|}.$$

Then the multivalued map F_B is norm to weak-star upper semi-continuous, $F_B(x)$ is norm closed and weak-star countably compact for every $x \in X$, and F_B has no selector in the first Baire class $B_1(X, X^*)$.

We now describe some concrete examples where the former corollary applies:

EXAMPLE 28. Let ω_1 be the first uncountable ordinal, and $[0, \omega_1]$ the ordinal interval with the order topology. Let X be the Banach space $C[0, \omega_1]$ of continuous real-valued functions on $[0, \omega_1]$. The dual space is identified with $l^1[0, \omega_1]$. If K is the dual unit ball, then a boundary B verifying the hypothesis of the former corollary is

$$B = \{\delta_\alpha : \alpha \in [0, \omega_1)\},$$

where δ_α is the Dirac measure at point α. ∎

More generally, we have:

EXAMPLE 29. If S is a scattered Valdivia compact space [6, 39] which is not angelic, then there is a set Γ such that S can be identified with a subspace of the cube $[0, 1]^\Gamma$ and

$$S(\Gamma) = \{x \in S : \text{support}(x) \text{ is countable}\}$$

is a dense subset of S. Let X be the Banach space $C(S)$ of continuous real valued functions on S. The dual space X^* is identified with $l^1(S)$. If K is the dual unit ball, then a boundary B verifying the hypothesis of Corollary 27 is $B = \{\delta_x : x \in S(\Gamma)\}$, where δ_x is the Dirac measure at the point $x \in S(\Gamma)$. Since S is not angelic, there exist $x \in S$ with $x \notin S(\Gamma)$. Consequently $\|\delta_x - y\| \geqslant 1$ for all $y \in co(B)$ and B is a weak-star countably compact subset of K with $\overline{co(B)}^{\|\ \|} \neq K$. ∎

Another example of the same nature is the following:

EXAMPLE 30. If X is an Asplund space whose weak topology is not realcompact, we take $f \in X^{**} \setminus X$ such that f is continuous when restricted to any separable subspace of X^* with its weak-star topology [4]. Then $H = \ker f$ is a norm closed hyperplane of X^* and $H \cap B_{X^*}$ is not weak-star closed by the Krein–Smulian Theorem [5, Theorem II.5.5]. However, $H \cap B_{X^*}$ is weak-star sequentially closed. If we take $K = \overline{H \cap B_{X^*}}^{\text{weak-star}}$ and B the boundary given by $H \cap B_{X^*}$, we are within the conditions of Corollary 27 and F_B has no selector in the first Baire class. Moreover, in this case $F_B(x)$ is convex for every x in X. ∎

Recall that a Banach space X has the property C of Corson if every family of closed convex subsets of X with the countably intersection property (i.e., every countable sub-family of the given family has non-empty intersection) has non-empty intersection. Every Banach space with property C of Corson is weakly realcompact [4]. A result of Pol [36] says that property C is equivalent to the following:

(P) For every subset $A \subset B_{X^*}$ and $f \in \bar{A}^{\text{weak-star}}$, there exists a countable subset D of A such that $f \in \overline{co(D)}^{\text{weak-star}}$.

Thus in a space X with the property C, a weak-star countably compact and convex subset of X^* is weak-star compact and we have the following:

THEOREM 31. *Let X be an Asplund space. Then the following conditions are equivalent*:

(i) *X has the property C;*

(ii) *if M is a metric space and F a weak-star upper semi-continuous multivalued map from M into X^*, for which $F(m)$ is a non-empty, convex and weak-star countably compact subset of X^* for each $m \in M$, then F has a selector in the first Baire class $B_1(M, X^*)$.*

Proof. (i) ⇒ (ii). This is Theorem 16 for upper semi-continuous compact valued maps. (ii) ⇒ (i). Suppose that X does not have property C. Then there exists A in B_{X^*}, $f \in \bar{A}^{\text{weak-star}}$ and $f \notin \overline{co(D)}^{\text{weak-star}}$ for every countable subset $D \subset A$. If K is the compact set $\overline{co(A)}^{\text{weak-star}}$, and B is the boundary defined by

$$B = \{g \in K\colon \text{there is a countable } D \subset A \text{ with } g \in \overline{co(D)}^{\text{weak-star}}\},$$

then B is weak-star countably compact, convex, and $B \neq K$. Corollary 27 says that F_B is upper semi-continuous, $F_B(x)$ is a convex and weak-star countably compact subset of X^* for each $x \in X$, and it has no selector in the first Baire class $B_1(X, X^*)$. ∎

REFERENCES

1. R. ARENS, Extension of functions on fully normal spaces, *Pacific J. Math.* **2** (1952), 11–22.
2. C. BESSAGA AND A. PEŁCZYŃSKY, "Selected Topics in Infinite-Dimensional Topology," Polish Sci. Publishers, Warsaw, 1975.
3. J. BOURGAIN AND M. TALAGRAND, Compacité extrêmale, *Proc. Amer. Math. Soc.* **80** (1980), 68–70.
4. H. H. CORSON, The weak topology of a Banach space, *Trans. Amer. Math. Soc.* **101** (1961), 1–15.
5. M. M. DAY, "Normed Linear Spaces," Springer-Verlag, New York, 1973.
6. R. DEVILLE AND G. GODEFROY, Some applications of projective resolutions of identity, *Proc. London Math. Soc.* **67** (1993), 183–199.
7. D. VAN DULST AND I. NAMIOKA, A note on trees in conjugate Banach spaces, *Indag. Math.* **46** (1984), 7–10.
8. G. A. EDGAR, Measurability in Banach spaces I and II, *Indiana Univ. Math. J.* **26** (1977), 663–677; **28** (1979), 559–579.
9. R. ENGELKING, "General Topology," Polish Sci. Publishers, Warsaw, 1977.
10. M. FABIAN AND G. GODEFROY, The dual of every Asplund space admits a projectional resolution of the identity, *Studia Math.* **91** (1988), 141–151.
11. M. FOSGERAU, When are Borel functions Baire functions?, *Fund. Math.*, to appear.
12. N. GHOUSSOUB, B. MAUREY, AND W. SCHACHERMAYER, Slicings, selections, and their applications, *Canad. J. Math.* **44** (1992), 483–504.
13. G. GODEFROY, Boundaries of a convex set and interpolation sets, *Math. Ann.* **277** (1987), 173–184.
14. G. GODEFROY, Five lectures in geometry of Banach spaces, *in* "Seminar on Functional Analysis, 1987, Universidad de Murcia," Notas de Matemática, Vol. 1, pp. 9–67, 1978.
15. G. GODEFROY AND M. TALAGRAND, Espaces de Banach representables, *Israel J. Math.* **41** (1982), 321–330.
16. R. W. HANSELL, Borel measurable mapping for non separable metric spaces, *Trans. Amer. Math. Soc.* **161** (1971), 145–169.
17. R. W. HANSELL, On Borel mapping and Baire functions, *Trans. Amer. Math. Soc.* **194** (1974), 195–211.
18. R. W. HANSELL, First class selectors for upper semi-continuous multifunctions, *J. Funct. Anal.* **75** (1987), 382–395.
19. R. W. HANSELL, Extended Bochner measurable selectors, *Math. Ann.* **277** (1987), 79–94.
20. R. W HANSELL, J. E. JAYNE, AND M. TALAGRAND, First class selector for weakly upper semi-continuous multivalued maps in Banach spaces, *J. Reine Angew. Math.* **361** (1985), 201–220; **362** (1986), 219–220.
21. R. W. HANSELL, J. E. JAYNE, I. LABUDA, AND C. A. ROGERS, Boundaries of and selectors for upper semi-continuous set-valued functions, *Math. Z.* **189** (1985), 297–318.
22. J. E. JAYNE, I. NAMIOKA, AND C. A. ROGERS, Topological properties of Banach spaces, *Proc. London Math. Soc.* **66** (1993), 651–672.
23. J. E. JAYNE, I. NAMIOKA, AND C. A. ROGERS, Norm fragmented weak* compact sets, *Collectanea Math.* **41** (1990), 133–163.
24. J. E. JAYNE, I. NAMIOKA, AND C. A. ROGERS, σ-fragmented Banach spaces, *Mathematika* **39** (1992), 161–188; 197–215.
25. J. E. JAYNE AND C. A. ROGERS, Upper semi-continuous set-valued functions, *Acta Math.* **149** (1982), 87–125; **155** (1985), 149–152.
26. J. E. JAYNE AND C. A. ROGERS, Borel selectors for upper semi-continuous multi-valued functions, *J. Funct. Anal.* **56** (1984), 279–299.
27. J. E. JAYNE AND C. A. ROGERS, Sélections boréliennes de multi-applications semi-continues supérieurement, *C.R. Acad. Sci. Paris Sér. I* **299** (1984), 125–128.

28. J. E. JAYNE AND C. A. ROGERS, Borel selectors for upper semi-continuous set-valued maps, *Acta. Math.* **155** (1985), 41–79.
29. K. KURATOWSKI, "Topology," Vol. 1, Academic Press, New York, 1966.
30. H. LEBESGUE, Une propriété caractéristique des fonctions de classe 1, *Bull. Soc. Math. de France* **32** (1904), 1–14.
31. I. NAMOKA, Radon–Nikodým compact spaces and fragmentability, *Mathematika* **34** (1987), 258–281.
32. I. NAMIOKA AND R. R. PHELPS, Banach spaces which are Asplund spaces, *Duke Math. J.* **42** (1975), 735–750.
33. J. ORIHUELA AND M. VALDIVIA, Resolutions of identity and first Baire class selectors in Banach spaces, preprint.
34. J. ORIHUELA, W. SCHACHERMAYER, AND M. VALDIVIA, Every Radon–Nikodým and Corson compact space is an Eberlein compact, *Studia Math.* **98** (1991), 157–174.
35. R. POL, A function space which is weakly Lindelöf but not weakly compactly generated, *Studia Math.* **64** (1979), 279–285.
36. R. POL, On a question of H. H. Corson and some related problems, *Fund. Math.* **109** (1980), 143–154.
37. C. A. ROGERS, Functions of the first Baire class, *J. London Math. Soc.* **37** (1988), 535–544.
38. V. V. SRIVATSA, Baire class 1 selectors for upper semi-continuous set-valued maps, typed manuscript, 1985.
39. M. VALDIVIA, Projective resolutions of identity in $C(K)$ spaces, *Archiv der Math.* **54** (1990), 493–498.
40. M. VALDIVIA, Complemented subspaces of certain Banach spaces, *in* "Seminar of Functional Analysis 1988–89 of the University of Murcia," Notas de Matematica, Vol. 4, to appear.