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Topologies Weaker Than the Weak Topology of a Banach Space

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In this paper we investigate some properties of the compact subsets of Banach spaces X endowed with topologies of the kind $\sigma(X, B)$ where B is a norming subset of the dual unit ball B_{X^*}. Assuming that B_{X^*} is sequentially compact we prove that the Krein–Smulian theorem holds for norm bounded $\sigma(X, B)$-compact subsets of X and we state that the convex $\sigma(X, B)$-compact subsets of X have the weak Radon–Nikodým property. When B_{X^*} is sequentially compact and X has either the separable complementation property or X is weakly Lindelöf (for instance, when B_{X^*} is Corson compact) we prove that the $\sigma(X, B)$-compact subsets (resp. $\sigma(X, B)$-compact convex subsets) of X are fragmented by the norm of X (resp. have the Radon–Nikodým property). So, if B_{X^*} is a Corson compact then the compact subsets of the space $X[\sigma(X; B)]$ are Radon–Nikodým compact and thus sequentially compact. We apply the previous results to prove that if B_{X^*} is sequentially compact and B is assumed to be a boundary of B_{X^*}, then the norm bounded $\sigma(X, B)$-compact subsets of X are weakly compact, which partially answers a problem posed by G. Godefroy. We give, among others, applications to spaces of vector-valued Bochner integrable functions as well as to spaces of countably additive measures.
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1. INTRODUCTION AND MAIN RESULTS

Dealing with Banach spaces the properties of the weak and weak* topologies have been intensively studied and there is a rich collection of results about them. The weak* topology $\sigma(X^*, X)$ of a dual Banach space X^* is, for non-reflexive spaces, strictly *weaker* than the weak topology of X^*. Of course, there are other situations where these kinds of *weaker* topologies appear: the topology of pointwise convergence in the space of continuous functions on a compact space K, or more generally, for a Banach space X, the topology $\sigma(X, \mathrm{Ext}(B_{X^*}))$ of pointwise convergence on the extreme points of the dual unit ball B_{X^*}. Relevant properties of

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$X^*[\sigma(X^*, X)]$, $C_p(K)$, and $X[\sigma(X, \mathrm{ext}(B_{X^*})]$ can be found in many papers, e.g., [1, 3, 8, 16, 20, 22, 32, 33, 38, 40, 41]. Sometimes, topologies $\sigma(X, Y)$ weaker than the weak topology on X appear, because it is easier to handle with a "reasonable" norming subset Y of X^* rather than all of X^*. Typical examples of the last situation are provided by the spaces $L_1(\mu, X)$ of X-valued Bochner μ-integrable functions and $\mathrm{ca}(\Omega, \Sigma)$ of countably additive measures on the measurable space (Ω, Σ) endowed with the variation norm. In these two cases the topologies $\sigma(L^1(\mu, X), L^\infty(\mu, X^*))$ and $\sigma(\mathrm{ca}(\Omega, \Sigma), S(\Sigma))$, where $S(\Sigma)$ stands for the Σ-simple functions on Ω, have been used to replace their respective weak topologies in the study of some interesting questions about weak compactness [4, 5, 18] (see also [11]).

The objective of this paper is to study certain properties of the compact subsets of X for topologies of the kind $\sigma(X, B)$, where X is a Banach space and B is a norming subset of B_{X^*}, and apply them to partially answer a problem posed by Godefroy in [21], as well as to improve a result of E. Saab [34] and M. Talagrand [39] and obtain many of the results in [4, 5] about the $\sigma(L^p(\mu, X), L^q(\mu, X^*))$-compact subsets of $L^p(\mu, X)$, in addition to new properties.

The space $X[\sigma(X, B)]$ can be considered as a subspace of a $C(K)$ space endowed with the topology $t_p(D)$ of pointwise convergence on a dense subset D of the compact space K. Therefore our approach lies in the context of $C(K)$ spaces. More concretely, we will be concerned with the study of $t_p(D)$-compact subsets of $C(K)$ looking for sufficient conditions which ensure that they have some of the well known properties of Eberlein compact spaces, for instance, sequential compactness, fragmentability, ... In order to have these properties we restrict our attention to a class of $t_p(D)$-compact subsets having properties analogous to the weak* compact subsets of dual Banach spaces X^* for which X does not contain l^1.

Our main tools are results of Talagrand coming from measure and vector integration theories [40], along with ideas of Namioka, who started the study of the compact spaces homeomorphic to weak*-compact subsets of an Asplund space [28].

Our notation is standard: X will be a real Banach space, X^* its dual, and B_X the unit ball of X. Most of the definitions are given when needed, although for some of them we refer elsewhere.

The main results we establish in this paper are summarized in the following:

THEOREM A. *Let* $(X, \|\ \|)$ *be a Banach space and* B *a norming subset of* B_{X^*} *(i.e.,* $\|x\| = \sup\{|x^*(x)| : x^* \in B\}$*). Consider the following statements:*

(i) *The Krein–Smulian theorem holds for each norm bounded* $\sigma(X, B)$*-compact subset* H *of* X, *i.e.,* $\overline{\mathrm{co}(H)}^{\sigma(X, B)}$ *is* $\sigma(X, B)$*-compact;*

(ii) *Every* $\sigma(X, B)$*-compact convex subset* H *of* X *is the norm closed convex hull of its extreme points, that is,* $H = \overline{\mathrm{co}(\mathrm{Ext}(H))}^{\|\ \|}$;

(iii) *Every* $\sigma(X, B)$*-compact convex subset of* X *has the weak Radon–Nikodým property*;

(iv) *Every norm bounded* $\sigma(X, B)$*-compact subset of* X *is norm fragmented*;

(v) *The* $\sigma(X, B)$*-compact subsets of* X *are Radon–Nikodým compact spaces*;

(vi) *The* $\sigma(X, B)$*-compact subsets of* X *are sequentially compact*;

(vii) *The* $\sigma(X, B)$*-compact convex subsets of* X *have the Radon–Nikodým property*;

(viii) $X[\sigma(X, B)]$ *is* σ*-fragmented by the norm of* X.

The properties (i), (ii), *and* (iii) *hold when* $(B_{X^*}, \mathit{weak}^*)$ *is sequentially compact. The properties* (iv), (v), (vi), *and* (vii) *hold when* $(B_{X^*}, \mathit{weak}^*)$ *is sequentially compact and* X *has either the separable complementation property or* X *is weakly Lindelöf* (*for instance, when* $(B_{X^*}, \mathit{weak}^*)$ *is Corson*). *Property* (viii) *holds when* X *is weakly countably determined.*

Theorem A also holds when the assumption (B_{X^*}, weak^*) sequentially compact is replaced for either of the conditions below:

(1) B is relatively sequentially compact in (B_{X^*}, weak^*);

(2) $\mathrm{co}(B)$ is sequentially dense in (B_{X^*}, weak^*).

Assuming that B is a boundary for B_{X^*} much more can be said about the norm bounded $\sigma(X, B)$-compact subsets of X.

THEOREM B. *Let* X *be a Banach space and* B *a boundary for* B_{X^*} (*i.e., for every* x *in* X *there exists* $b^* \in B$ *such that* $\sup\{x^*(x) : x^* \in B_{X^*}\} = b^*(x)$). *For any norm bounded* $\sigma(X, B)$*-compact subset* H *of* X, *the following are equivalent*:

(i) H *is weakly compact*;

(ii) H *is weakly K-analytic* (*or weakly countably determined*);

(iii) *For every sequence* (x_n^*) *in* B *there exists a subsequence* $(x_{n_j}^*)$ *such that* $(x_{n_j}^*(x))$ *converges for every* x *in* H;

(iv) $\overline{\mathrm{co}(H)}^{\sigma(X, B)}$ *is* $\sigma(x, B)$*-compact*;

(v) *The* $\sigma(X, B)$*-separable subsets of* H *are weakly separable*;

(vi) H *is fragmented by the norm of* X;

(vii) H *is* $\sigma(X, B)$*-sequentially compact.*

The equivalence between conditions (i), (iv), and (vii) is classical and can be found in [17].

The following question of Godefroy was formulated in [21]: let X be a Banach space, B a boundary for B_{X^*}, and H a norm bounded $\sigma(X, B)$-compact subset of X. Is H weakly compact? That this question has a positive answer when X does either not contain l^1 or B is the set of extreme points of B_{X^*} has been stated in [21] and [8], respectively. The new equivalences in Theorem B enable us to add some more cases for which the previous problem has a positive answer:

COROLLARY C. *Let X be a Banach space, B a boundary for B_{X^*}, and H a norm bounded $\sigma(X, B)$-compact subset of X. H is weakly compact if one of the following conditions holds*:

(a) *$(B_{X^*}$, weak*$)$ is sequentially compact*;

(b) *B is relatively sequentially compact in $(B_{X^*}$,weak*$)$*;

(c) *$\mathrm{co}(B)$ is sequentially dense in $(B_{X^*}$, weak*$)$.*

In Section 2 we present the definitions, examples, and preliminary results that are needed for the subsequent sections. In Section 3, we establish the validity of the Krein–Smulian theorem for certain norm bounded $t_p(D)$-compact subsets H of $C(K)$, we study when it is possible to have formulas of the kind $\overline{\mathrm{co}(H)}^{t_p(D)} = \overline{\mathrm{co}(H)}^{\mathrm{norm}}$, and we characterize the convex $t_p(D)$-compact subsets of $C(K)$ having the Weak Radon–Nikodým Property. In Section 4, we give sufficient conditions for the fragmentability by the norm of the $t_p(D)$-compact subsets of $C(K)$ and we characterize it through the Radon–Nikodým Property. In Section 5, we prove Theorem A, Theorem B, and Corollary C and we use them to give some applications to spaces $L^p(\mu, X)$ endowed with the topology $\sigma(L^p(\mu, X), L^q(\mu, X^*))$ and to spaces of countably additive measures. We finish the paper improving a result by E. Saab [34] and M. Talagrand [39].

2. DEFINITIONS, EXAMPLES, AND PRELIMINARY RESULTS

Throughout, K denotes a compact Hausdorff space and $C(K)$ the Banach space of continuous real-valued functions on K endowed with the supremum norm, $\|\ \|$. If F is a subset of K, we denote by $t_p(F)$ the topology in $C(K)$ of pointwise convergence on F. In what follows D always will be a dense subset in K; in this case $t_p(D)$ is a Hausdorff locally convex topology in $C(K)$.

Note that if $(X, \|\ \|)$ is a Banach space and B is a norming subset of B_{X^*} (that is, $\|x\| = \sup\{|x'(x)| : x' \in B\}$), then the convex hull of B, $D = \mathrm{co}(B)$, is a dense subset of the compact space $K = (B_{X^*}$, weak*$)$, and so considering

X as a subspace of $C(B_{X^*})$ the topology $\sigma(X, B)$ is the one induced by $t_p(D)$. In particular, dealing with a dual Banach space X^* the weak* topology is obtained when we take $K = B_{X^{**}}$ and $D = B = B_X$. With this identification for dual Banach spaces, our definition below can be thought of as an extension of the weak* compact Pettis sets considered by Talagrand in [40].

DEFINITION 1. Let K be a compact Hausdorff space, D a dense subset of it, and H a uniformly bounded $t_p(D)$-compact subset of $C(K)$. We say that H is a $P(D)$-set if every sequence (d_n) in D has a subsequence (d_{n_j}) such that $h(d_{n_j})$ converges for each h in H.

The first examples of $P(D)$-sets are provided by the $t_p(K)$-compact subsets of $C(K)$. The above definition can be realized as a compactness requirement on D when we look at D as a set of functions on H. Indeed, for each point x in K we denote by δ_x the "point mass" at x, that is, $\delta_x(f) := f(x)$ for every $f \in C(K)$ and by $\hat{x}$ the restriction of δ_x to H. It is clear that $\hat{D} = \{\hat{d} : d \in D\}$ is a uniformly bounded set of continuous functions on the compact space $(H, t_p(D))$, and $\hat{K} = \{\hat{x} : x \in D\}$ is a uniformly bounded set of continuous functions on the topological space $(H, t_p(K))$. According to the definition, H is a $P(D)$-set if, and only if, $\hat{D}$ is relatively sequentially compact in $\mathbb{R}^H$.

Given a compact Hausdorff space H we will use the following spaces of functions defined on it:

(i) $B_r(H)$, the space of real-valued functions f on H such that for each non-empty closed subset C of H the restriction of f to C has at least one point of continuity;

(ii) $M_\mu(H)$, the space of μ-measurable real-valued functions on H, where μ is a Radon probability on H.

We use some of Talagrand's results concerning stable subsets of $M_\mu(H)$. This is a technical notion stronger than pointwise compactness in $M_\mu(H)$. Stable subsets of $M_\mu(H)$ are reasonable pointwise compact subsets which in a sense are "small," namely totally bounded for the (pseudo-metric) topology of convergence in measure [40, Chap. 9, 15, Chap. II]. We recall that a bounded sequence (f_n) in $C(H)$ is said to be equivalent to the standard unit vector basis of l^1 (briefly, an l^1-sequence) if there is a $\delta > 0$ such that

$$\sup_{t \in H} \left| \sum \alpha_k f_k(t) \right| \geqslant \delta \sum |\alpha_k|$$

for all finite sequences (α_k) of real numbers.

The following theorem is essentially a combination of Rosenthal's l^1-theorem and some results of [9, 40] (see also [15]). It provides us with equivalent conditions to the one in Definition 1 which will be the key to proving the main results of this paper.

THEOREM 1. (Rosenthal, Bourgain, Fremlin, Talagrand). *Let H be a compact Hausdorff space and $Z \subset C(H)$ a uniformly bounded subset. The following are equivalent*:

(a) *Z does not contain an l^1-sequence*;

(b) *Each sequence in Z has a pointwise convergent subsequence*;

(c) *Z is pointwise relatively compact in $B_r(H)$*;

(d) *For each Radon probability μ on H, Z is pointwise relatively compact in $M_\mu(H)$*;

(e) *For each Radon probability μ on H, Z is a stable subset of $M_\mu(H)$*.

For the proof of this theorem we refer to [9, 40, 15] where more equivalences are stated and the relative compactness of Z in different spaces of "regular" functions on H has been studied. Note that H is a $P(D)$-subset of $C(K)$ if, and only if, $Z = \hat{D}$ satisfies the equivalent conditions of Theorem 1. Since $\hat{K}$ is the pointwise closure of $\hat{D}$ in $\mathbb{R}^H$, $\hat{K}$ is a pointwise compact subset of $B_r(H)$ and $M_\mu(H)$, for every Radon probability μ on H.

EXAMPLE A. Let X be a Banach space and X^* its dual. It was proved in [33, 35] that a Banach space X does not contain a copy of l^1 if, and only if, for every non-empty weak*-compact subset H of X^* and every x^{**} in X^{**}, the restriction of x^{**} to (H, weak^*) has a point of continuity. So for spaces not containing l^1 it follows from Theorem 1 that each weak*-compact subset of X^* is a $P(B_X)$-subset of $C(B_{X^{**}})$. More generally, Theorem 1 shows that the Pettis sets introduced by Talagrand in [40] are $P(B_X)$-sets: A weak* compact subset H of X^* is called a Pettis set if for every x^{**} in X^{**} the restriction $x^{**}|_H$ is μ-measurable for every weak* Radon measure on H (see also [31–33, 35]).

EXAMPLE B. Recall that a subset D of the space K is said to be *relatively sequentially compact* (shortly, RSC) if every sequence in D has a subsequence converging to a limit in K. If D is a dense and RSC subset of the compact space K, every uniformly bounded $t_p(D)$-compact subset H of $C(K)$ is a $P(D)$-set. In particular, if K is a compact and sequentially compact space, then for every dense subset D of K all the uniformly bounded and $t_p(D)$-compact subsets of $C(K)$ are $P(D)$-sets. As is well known the class of compact and sequentially compact spaces is a wide class which includes the class of angelic compact spaces [17], and therefore the classes of Eberlein compact [2], Talagrand compact [38], Gul'ko compact [38], Corson compact [12], Rosenthal compact [9], and Radon–Nikodým compact [28]. However, there are compact spaces which are not sequentially compact and have a dense RSC subset: take a set I of cardinality $2^{\aleph_0}$ and consider $K := [0, 1]^I$. K is not sequentially compact but it has a dense RSC subset. The last kind of compact spaces are a special type of Valdivia compact spaces [3, 41]: a compact space K is said to be Valdivia compact if K is homeomorphic to a compact subset of $\mathbb{R}^\Gamma$ (for some set Γ) with the property that

$$D = K \cap \{(x_\gamma) \in \mathbb{R}^\Gamma : \{\gamma : x_\gamma \neq 0\} \text{ is countable}\}$$

is dense in K. In general these K's are not sequentially compact but D is always a dense RSC subset.

Example C. In the case for which D is a sequentially dense subset of K, given a uniformly bounded $t_p(D)$-compact subset H of $C(K)$ it is clear that $\hat{D}$ is sequentially dense in $\hat{K}$. As $\hat{D}$ is made up of continuous functions on $(H, t_p(D))$, $\hat{K}$ is a compact subset of the space of real-valued functions of the first Baire class on $(H, t_p(D))$. Thus the set $Z = \hat{D}$ verifies condition (c) of Theorem 1 and so H is a $P(D)$-set in $C(K)$.

Example D. Given a Banach space X and a norming subset B of B_{X^*} let $D = \text{co}(B)$. By definition a norm bounded and $\sigma(X, B)$-compact subset H of X is a $P(D)$-set if, and only if, $\hat{D}$ is a subset of $C(H)$ that verifies condition (b) of Theorem 1, and the last is the case if, and only if, $\hat{B}$ is a subset of $C(H)$ that also verifies condition (b) of Theorem 1, after Theorem 11.2.1 of [40]. Consequently, in what follows we will talk about $P(B)$-sets with the meaning of $P(D)$-sets. With this terminology, if $(B_{X^*}$, weak*) is sequentially compact or if B is RSC in $(B_{X^*}$, weak*), then H is a $P(B)$-set. Recall that the class of Banach spaces having a weak* sequentially compact unit dual ball contains the weakly countably determined Banach spaces [38] and the weak Asplund Banach spaces [14, p. 239].

In the rest of this section we will present more situations in which $P(D)$-sets naturally appear. We start with the following proposition which is an easy but useful observation.

Proposition 2. *Let K be a compact space and $D \subset K$ a dense subset. Then:*

(i) *Given any subset $A \subset D$ the restriction $R_{\bar{A}}: C(K) \to C(\bar{A})$ sends $P(D)$-sets into $P(A)$-sets.*

For any uniformly bounded $t_p(D)$-compact subset H of $C(K)$ we have:

(ii) *H is a $P(D)$-set in $C(K)$ if, and only if, the set of restrictions $R_{\bar{A}}(H)$ is a $P(A)$-set in $C(\bar{A})$, for every countable subset $A \subset D$.*

(iii) *H is a $P(D)$-set in $C(K)$ if, and only if, every $t_p(D)$-separable subset F of H is a $P(D)$-set in $C(K)$.*

Proof. Parts (i) and (ii) are clear. We only prove the sufficiency in (iii): take $A \subset D$ countable and consider $R_{\bar{A}}(H) \subset C(\bar{A})$. $R_{\bar{A}}(H)$ is $t_p(A)$-compact and metrizable and so separable. Take a countable subset C of H such that $R_{\bar{A}}(C)$ is $t_p(A)$-dense in $R_{\bar{A}}(H)$. If F is the $t_p(D)$-closure of C in H, we have $R_{\bar{A}}(F) = R_{\bar{A}}(H)$. Since F is a $P(D)$-set we know that $R_{\bar{A}}(H)$ is a $P(A)$-set from which an appeal to (ii) allows us to conclude the proof. ∎

The condition of H being a $P(D)$-set is a sequential compactness requirement on $\hat{D} \subset C(H)$. A way to obtain this is have $\hat{K}$ sequentially compact in $C(H, t_p(K))$. We reach the following:

PROPOSITION 3. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$, and denote by $C(H[t_p(K)])$ the space of continuous functions on the topological space $H[t_p(K)]$. If the uniformly bounded and pointwise compact subsets of $C(H[t_p(K)])$ are sequentially compact, then H is a $P(D)$-set. In particular, if the topological space $(H, t_p(K))$ is separable, or more generally, if it has a dense countably determined subset then H is a $P(D)$-sets.*

Proof. The first part of the proposition is clear by the previous observation. The proof of the second part follows from the fact that for a separable space Y the pointwise compact subsets of $C(Y)$ are metrizable. If Y has a dense and countably determined subspace we can use the fact that the space of continuous functions $C(Y)$ is angelic for the pointwise convergence topology [29]. ∎

A combination of Propositions 2 and 3 enables us to establish that:

> If H is a $t_p(D)$-compact and uniformly bounded subset of $C(K)$ such that every $t_p(D)$-separable subset F of H is $t_p(K)$-separable ($\Leftrightarrow$ norm separable), then H is a $P(D)$-set.

As we shall see in Example E the assumption

$$t_p(D)\text{-separable} \Rightarrow t_p(K)\text{-separable}$$

is satisfied in some very important kinds of spaces. Now we are to prove that this assumption implies that H is in fact *fragmentable* which is a stronger condition than the property of H being a $P(D)$-set.

The following definition can be found in the paper by Jayne and Rogers [24]:

DEFINITION 2. A $t_p(D)$-compact subset H of $C(K)$ is said to be fragmented by the norm, $\| \|$, on $C(K)$ if for each non-empty subset C of H and for each positive ε, there exists a $t_p(D)$-open subset U of H such that $U \cap C \neq 0$ and $\| \|$-diameter $(U \cap C) \leqslant \varepsilon$.

A straightforward adaptation of the proof of Theorem 3.4 of [28] gives us the following:

PROPOSITION 4. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$, Then $(H, t_p(D))$ is fragmented by the norm of $C(K)$ if, and only if, for each countable subset A of D, the set of restrictions $R_{\bar{A}}(H) = \{f|_{\bar{A}} : f \in H\}$ is separable in the Banach space $C(\bar{A})$.*

COROLLARY 4.1. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. If $(H, t_p(D))$ is fragmented by the norm of $C(K)$, then H is a $P(D)$-set.*

Proof. By Proposition 2 it is enough to prove that for every countable subset A of D the set of restrictions $R_{\bar{A}}(H)$ is a $P(A)$-set in $C(\bar{A})$. Given such an A, $R_{\bar{A}}(H)$ is a norm separable subset of $C(\bar{A})$ by Proposition 4, and so Proposition 3 can be applied to get that $R_{\bar{A}}(H)$ is a $P(A)$-set. ∎

COROLLARY 4.2. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$ such that each $t_p(D)$-separable subset F of H is $t_p(K)$-separable ($\Leftrightarrow$ norm separable in $C(K)$). Then $(H, t_p(D))$ is fragmented by the norm of $C(K)$ and so, in particular, H is a $P(D)$-set.*

Proof. To see that $(H, t_p(D))$ is fragmented by the norm of $C(K)$ we will use Proposition 4. Take a countable subset A of D and consider the set of restrictions $R_{\bar{A}}(H)$. Proceeding as we did in the proof of Proposition 2 we obtain a $t_p(D)$-separable subset F of H such that $R_{\bar{A}}(F) = R_{\bar{A}}(H)$. By assumption F is $t_p(K)$-separable ($\Leftrightarrow$ norm separable in $C(K)$), so $R_{\bar{A}}(H)$ is a norm separable subset of $C(\bar{A})$. ∎

The fragmentable compact spaces we are dealing with here are in fact Radon–Nikodým compact: *A compact space is said to be a Radon–Nikodým compact (briefly,* RN*-compact) if it is homeomorphic to a weak*-compact subset of a dual Banach space with the Radon–Nikodým property (briefly,* RNP). As has been proved in [28, Corollary 6.7], a compact Hausdorff space is RN-compact if, and only if, it is fragmented by a lower semicontinuous metric. This is the case in our Definition 2 and so the compact H is a RN-compact. In [28, Theorem 4.3] Namioka proved that

> H is a RN-compact if, and only if, there exists a compact Hausdorff space X and a dense subset $B \subset X$ such that H is homeomorphic to a $t_p(B)$-compact subset of $C(X)$, where B has the property that for each countable subset $A \subset B$ the closure $\bar{A}$ is metrizable.

It is not difficult to establish that the counterpart to the above result for $P(D)$-sets is

> If a uniformly bounded $t_p(D)$-compact subset H of $C(K)$ is a $P(D)$-set, then there exists a compact space X and a dense subset $B \subset X$ such that H is homeomorphic to a $t_p(B)$-compact subset of $C(X)$ where B has the property that for each countable subset $A \subset B$ the closure $\bar{A}$ is a Rosenthal compact.

Recall that a Rosenthal compact is a compact Hausdorff space which is homeomorphic to a pointwise compact subset of a space of first Baire class real-valued functions on a Polish space.

COROLLARY 4.3. *Let $(X, \| \|)$ be a Banach space, B a norming subset of B_{X^*}, and H a norm bounded $\sigma(X, B)$-compact subset of X. If every $\sigma(X, B)$-separable subset F of H is weakly separable then H is $\| \|$-fragmented and so it is a $P(B)$-set.*

Proof. Apply Corollary 4.2 to $C(B_{X^*}[\text{weak}^*])$ bearing in mind that the weak topology of X is just the one induced by the topology $t_p(B_{X^*})$ of $C(B_{X^*}[\text{weak}^*])$. ∎

We will finish this section by describing an important example where the assumptions of Corollary 4.3 are satisfied.

EXAMPLE E. Given a probability space (Ω, Σ, μ) and a Banach space $(X, \| \|)$ we will denote by $L^p(\mu, X)$, $1 \leqslant p < +\infty$, the Banach space of μ-strongly measurable X-valued Bochner integrable functions $f: \Omega \to X$ normed by

$$\|f\|_p = \left(\int_\Omega \|f\|^p \, d\mu \right)^{1/p}.$$

The dual $L^p(\mu, X)^*$ of $L^p(\mu, X)$ is a space of weak* measurable functions [36]. The space $L^q(\mu, X^*)$, $1 = 1/p + 1/q$, can be identified isometrically with a subspace of $L^p(\mu, X)^*$ and it is not difficult to prove that the unit ball of $L^q(\mu, X^*)$ is a norming subset of $B_{L^p(\mu, X)^*}$. So $\sigma' = \sigma(L^p(\mu, X), L^q(\mu, X^*))$ is a Hausdorff topology which is coarser than the weak topology of $L^p(\mu, X)$; these two topologies coincide if, and only if, X^* has the RNP [13, IV. 1.1]. Now we will show that every σ'-compact and σ'-separable subset H of $L^p(\mu, X)$ is norm separable. Indeed, given a countable σ'-dense subset M of H, there exists a closed separable subspace Y of X and a set $A \in \Sigma$ such that $\mu(A)=0$ and $f(\Omega \setminus A) \subset Y$ for every $f \in M$. Let $\Sigma_0 \subset \Sigma$ be a countably generated σ-field such that every f in M is measurable with respect to Σ_0. It is easy to check that $L^p(\mu, \Sigma_0, Y)$ is a separable Banach space which can be identified with the closed subspace Z of $L^p(\mu, X)$ made up by functions which are μ-equivalent to some function in $L^p(\mu, \Sigma_0, Y)$. Because the space $L^p(\mu, X)[\sigma']$ is angelic [40, 16.5.6], given $f \in H$ there is a sequence (f_n) in M such that f is the σ'-limit of (f_n). For every $E \in \Sigma$ the sequence $(\int_E f_n \, d\mu)$ is in Y and weakly converges to $\int_E f \, d\mu$. From this fact we conclude that f is μ-equivalent to some $g \in L^p(\mu, \Sigma, Y)$. On the other hand, for every $x^* \in X^*$ the sequence $(x^* \circ f_n)$ weakly converges to $x^* \circ g$ in $L^p(\mu, \Sigma)$. Since the sequence $(x^* \circ f_n)$ lies in $L^p(\mu, \Sigma_0)$ we obtain that $x^* \circ g \in L^p(\mu, \Sigma_0)$ and the Pettis measurability criterion [13, II.1.2] gives us that $g \in L^p(\mu, \Sigma_0, Y)$. Hence $H \subset Z$ and so H is norm separable.

3. The Krein–Smulian Theorem and the Weak Radon–Nikodým Property

Our aim here is to show that the behavior of the $t_p(D)$-compact subsets of $C(K)$ which are $P(D)$-sets is like the behavior of the weak* compact subsets of a dual Banach space X^* for which X does not contain l^1. By the way we will establish a "Krein–Smulian Theorem" for $(C_p(K), t_p(D))$ that we will also use in the applications in the subsequent sections. For doing this we use some deep results by Talagrand about universal Pettis integrability.

Recall that a function f from a probability space (Ω, Σ, μ) into a Banach space X is said to be scalarly measurable if $x^* \circ f$ is measurable for every $x^* \in X^*$. The function f is said to be Pettis integrable if $x^* \circ f$ is integrable for every $x^* \in X^*$ and for every $E \in \Sigma$ there exists $x_E \in X$ such that

$$x^*(x_E) = \int_E x^* \circ f \, d\mu \qquad \text{for each} \quad x^* \in X^*$$

(in this case we write $x_E = P - \int_E f \, d\mu$). If H is a compact Hausdorff space, a function $f: H \to X$ is said to be universally scalarly measurable (resp. universally Pettis integrable) if f is scalarly measurable (resp. Pettis integrable) with respect to all Radon measures on H. With this terminology the Pettis sets H of X^* (see Example A) are those weak* compact subsets of X^* for which the canonical injection $i: H \hookrightarrow X^*$ is universally measurable. As Talagrand has shown in [40, 7.3.3] the last is the case if, and only if, the canonical injection $i: H \hookrightarrow X^*$ is universally Pettis integrable. Using these ideas we are going to point out that the above result extends to the more general situation of dealing with $P(D)$-sets.

THEOREM 5. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. If H is a $P(D)$-set then the canonical injection $i: H \hookrightarrow C(K)$ is universally Pettis integrable.*

Proof. The dual ball $B_{C(K)^*}$ is the weak* closed convex hull of the set of its extreme points $\{\pm\delta_x : x \in K\}$. Since D is dense in K we obtain that $B_{C(K)^*}$ is also the weak* closed convex hull of the set $\{\pm\delta_d : d \in D\}$ and so the set of restriction

$$Z = \{\phi|_H : \phi \in C(H)^*, \|\phi\| \leqslant 1\}$$

is the pointwise closure of co(F) in $\mathbb{R}^H$ where $F = \{\pm\delta_d|_H : d \in D\}$. The set F is contained $C(H, t_p(D))$ and verifies condition (b) of Theorem 1 and so for each Radon probability μ on $(H, t_p(D))$ it is a stable subset in $M_\mu(H)$. The convex hull co(F) is a stable set in $M_\mu(H)$ after Theorem 11.2.1 of [40]. Since the pointwise closure of a stable subset of $M_\mu(H)$ is stable [40, p. 98], we get that Z is a stable subset of $M_\mu(H)$. In the language of Definition 6.1.1 of [40] that means that the canonical injection $i: H \hookrightarrow C(K)$ is properly measurable with respect to any Radon probability on $(H, t_p(D))$. Applying now Theorem 6.1.2 of [40] we conclude that i is universally Pettis integrable. ∎

Combining Theorem 5 with Theorem 1 we get:

COROLLARY 5.1. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. Then the following are equivalent:*

(i) *H is a $P(D)$-set.*

(ii) *For every element $\theta \in C(K)^*$ the restriction $\theta|_H$ belongs to $B_r(H)$.*

(iii) *For every element $\theta \in C(K)^*$ the restriction $\theta|_H$ is universally measurable on H.*

(iv) *Every Radon probability μ on $(H, t_p(D))$ has a barycentre in $C(K)$, that is, the functions $\{\phi|_H : \phi \in C(K)^*\}$ are μ-measurable and there exists $f_\mu \in C(K)$ such that*

$$\phi(f_\mu) = \int_H \phi(h)\, d\mu(h)$$

for every $\phi \in C(K)^*$.

COROLLARY 5.2. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. If H is a $P(D)$-set then its $t_p(D)$-closed convex hull $\overline{\text{co}(H)}^{t_p(D)}$ is $t_p(D)$-compact.*

Proof. After Corollary 5.1 every Radon probability μ on H has a barycentre f_μ in $C(K)$. The map $\mu \to f_\mu$ from the weak* compact subset $P(H)$ of all Radon probabilities on H into the Banach space $C(K)$ is weak* $-t_p(D)$-continuous and its range contains $\overline{\text{co}(H)}^{t_p(D)}$. ∎

For the locally convex spaces $(C(K), t_p(D))$ for which all the $t_p(D)$-compact subsets of $C(K)$ are $P(D)$-sets, Corollary 5.2 is a Krein–Smulian Theorem.

COROLLARY 5.3. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$.*

(i) *If H is a $P(D)$-set, then $\overline{\text{co}(H)}^{t_p(D)} = \overline{\text{co}(H)}^{\|\,\|}$.*

(ii) *H is a $P(D)$-set if, and only if, $\overline{\text{co}(H)}^{t_p(D)}$ is a $P(D)$-set.*

(iii) *H is fragmented by the norm of $C(K)$ if, and only if, $\overline{\text{co}(H)}^{t_p(D)}$ is fragmented too.*

Proof. (i) We only have to prove that $\overline{\text{co}(H)}^{t_p(D)} \subset \overline{\text{co}(H)}^{\|\,\|}$. Assume that this is not the case. Take $f \in \overline{\text{co}(H)}^{t_p(D)} \setminus \overline{\text{co}(H)}^{\|\,\|}$ and use the Hahn–Banach Theorem to find a $\phi_0 \in C(K)^*$ so that

$$\sup_{g \in H} \phi_0(g) < \phi_0(f). \tag{1}$$

Let now μ be a Radon probability on H representing f in $(C(K), t_p(D))$, that is, satisfying

$$f(d) = \int_H g(d)\, d\mu(g)$$

for every $d \in D$ [30, Proposition 1.2]. By the Corollary 5.1 we have that ϕ_0 satisfies the formula

$$\phi_0(f) = \int_H \phi_0(g)\, d\mu(g)$$

which clearly contradicts (1).

(ii) Being one implication obvious, it remains to prove that $\overline{\text{co}(H)}^{t_p(D)}$ is a $P(D)$-set when H is a $P(D)$-set. If the last is the case, it is clear that the convex hull, $\text{co}(H)$, is a $P(D)$-set too, and so given a sequence (d_n) in D there exists a subsequence (d_{n_j}) such that $(h(d_{n_j}))$ converges for each $h \in \text{co}(H)$. If we take h in $\overline{\text{co}(H)}^{t_p(D)} = \overline{\text{co}(H)}^{\|\,\|}$ there exists a sequence (h_m) in $\text{co}(H)$ such that $\lim_m h_m = h$ uniformly on K, from which we obtain the existence of the limit

$$\lim_j \lim_m h_m(d_{n_j}) = \lim_j h(d_{n_j})$$

that finishes the proof.

(iii) We only have to prove that if H is norm fragmented then $\overline{\text{co}(H)}^{t_p(D)}$ is norm fragmented. To see this we will use Proposition 4. If A is a countable subset of D, then the set of restrictions $R_{\bar{A}}(H)$ is separable in the Banach space $C(\bar{A})$ by Proposition 4. On the other hand as H is a $P(D)$-set by Corollary 4.1 we can use the formula obtained in (i) together with the fact that the operator $R_{\bar{A}}\colon C(K) \to C(\bar{A})$ is linear and bounded to deduce that

$$R_{\bar{A}}(\overline{\text{co}(H)}^{t_p(D)}) = R_{\bar{A}}(\overline{\text{co}(H)}^{\|\ \|}) \subset \overline{R_{\bar{A}}\,\text{co}(H)}^{\|\ \|_{C(\bar{A})}} = \overline{\text{co}(R_{\bar{A}}(H))}^{\|\ \|_{C(\bar{A})}}$$

from which it follows that $R_{\bar{A}}(\overline{\text{co}(H)}^{t_p(D)})$ is separable in the Banach space $C(\bar{A})$ and thus a new application of Proposition 4 allows us to conclude that $\overline{\text{co}(H)}^{t_p(D)}$ is norm fragmented. ∎

Let us remark that Namioka proved in [28, Theorems 2.3, 2.5] that if H is a norm fragmented weak*-compact subset of a dual Banach space X^*, then

$$\overline{\text{co}(H)}^{\text{weak}^*} = \overline{\text{co}(H)}^{\|\ \|}$$

being this closed convex hull fragmented too. After Corollary 4.1, Namioka's results are particular cases of properties (i) and (iii) in the above corollary. Namioka's proof of the fragmentability of $\overline{\text{co}(H)}^{\text{weak}^*}$ by the norm of X^* is based upon the weak* compactness of the unit ball (B_{X^*}, weak^*) which is an argument that can not be used in the proof of the general situation of Corollary 5.3.

The above result enables us to give another condition equivalent to the ones in Corollary 5.1.

PROPOSITION 6. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. The following are equivalent*:

(i) *H is a $P(D)$-set.*

(ii) *For every $\phi \in C(K)^*$ and for every Radon probability μ on $(H, t_p(D))$ there exists a sequence (ϕ_n) in $C(H, t_p(D))$ such that*

$$\phi_n \to \phi|_H$$

μ-almost everywhere.

Proof. The implication (ii)$\Rightarrow$(i) follows from Theorem 1. In order to prove that (i)$\Rightarrow$(ii) it is not restrictive to assume that H is convex after Corollary 5.3. With the notation of the proof of Theorem 5, if ϕ belongs to $B_{C(K)^*}$ then the restriction $\phi=\phi|_H$ belongs to $\overline{\text{co}(F)}^{\mathbb{R}^H}$ and so there exists a net $\{\varphi_\alpha : \alpha\in A, \geqslant\}\subset \text{co}(F)$ such that $\varphi_\alpha(h)\to\phi(h)$ for every $h\in H$. Given any Radon probability μ on $(H, t_p(D))$ the set co(F) is a uniformly integrable subset of $L^1(\mu)$ because co(F) is a uniformly bounded subset of the space $C(H, t_p(D))$. For the net $\{\varphi_\alpha : \alpha\in A, \geqslant\}$ there exists a subnet $\{\varphi_\lambda : \lambda\in L, \geqslant\}$ and a point $\varphi\in L^1(\mu)$ such that $\{\varphi_\lambda : \lambda\in L, \geqslant\}$ weakly converges to φ in $L^1(\mu)$. Now we prove that $\varphi=\phi|_H$ μ-almost everywhere. To see this we will show that $\int_E \varphi\, d\mu=\int_E \phi\, d\mu$ for every Borel subset $E\subset H$. Indeed, given the Borel subset $E\subset H$ we have

$$\int_E \varphi\, d\mu=\lim_{\lambda\in L}\int_E \varphi_\lambda\, d\mu=\lim_{\lambda\in L}\int \varphi_\lambda\, d\mu_E=\lim_{\lambda\in L}\varphi_\lambda(f_E)$$

$$=\phi(f_E)=\int \phi\, d\mu_E=\int_E \phi\, d\mu,$$

where μ_E is the Radon measure given by $\mu_E(A)=\mu(E\cap A)$ for every Borel $A\subset H$, and $f_E\in H$ is the barycentre of μ_E which is ensured by (iv) of Corollary 5.1. The next step in the proof is to change the net $\{\varphi_\lambda : \lambda\in L, \geqslant\}$ for a sequence: φ is in the weak closure of the relatively weakly compact subset co(F) of $L^1(\mu)$. Because of the angelic character of the weak topology of $L^1(\mu)$ we can find a sequence $(\phi_n)_{n\in\mathbb{N}}$ in co(F) such that $(\phi_n)_{n\in\mathbb{N}}$ weakly converges to φ in $L^1(\mu)$. As the set co(F) satisfies condition (b) of Theorem 1 there exists a subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$ which is pointwise convergent in $\mathbb{R}^H$. If we define $\psi(h)=\lim_k \phi_{n_k}(h)$, $h\in H$, then $(\phi_{n_k})_{k\in\mathbb{N}}$ converges to ψ in the norm of $L^1(\mu)$ and so we reach the equality $\psi=\varphi$. Since $\varphi=\phi$ μ-almost everywhere the proof is finished. ∎

Note that the equivalences stated in Corollary 5.1 and the above proposition are also equivalent to the following ones:

(vi) For every $x\in K$, $\hat{x}\in B_r(H)$.

(vii) For every $x\in K$, $\hat{x}\in M_\mu(H)$ for each Radon probability μ on H.

(viii) For every Radon probability μ on H the functions $\{\hat{x} : x\in K\}$ are μ-measurable and there exists $f_\mu\in C(K)$ such that

$$f_\mu(x)=\int_H h(x)\, d\mu(h)$$

for every $x\in K$.

(ix) For every $x\in K$ and every Radon probability μ on H, there exists a sequence $(d_n)_{n\in\mathbb{N}}$ in D such that $\hat{d}_n\to\hat{x}$ μ-almost everywhere.

When we deal with a Banach space X if we specialize the above equivalences for $K = B_{X^{**}}$, $H = B_{X^*}$, and $D = B_X$ we obtain some of the different charaterizations of when X does not contain l^1.

Now we are ready to characterize the $P(D)$-sets through the weak Radon–Nikodým property. Recall that a closed bounded convex subset H of a Banach space X is said to have the weak Radon–Nikodým property (resp. Radon–Nikodým property) if for every complete probability space (Ω, Σ, μ) and every vector measure $m: \Sigma \to X$ such that the average range of m

$$AR(m) = \left\{ \frac{m(E)}{\mu(E)} : E \in \Sigma,\ \mu(E) > 0 \right\}$$

is contained in H, there exists a Pettis (resp. Bochner) integrable function $f: \Omega \to X$ such that

$$m(A) = \int_A f\,d\mu$$

for every A in Σ [7, 40]. We will write WRNP (resp. RNP) as the abbreviation of "weak Radon–Nikodým property" (resp. "Radon–Nikodým property").

THEOREM 7. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. The following are equivalent*:

(i) *H is a $P(D)$-set.*

(ii) *$\overline{\mathrm{co}(H)}^{t_p(D)}$ is a uniformly bounded, $t_p(D)$-compact subset of $C(K)$ which has the WRNP.*

Proof. Let us observe that Corollaries 5.2 and 5.3 give us that H is a uniformly bounded, $t_p(D)$-compact, and $P(D)$-subset of $C(K)$ if, and only if, $\overline{\mathrm{co}(H)}^{t_p(D)}$ has the same properties. So for proving the theorem we can restrict ourselves to the case in which H is convex, that is, to the case $H = \overline{\mathrm{co}(H)}^{t_p(D)}$.

(i) ⇒ (ii). Assume H is a convex $P(D)$-set. Let (Ω, Σ, μ) be a complete probability space and $m: \Sigma \to C(K)$ a vector measure such that $AR(m)$ is contained in H. Take a lifting ρ on $L^\infty(\mu)$ [15, Appendix G]. By standard arguments, similar to the ones used to establish the equivalence between the WRNP for a dual Banach space X^* and the fact that X does not contain l^1 [23], a function $f: \Omega \to H$ can be obtained which is measurable for the Baire σ-field of $(H, t_p(D))$ and such that

(α) $\rho(x' \circ f) = x' \circ f$

(β) $x'(m(E)) = \int_E x' \circ f \, d\mu$

for every $x' \in (C(K), t_p(D))'$. Now, if $\rho_H(f): \Omega \to H$ is the abstract lifting of f [6, Sect. 2] defined by the formula

$$\rho(h \circ f)(\omega) = h \circ \rho_H(f)(\omega) \qquad \text{for each} \quad h \in C(H),\ \omega \in \Omega,$$

we have the equality $\rho_H(f) = f$ because of (α). So, Theorem 2.1 of [6] tells us that f is measurable with respect to the Borel σ-field of $(H, t_p(D))$ and that the Borel measure image $\nu = \mu f^{-1}$ is a Radon measure. As H is a $P(D)$-set we conclude that $f: \Omega \to C(K)$ is scalarly μ-measurable and that the inclusion mapping $i: H \hookrightarrow C(K)$ is universally Pettis integrable (Theorem 5) from which it follows that f is μ-integrable and $m(E) = P - \int_E f \, d\mu$.

(ii) $\Rightarrow$ (i). Assume H is a convex set with the WRNP. To see that H is a $P(D)$-set it is enough to prove, after Proposition 2, that for every countable subset A of D the restriction map $R_{\bar{A}}: C(K) \to C(\bar{A})$ sends H onto a $P(A)$-set of $C(\bar{A})$. Take a Radon probability ν on the $t_p(A)$-compact $R_{\bar{A}}(H)$. Because $(R_{\bar{A}}(H), t_p(A))$ is a polish space and $R_{\bar{A}}$ is onto and $t_p(D) - t_p(A)$-continuous, there exists a Radon probability μ on $(H, t_p(D))$ such that ν is the image measure of μ under $R_{\bar{A}}|_H$ [37, p. 39]. For each Borel subset B of $(H, t_p(D))$, let $m(B)$ be the element of $C(K)$ defined by

$$m(B)(d) = \int_B h(d) \, d\mu(h).$$

It is clear that m is a μ-continuous vector measure that satisfies $m(B)/\mu(B) \in H$ for every Borel B of $(H, t_p(D))$ with $\mu(B) > 0$. Since H has the WRNP there exists a scalarly measurable function $f: H \to C(K)$ such that $f(H) \subset H$ and

$$m(B) = P - \int_B f \, d\mu.$$

If we put $g = R_{\bar{A}} \circ f$ then $R_{\bar{A}}(m(B)) = \int_B g \, d\mu$. For each $d \in A$ we have

$$\int_B \hat{d}(h) \, d\mu(h) = \int_B h(d) \, d\mu(h) = m(B)(d) = R_{\bar{A}}(m(B))(d) = \hat{d}(R_{\bar{A}}(m(B)))$$
$$= \int_B \hat{d}(g(h)) \, d\mu(h)$$

for every Borel subset B of $(H, t_p(D))$. Then we obtain that $\hat{d}(h) = \hat{d}(g(h))$ μ-almost everywhere h and for $d \in A$. Since A is countable there exists a μ-null set $N \subset H$ such that $\hat{d}(h) = \hat{d}(g(h))$ for every $d \in A$ and every h in

$H \setminus N$. So, $R_{\bar{A}}|_H : H \to C(\bar{A})$ coincides with g on $H \setminus N$. Since g is scalarly measurable we conclude that $R_{\bar{A}}|_H$ is also scalarly measurable and so for any $\phi \in C(\bar{A})^*$ the composition $\phi \circ R_{\bar{A}}|_H$ is μ-measurable. Now, Theorem I.5.9 of [37] allows us to conclude that $\phi|_{R_{\bar{A}}(H)}$ is ν-measurable and thus applying Corollary 5.1 we obtain that $R_{\bar{A}}(H)$ is a $P(A)$-set in $C(\bar{A})$ and the proof is complete. ∎

It should be noted that, as a consequence of the former theorem, if D_1 and D_2 are dense subspaces of K and H is a uniformly bounded subset of $C(K)$ which is compact for $t_p(D_1)$ and $t_p(D_2)$, then H is a $P(D_1)$-set if, and only if, H is a $P(D_2)$-set. In particular, dealing with a Banach space X, if $B \subset B_X$ is a fixed norming subset then X does not contain l^1 if, and only if, B does not contain an l^1-sequence.

We finish this section highlighting the extension for $P(D)$-sets of Haydon's well known result [22], saying that any convex weak* compact subset of a dual Banach space X^* is the norm closed convex hull of its extreme points when X does not contain l^1. See [33] for a localized version of Haydon's result.

THEOREM 8. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. If H is a convex $P(D)$-set, then H is the norm closed convex hull of its extreme points.*

Proof. The proof of this theorem is completely analogous to the proof of the Haydon's result given in 6.12 of [15]. The only point that must be kept in mind is that the elements of the bidual x^{**} in the aforesaid proof of [15] have to be replaced, in our situation here, by the elements $\phi \in C(K)^*$ that now are in $B_r(H)$ after Corollary 5.1. ∎

4. Fragmentability and the Radon–Nikodým Property

Given a uniformly bounded $t_p(D)$-compact convex subset H of $C(K)$ we already know that the following implications hold between the different concepts used in this paper:

$$\begin{array}{ccc} H \text{ is a } P(D)\text{-set} & \Leftrightarrow & H \text{ has the WRNP} \\ \Uparrow & & \Uparrow \\ H \text{ is fragmentable} & & H \text{ has the RNP} \end{array}$$

In this section we will complete the picture above proving that "fragmentable" is equivalent to the "RNP." This result is an extension of the well known characterization of the weak* compact subsets with the RNP [7, 4.2.13]. We also add reasonable conditions to the statement "H is a

$P(D)$-set" in order to get that "H is fragmentable." Our tools for this study come from measures theory. We start with the following observation: if H is a $P(D)$-set and μ is a Radon probability on $(H; t_pD))$ the natural inclusion $i: H \subsetneq C(K)$ is μ-scalarly measurable (Theorem 5), and so by a result of [16], i is μ-measurable with respect to the Baire σ-field of $(C(K)$, weak). Thus we can consider the Baire measure image $\lambda = \mu i^{-1}$ defined on Baire($C(K)$, weak). More precisely, looking at μ as a Radon measure on Borel($C(K), t_p(D)$) supported by H, if $\tilde{\mu}$ is the associated complete measure, then λ is the restriction of $\tilde{\mu}$ to Baire($C(K)$, weak). For the concepts about topological measures we refer to [16, 39].

LEMMA 9. *Let H be a uniformly bounded $t_p(D)$-compact and $P(D)$-subset of $C(K)$. Given a Radon probability μ on $(H, t_p(D))$ let us denote by λ be Baire image measure on $(C(K)$, weak$)$ under the inclusion $i: H \subsetneq C(K)$. If λ is tight and $\bar{\lambda}$ is its canonical extension to the Borel σ-field of $(C(K)$, weak$)$, then*

$$\bar{\lambda}(F) = \mu(F)$$

for every $t_p(D)$-compact subset F of H.

Proof. We can consider μ as a Borel measure on the Borel σ-field of $(C(K), t_p(D))$. Let $\mathscr{Z}'$ (resp. $\mathscr{Z}$) be the family of zero subsets of $(C(K), t_p(D))$ (resp. $(C(K)$, weak$)$). If $V \subset C(K)$ is a $t_p(D)$-open subset, then

$$\mu(V) = \sup\{\mu(Z) : V \supset Z \in \mathscr{Z}'\} = \sup\{\lambda(Z) : V \supset Z \in \mathscr{Z}'\} \leqslant \bar{\lambda}(V)$$

$$\bar{\lambda}(V) = \sup\{\lambda(Z) : V \supset Z \in \mathscr{Z}\} = \sup\{\tilde{\mu}(Z) : V \supset Z \in \mathscr{Z}\} \leqslant \mu(V).$$

Since $\mu(V) = \bar{\lambda}(V)$ for each open subset V of $(C(K), t_p(D))$ then $\bar{\lambda}(F) = \mu(F)$ for each $t_p(D)$-compact subset F of H. ∎

THEOREM 10. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. Then H is fragmented by the norm of $C(K)$, if, and only if, $\overline{\mathrm{co}(H)}^{t_p(D)}$ has the RNP.*

Proof. Let us observe that Corollaries 5.2 and 5.3 give us that H is uniformly bounded, $t_p(D)$-compact, and fragmented by the norm of $C(K)$ if, and only if, $\overline{\mathrm{co}(H)}^{t_p(D)}$ has the same properties. So for proving the theorem we can restrict ourselves to the case in which H is convex, that is, to the case $H = \overline{\mathrm{co}(H)}^{t_p(D)}$.

Assume H is convex and has the RNP. To prove that H is norm fragmented we will show that for any Radon probability μ on $(H, t_p(D))$ there is a norm compact subset C of H with $\mu(C) > 0$ [25]. By

Corollary 4.1, H is a $P(D)$-set and so the inclusion $i: H \subsetneq C(K)$ is universally Pettis integrable. In particular, defining

$$m(B) = P - \int_B i \, d\mu$$

for every Borel subset B of $(H, t_p(D))$, we obtain a μ-continuous vector measure that satisfies $AR(m) \subset H$. By hypothesis m has a Bochner derivative with respect to μ and thus i is scalarly equivalent to a Bochner measurable function. By [16], the Baire image measure λ of μ under i is tight and Lemma 9 gives us that $\bar{\lambda}(H) = \mu(H) = 1$, from which we can be sure of the existence of a norm compact subset C of H such that $\mu(C) = \bar{\lambda}(C) > 1/2 > 0$.

Conversely, assume $(H, t_p(D))$ is fragmented by the norm of $C(K)$. By Corollary 4.1 and Theorem 7 the set H has the WRNP. Given a complete probability space (Ω, Σ, μ) and a vector measure $m: \Sigma \to C(K)$ such that $AR(m) \subset H$, for every $B \in \Sigma$, there exists a Pettis integrable function $f: \Omega \to H$ such that $m(E) = \int_E f \, d\mu$ for each $E \in \Sigma$. Bearing in mind the proof of Theorem 7 we can assume that f is measurable with respect to the Borel σ-field of $(C(K), t_p(D))$ and that the Borel image ν of μ under f is a Radon probability on $(H, t_p(D))$. Theorem 4.1 of [25] tells us that ν is regular with respect to norm compact subsets of H. Then there exists a set $F \subset H$ which is a countable union of norm compact subsets of H such that $\nu(H) = \nu(F)$. This set F is an $\mathscr{F}_\sigma$-subset of $(H, t_p(D))$ and thus $N = f^{-1}(H \setminus F)$ belongs to Σ and $\mu(N) = 0$. Since $f(\Omega \setminus N) \subset F$ and F is norm separable, we obtain that f is Bochner-measurable by the Pettis measurability theorem. ∎

Similar arguments to the ones used in the theorem above lead us to the following

PROPOSITION 11. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. If H is a $P(D)$-set and $(H, t_p(K))$ is Lindelöf, then H is fragmented by the norm of $C(K)$.*

Proof. Take a Radon probability μ on $(H, t_p(D))$ and consider its Baire image λ on $(C(K), t_p(K))$ under the natural inclusion i. Since $(H, t_p(K))$ is Lindelöf, the measure λ is τ-smooth by [42, Corollary 5, p. 175]. Lemma 9 allows us to establish the equality $\bar{\lambda}(F) = \mu(F)$ for each $t_p(D)$-compact subset F of H, where $\bar{\lambda}$ is the canonical extension of λ to the Borel σ-field of $(C(K), t_p(K))$. By [10, Theorem 4.5] the measure $\bar{\lambda}$ is a norm Radon measure on $C(K)$ and the proof finishes with the same arguments we gave in the first part of the proof of Theorem 10. ∎

Recall that a compact space K is said to be Corson compact if it is (homeomorphic to) a compact subset of $\mathbb{R}^{\Gamma}$ (for some set Γ) such that for every $x = (x(\gamma))$ in K the set $\{\gamma : x(\gamma) \neq 0\}$ is countable.

COROLLARY 11.1 *Let K be a Corson compact space and D a dense subset. Every uniformly bounded $t_p(D)$-compact subset H of $C(K)$ is fragmented by the norm of $C(K)$.*

Proof. A result of Alster and Pol [1] states that $(C(K), t_p(K))$ is Lindelöf. On the other hand, K is sequentially compact and so H is always a $P(D)$-set. Hence Proposition 11 can be applied and the proof is concluded. ∎

COROLLARY 11.2. *Let K be a Corson compact space and D a dense subset. Every $t_p(D)$-compact subset H of $C(K)$ is a Radon–Nikodým compact. In particular H is $t_p(D)$-sequentially compact.*

Proof. Without loss of generality we can assume that H is also uniformly bounded. In this situation Corollary 11.1 states that H is fragmented by the norm, $\| \|$, of $C(K)$. As $\| \|$ is $t_p(D)$-lower semi-continuous we obtain that H is a Radon–Nikodým compact after [28, Corollary 6.7]. The fact that H is $t_p(D)$-sequentially compact follows from [28, Corollary 5.4]. ∎

We say that a Banach space X has the separable complementation property (briefly, SCP) if for every separable subspace Z in X there exists a separable subspace Y containing Z and complemented in X. Examples of spaces having the SCP are the spaces $C(K)$ for K Corson compact [41], the Banach lattices not containing c_0 [19], and the preduals of Von Neumann algebras [20].

PROPOSITION 12. *Let H be a uniformly bounded $t_p(D)$-compact subset of $C(K)$. If $C(K)$ has the separable complementation property, then H is a $P(D)$-set if, and only if, H is fragmented by the norm of $C(K)$.*

Proof. H fragmentable implying that H is a $P(D)$-set follows from Corollary 4.1. Conversely, let us suppose that H is a $P(D)$-set. By Theorem 7, $\overline{\text{co}(H)}^{t_p(D)}$ has the WRNP which implies that $\overline{\text{co}(H)}^{t_p(D)}$ has the WRNP because in spaces with the SCP every Pettis integrable function is scalarly equivalent to a Bochner measurable function [7, 27]. Now apply Theorem 10 to finish the proof. ∎

Let us observe that Corollary 11.1 can also be obtained as a consequence of Proposition 12.

5. Applications

Our first task here will be to prove Theorems A and B and their Corollary C enunciated in Section 1.

Proof of Theorem A. Assume that (B_{X^*}, weak^*) is sequentially compact. Every norm bounded $\sigma(X, B)$-compact subset H of X is a $P(B)$-set after Example D and so the validity of (i), (ii), and (iii) follows from Corollary 5.2 and Theorems 8 and 7, respectively.

If we assume that (B_{X^*}, weak^*) is sequentially compact and X has the separable complementation property (resp. (B_{X^*}, weak^*) sequentially compact and X weakly Lindelöf) the validity of property (iv) is obtained from Proposition 12 (resp. Proposition 11), (v) is a consequence of (iv) together with [28, Corollary 6.7], (vi) is a result of combining (iv) and Corollary 5.4 of [28], and (vii) follows from Theorem 10.

If X is weakly countably determined the unit ball (B_X, weak) is countably determined and so Čech-analytic [25]. In this situation a result of [25] states that the σ-fragmentability of (B_X, weak) is equivalent to the norm fragmentability of the $\sigma(X, B)$-compact subsets of B_X, and the last is the case because (B_{X^*}, weak^*) is a Corson compact and we can apply (iv). ∎

As we wrote in the Introduction, the assumption (B_{X^*}, weak^*) sequentially compact in Theorem A can be replaced by either:

(1) B is conditionally sequentially compact in (B_{X^*}, weak^*), or

(2) $\text{co}(B)$ is sequentially dense in (B_{X^*}, weak^*)

keeping the validity of the statements of the theorem. This follows from the considerations we have done in the Examples B, C, and D in Section 2.

Proof of Theorem B. (i) $\Rightarrow$ (ii). This is clear. (ii) $\Rightarrow$ (iii). This follows from the fact that $C(H, [\text{weak}])$ is angelic for the pointwise convergence topology (see Proposition 3). The implication (iii) $\Rightarrow$ (iv) is a consequence of Corollary 5.2. (iv) $\Rightarrow$ (i). This is obtained using the Theorem on page 99 of [17]. (i) $\Rightarrow$ (v). This is obvious. (v) $\Rightarrow$ (vi). This follows from Corollary 4.3. (vi) $\Rightarrow$ (vii). This follows from [28, Corollary 5.4] because in assuming that (vi) holds we know that H is a Radon–Nikodým compact. (vii) $\Rightarrow$ (i). This follows from [17, Corollary 4, p. 101]. ∎

Proof of Corollary C. It is enough to use Theorem B bearing in mind that condition (iii) there is nothing else than saying that H is a $P(B)$-set. Now the proof can be concluded using the considerations we did in the examples of Section 2. ∎

A natural context for applying the techniques developed in this paper are the spaces $L^p(\mu, X)$ provided with the topology $\sigma' = \sigma(L^p(\mu, X), L^q(\mu, X^*))$, $1 \leqslant p < \infty$, and $1/p + 1/q = 1$, that we have already found in Example E.

THEOREM 13. *Let (Ω, Σ, μ) be a probability space, $1 \leqslant p < \infty$, and X a Banach space. For any σ'-compact subset H of $L^p(\mu, X)$ we have the following properties:*

(i) *(H, σ') is an Eberlein compact;*

(ii) *If $L = \overline{\text{span}(H)}^{\sigma'}$, then there exists a set Γ and a linear bounded one-to-one operator $T: L \to c_0(\Gamma)$ which is σ'-to-pointwise continuous;*

(iii) *The closed convex hull $\overline{\text{co}(H)}^{\sigma'}$ is σ'-compact;*

(iv) *$\overline{\text{co}(H)}^{\sigma'} = \overline{\text{co}(H)}^{\|\,\|_p}$;*

(v) *H is fragmented by the norm of $L^p(\mu, X)$;*

(vi) *If H is convex, then H has the Radon–Nikodým property.*

Proof. In Example E of Section 2 we have proved that every σ'-compact and σ'-separable subset of $L^p(\mu, X)$ is norm separable. On the other hand, it is not difficult to prove that the σ'-compact subsets of $L^p(\mu, X)$ are norm bounded. So, property (v) follows from Corollary 4.3, (vi) is a consequence of Theorem 10, (iii) is obtained combining Corollary 4.1 and Theorem 5, and (iv) follows from Corollary 5.2. To finish the proof it remains to prove (i) and (ii). The proof of (i) will provide us with the tools for proving (ii). Consider the unit ball $B_{L^q(\mu)}$ endowed with the weak* topology $\sigma(L^q(\mu), L^p(\mu))$. The linear map

$$\phi: L^p(\mu, X) \to \mathbb{R}^{B_{L^q(\mu)} \times B_{X^*}}$$

given by the formula

$$\phi(f)(g, x^*) = \int_\Omega (x^* \circ g) \cdot f \, d\mu$$

for $f \in L^p(\mu, X)$, $g \in L^q(\mu)$, and $x^* \in B_{X^*}$, is injective and its range is contained in the space of continuous functions $C(B_{L^q(\mu)} \times B_{X^*})$ [13, II.3.8]. It is clear that the map $\phi: L^p(\mu, X) \to C(B_{L^q(\mu)} \times B_{X^*})$ is σ'-to-pointwise continuous and so we conclude the proof of (i). Taking advantage of our knowledge of the proof of (i), the proof of (ii) is a consequence of the lemma below. ∎

LEMMA 14. *Let K be a compact space, H a $t_p(K)$-compact subset of $C_p(K)$, and $Y = \overline{\text{span}\, H}^{t_p(K)}$. Then there is a set Γ and bounded linear one-to-one operator*

$$\Phi: Y \to c_0(\Gamma)$$

which is pointwise-to-pointwise continuous.

Proof. Consider Y with the topology induced by the supremum norm of $C(K)$. The set

$$K^* = \{\delta_x|_Y : x \in K\}$$

is a $\sigma(Y^*, Y)$-compact subset of Y^*. The mapping $\psi: H \times K^* \to \mathbb{R}$ given by

$$\psi(f, \delta_x|_Y) = f(x)$$

is separately continuous and distinguishes the points of H and K^*. Therefore K^* is an Eberlein compact which is norming for the Banach space Y. The natural restriction map

$$R: Y \to C(K^*),$$

given by $R(f)(\delta_x|_Y) = f(x)$, is injective and pointwise-to-pointwise continuous. Now, $C(K^*)$ can be embedded to some $c_0(\Gamma)$ through a linear pointwise-to-pointwise continuous operator [2], and so the proof is concluded. ∎

A few comments about Theorem 13 are needed: for $p = 1$ statement (ii) is the main result of [5] where the proof given is very different than the one given here. Part (i) can be found in [40, 16-5-6]; we have put it here for a good understanding of the proof of (ii). Part (iii) was stated in [4] using a characterization of the σ'-compact subsets of $L^p(\mu, X)$. Because the closed bounded convex subsets with the RNP are the norm closed convex hull of the set of its strongly exposed points, statement (vi) is stronger than (iv). In [11] the results of this theorem are used to obtain some characterization of the RNP for dual Banach spaces. It should be noted that our proof of these properties of $L^p(\mu, X)[\sigma']$ can be used for other more general spaces.

Our results about boundaries can be used to give a classical characterization of the weakly compact sets in spaces of countably additive measures. Let (Ω, Σ) be a measurable space and denote by $\text{ca}(\Omega, \Sigma)$ the Banach space of all countably additive measures on Σ endowed with the variation norm, $\|\mu\| = |\mu|(\Omega)$. There are well known and useful characterizations of the (relatively) weakly compact subsets of $\text{ca}(\Omega, \Sigma)$ [14]. Being rather difficult to deal with the dual $\text{ca}(\Omega, \Sigma)^*$, it seems to be reasonable to try studying properties of the weak topology of $\mathrm{ca}(\Omega, \Sigma)$ through properties of certain topologies of the kind $\sigma(\mathrm{ca}(\Omega, \Sigma), B)$ for adequate norming subsets B. Taking $B = B_{S(\Sigma)}$, the unit ball of the spaces of Σ-simple functions on Ω we arrive at the following:

THEOREM 15. (Gänssler [18]). *Given a bounded subset H of* $\mathrm{ca}(\Omega, \Sigma)$ *the following are equivalent*:

(i) *H is $\sigma(\mathrm{co}(\Omega, \Sigma), S(\Sigma))$-relatively compact in* $\mathrm{ca}(\Omega, \Sigma)$;

(ii) *H is relatively weakly compact in* $\mathrm{ca}(\Omega, \Sigma)$.

Proof. The unit ball $B_{S(\Sigma)}$ is a boundary for the unit ball of $\mathrm{ca}(\Omega, \Sigma)^*$ after the Hahn decomposition theorem for signed measures. If we prove that the bounded $\sigma(\mathrm{ca}(\Omega, \Sigma), S(\Sigma))$-compact separable subsets are norm separable, then the arguments in Theorem B will give us the equivalence between (i), (ii). Let F be a bounded $\sigma(\mathrm{ca}(\Omega, \Sigma), S(\Sigma))$-compact separable subset of $\mathrm{ca}(\Omega, \Sigma)$. Take $\{\mu_n : n \in \mathbb{N}\}$ a countable dense subset of F and define

$$\lambda(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_n(E)|$$

for every E in Σ. λ is a positive measure of $\mathrm{ca}(\Omega, \Sigma)$ and every μ in F is λ-continuous. Given μ in F let f_μ be the Radon–Nikodým derivative of μ with respect to λ and denote by $\tilde{F}$ the subset of $L^1(\lambda)$ defined by $\tilde{F} = \{f_\mu : \mu \in F\}$. The map $\mu \to f_\mu$ from $(F, \sigma(\mathrm{ca}(\Omega, \Sigma), S(\Sigma)))$ onto $(\tilde{F}, \sigma(L^1(\lambda), S(\Sigma)))$ is continuous. So, $\tilde{F}$ is a weakly compact [17, p. 108], norm separable subset of $L^1(\lambda)$ and thus F is norm separable in $\mathrm{ca}(\Omega, \Sigma)$. ∎

The last application we give here improves a result obtained independently by E. Saab [34, Theorem 9] and M. Talagrand [39, Theorem 4.3], saying that if H is a weak* compact convex subset of a dual Banach space X^* and the set of extreme points of H is contained in a weakly K-analytic subset W, then H is the norm closed convex hull of its extreme points.

THEOREM 16. *Let $(X, \| \|)$ be a Banach space and B a norming subset of $B_{X^{**}}$. Let H be a $\sigma(X, B)$-compact convex subset of X and let W be a subset of H containing the set of the extreme points* $\mathrm{Ext}(H)$ *of H. If the compact subsets of the space of continuous functions $C(W[\sigma(X, X^*)])$ are sequentially compact, then,*

(i) *H has the weak Radon–Nikodým property.*

(ii) $H = \overline{\mathrm{co}(\mathrm{Ext}(H))}^{\| \|}$.

The last applies for $W[\sigma(X, X^*)]$ countably determined and in this case we also have that

(iii) *H is weakly countably determined and has the Radon–Nikodým property.*

Proof. To prove (i) and (ii) it is enough to prove that H is a $P(B)$-set and use Theorems 7 and 8. B_{X^*} is a compact subset of $C_p(H[\sigma(X, X^*)])$ and so $\{x^*|_W : x^* \in B_{X^*}\}$ is a compact subset of $C_p(W)$. Hence, by the assumptions, given a sequence (x_n^*) in B there is a subsequence $(x_{n_j}^*)$ such that $(x_{n_j}^*(x))$ converges for all x in W. If $\mathrm{Ba}(H)$ is the Baire σ-field of $(H, \sigma(X, B))$ and $\Sigma = \{E \cap \mathrm{Ext}(H) : E \in \mathrm{Ba}(H)\}$ then the Bishop–de Leeuw Theorem [30, p. 30] assures us that for every x in H there exists a probability measure μ on Σ such that

$$\phi(x) = \int_{\mathrm{Ect}(H)} \phi(y)\, d\mu(y)$$

for every continuous affine function ϕ on H. Applying this formula to $\phi = x_{n_j}^*$, the Lebesgue Convergence Theorem tells us that $(x_{n_j}^*(x))$ converges and so H is a $P(B)$-set.

For proving (iii) we use Proposition 11 to obtain that H has the Radon–Nikodým property. The fact that H is weakly countably determined follows from the fact that $Y = \overline{\mathrm{span}(W)}^{\|\ \|}$ is weakly countably determined [39], and $H = \overline{\mathrm{co}(\mathrm{Ext}(H))}^{\|\ \|}$ is weakly closed in Y. ∎

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