# First-price auctions where one of the bidders' valuations is common knowledge.*

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**Abstract**

We analyze a private and independent valuation first-price auction under the assumption that one of the bidders' valuations is common knowledge. We show that no pure strategy equilibrium exists and we characterize a mixed strategy equilibrium in which the bidder whose valuation is common knowledge randomizes her bid while the other bidders play pure strategies. In an example with the uniform distribution, we compare the expected profits of seller and buyers in this auction with those in a standard symmetric private valuation model.

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## 1 Introduction

In this paper we study a first-price private-value auction in which the valuation of one of the bidders is common knowledge. There are many situations in which some or all of the players in a game may have information about the private valuation of other player(s). For example, an auction may be held in order to renew a license to run a business (from university cafeterias to radio licenses). Information about the current incumbent (sometimes himself a winner of a previous auction) may be available to his rivals and/or to the seller. Actually, sequential auctions are a source of information revelation: in first-price auctions, the seller can learn information about the buyers' preferences through the losing bids. On the other hand, when the buyers have multi-unit demand, the first auction winner's bid may reveal information about his preferences for the rest of the objects. Of course, the use of this information in subsequent stages of a game will affect the strategic behavior of the players in the first stage. While it is outside the scope of this paper to analyze these multi-stage games, we think that studying this auction may be useful to understand the costs and/or possible advantages of this information release.

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In his 1961 seminal paper, Vickrey compared first-price and second-price auctions of a single object and stated an early version of the revenue equivalence theorem. To give an example of an asymmetric case, he also studied an auction where two asymmetric bidders competed for an object. One of the bidders' valuations was fixed, while the other one was uniformly distributed. Vickrey solved for the first-price and second-price sealed-bid auctions and compared the seller's expected revenue.

While in the second-price auction a dominant strategy is (as usual) to bid one's own valuation, the equilibrium in the first-price auction is more involved. The bidder with known valuation must randomize her bid, while her rival uses a strictly increasing bidding function. The difference in the seller's revenue across auctions depends on the value of the fixed valuation: while for low valuations the second-price auction is superior, for high valuations the converse holds.

Already in 2000, Kaplan and Zamir (henceforth, KZ) analyze a two-bidder model in which the seller has information about the bidders' valuations, and can exploit it or not depending on his capacity to commit himself to some specific revelation policy. In this framework they analyze Vickrey's two-bidder problem for a general distribution of the unknown valuation.

We generalize Vickrey's analysis of the asymmetric first-price auction to an arbitrary number of bidders and a general distribution function. We will see that extending the equilibrium analysis is not trivial: while our equilibrium shares some features with the two-bidder equilibrium, most of the techniques used to compute the latter are not useful when more players are involved. We show that no pure strategy equilibrium exists in the auction we analyze and we characterize an equilibrium in which the bidder whose valuation is common knowledge plays a mixed strategy and the other bidders play pure strategies. Our equilibrium shares these features with the one described in Vickrey's example (and in KZ's), and the outcome of the auction is, like in theirs, inefficient, as is usual in asymmetric first-price auctions. In our auction, the random bid of one of the bidders operates as a random reserve price from the point of view of the other bidders.

The expected profit of the bidder whose valuation is known is lower than in a standard auction, while the effect on the other players' expected profits is ambiguous: on the one hand they benefit from an informational advantage, but on the other hand they face a bidder (the one whose valuation is common knowledge) who may bid more aggressively than she would in a standard private-value auction. This may force all of her rivals to bid more aggressively too, but this will not always be the case. The bidders will bid more or less aggressively depending on the valuation which has been revealed. If it is low, they will bid less than in the standard auction: it is easy to see that in the limit, when the valuation which is revealed is zero, the rival bidders will bid as in a standard auction with one less bidder, that is, less than if all valuations were private knowledge. By contrast, if the valuation which has been revealed is high, all bidders may bid more aggressively than they would in a standard auction.[1]

[1] Actually, we will see that in equilibrium the low-valuation bidders bid more in our auction

Regarding the question of the seller's (and buyers') expected revenues, we would like to know if the seller has incentives to announce one of the bidders' valuations when he has this information. Because the answer to this question is ambiguous, we analyze an example with the uniform distribution. The analysis we undertake differs slightly from those of Vickrey and KZ. Vickrey focuses on seller's revenue comparisons between first-price and second-price auctions when the valuation of one of the bidders is fixed. KZ assume that the seller knows either both valuations or their rankings and explore the seller's best strategy given restrictions on his capacity to convey informative signals and on his commitment power. In that framework, they analyze the case in which the seller reveals the highest of the two valuations, and compute the seller's and the buyers' expected profits for the case in which both valuations are uniformly distributed.

In our example we assume that, while ex-ante all valuations are identically and uniformly distributed, the seller can commit himself to announce one of them (which is not necessarily the highest one). We compute the seller's expected revenue for the case of two and three bidders and we find out that in both cases the seller's expected revenue is higher if he follows the policy of announcing one of the bidders' valuations. Note that if we drop the ex-ante symmetry assumption, and assume that it is likely that the valuation which is revealed is high, the seller will have more incentives to find out and announce that valuation. This could be the case, for example, if an authority holds an auction to renew a licence to run some business, and the current incumbent is taking part in it (the incumbent could be supposed to have a high valuation because of know-how, no entry costs, etc.) The authority could order an auditing of the firm and make the results public.

With this analysis, this paper contributes to the recent literature on asymmetric first-price auctions under the private-value assumption. Indeed, after Vickrey's analysis in the early sixties and Griesmer, Levitan, and Shubik's (1967) generalization to asymmetric uniform distributions (i.e. different supports), little work had been published before the nineties, when asymmetric auctions have received renewed attention.[2] Plum (1992) characterizes Nash equilibria in a particular asymmetric sealed-bid auction. Marshall, Meurer, Richard, and Stromquist (1994) propose an algorithm to solve some class of asymmetric auctions and provide some numerical analysis. Lebrun (1996, 1999, and 2002) studies asymmetric first-price auctions with an arbitrary number of bidders, proves existence and uniqueness of equilibrium when the distributions of the valuations have the same support, and studies the continuity of equilibria. A more recent paper, Lebrun (2004), proves uniqueness of equilibrium allowing for different supports in the valuations' distributions.[3] Maskin and Riley's (2000a, 2000b, and 2003) study, respectively, the optimal bidding strategies and

than they bid in a standard auction. However, they win with probability zero, so their bids are not relevant in order to discuss the seller's expected revenue.

[2] Although we only refer here to private-value auctions, there have been also numerous works on asymmetric auctions under the affiliated values assumption over the last years.

[3] His model does not allow, however, for a degenerated distribution of valuations.

the seller's expected revenues for several two-asymmetric-bidder examples, the existence of equilibrium in asymmetric first-price auctions, and its uniqueness. Li and Riley (1999) generalize some of those results to an arbitrary number of bidders using numerical methods.

Also, a series of recent works address the question of the seller's expected revenues in asymmetric first-price auctions under the private-valuation assumption. Landsberger, Rubinstein, Wolfstetter, and Zamir (2002) study a two-bidder auction where the ranking of valuations is common knowledge. They find that the lower valuation bidder bids more aggressively than his rival, and that, in spite of the induced inefficiency, the seller's expected revenue is higher than in the standard auction for a class of distributions (including the uniform). Fibich, Gavious and Sela (2004) show that weakly asymmetric auctions are "essentially" revenue equivalent, since the differences in the seller's revenue across auction mechanisms are only of second order. To finish, Cantillon (2005) compares the expected revenue of asymmetric auctions with those in a symmetric benchmark where the expected valuation of the highest bidder is the same as in the original asymmetric auction, (that is, the social surplus is identical). For some classes of distributions and 2 bidders, she finds that asymmetries reduce the seller's revenue.

In section 2 we present the model and characterize the equilibrium. In section 3 we analyze an example where the bidders' valuations are uniformly distributed, and compare the seller's expected revenue with that in the standard auction. Conclusions are found in section 4.

# 2 The model

There are $n \geq 3$ buyers with a positive valuation for one object which is to be sold in a first-price auction or in an oral descending auction. We assume that the bidders draw their valuations independently from a twice differentiable distribution function $F$, with $F(0) = 0$ and $F(1) = 1$ and strictly positive density $f$ on $[0, 1]$. All players are risk neutral. We assume that the valuation of one of the bidders is common knowledge. For simplicity, we will refer to this bidder as bidder 1, and in the feminine, and to the rest of the bidders, in the masculine, by using the subindex $i$, where $i = 2, ..., n$. In this section we characterize an equilibrium to this game. First, we show that any equilibrium necessarily involves mixed strategies.

## 2.1 (Non existence of) a pure strategy equilibrium

Suppose bidder 1 plays a pure strategy. Her valuation is common knowledge, so her rivals anticipate her bid. Denote this bid by $\overline{r}$. Note that $\overline{r} > 0$ unless $v_1 = 0$.[4] The best response of bidder 1's rivals will be to bid as in an auction

[4] If bidder 1 bids zero, the rest of the bidders will bid as in a standard symmetric auction with $n-1$ bidders, so that bidder 1 obtains an expected profit of zero. Therefore, as long as her valuation is strictly positive, she will submit a positive bid, $\overline{r} > 0$, and will obtain a

with $n-1$ bidders and a common knowledge reserve price equal to $\overline{r}$. It is well known that the equilibrium of this auction is symmetric and unique.[5] Here we show that if the bidders $2, ..., n$ bid according to this equilibrium, bidder 1 has incentives to deviate from her pure strategy bid.

**Proposition 1** *There is no pure strategy equilibrium in undominated strategies in the game described above.*

**Proof.** The equilibrium bidding strategy in a first-price auction with $n-1$ bidders and a reservation price of $\overline{r}$ is[6]

$$b(v_i) = v_i - \frac{\int_{\overline{r}}^{v_i} F(x)^{n-2} dx}{F(v_i)^{n-2}} \quad \forall v_i \geq \overline{r} \tag{1}$$

It is easy to see that $lim_{v_i \to \overline{r}^+} b'(v_i) = 0$, which implies that the rate at which a bidder with valuation $r$ is willing to increase his bid in return for a greater probability of winning is zero (although increasing in $v_i$). But as long as $v_1 > \overline{r}$, bidder 1's willingness to increase her bid is strictly positive, so that she has an incentive to deviate upwards. Formally, given (1), bidder 1's maximization problem is

$$Max_{\ r}\ P[b(v_i) \leq r]^{n-1}(v_1 - r) \tag{2}$$

In order to be an optimal solution to the above problem, $\overline{r}$ must satisfy:

$$\overline{r} = ArgMax_{\ r \geq \overline{r}}\ F[b^{-1}(r)]^{n-1}(v_1 - r) \tag{3}$$

Substituting $r$ by $b(z)$ we can rewrite condition (3) as

$$\overline{r} = ArgMax_{\ z \geq b^{-1}(\overline{r})}\ F(z)^{n-1}(v_1 - b(z)) \tag{4}$$

The marginal benefit of bidding $b(z)$ above $\overline{r}$ in (4) must be non positive, thus

$$(n-1)F(z)^{n-2} f(z)(v_1 - b(z)) - b'(z)F(z)^{n-1} \mid_{z=\overline{r}} \leq 0. \tag{5}$$

Since $lim_{z \to \overline{r}^+} b'(z) = 0$, and $f > 0$, the term above is positive unless $v_1 - \overline{r} \leq 0$, which would leave a non positive profit to bidder 1. But she can obtain a positive profit bidding less than $v_1$. Therefore, condition (5) cannot be satisfied.[7] ■

positive expected profit.

[5] A proof of the uniqueness of equilibrium can be found in Lebrun (1999).

[6] This bidding strategy can be easily computed by standard methods, and can be found, for example, in Riley and Samuelson (1981).

[7] A pure strategy equilibrium where bidder 1 bids $v_1$ could be sustained if she won with probability zero bidding below that point, that is, if one (or more) of her rivals bid always at least $v_1$. Of course, this "overbidding" is a dominated strategy for the rival(s).

## 2.2 A mixed strategy equilibrium

In this section we characterize an equilibrium in which, given $v_1$, player 1 randomizes her bid in some interval $[\underline{b}, \overline{b}]$, while the other players bid according to a strictly increasing bidding function $b(v_i)$. Bidders 2 to $n$ with valuation below $\underline{b}$ do not have any chance to win the auction, and we assume that they bid their own valuation. We can distinguish two kinds of equilibria depending on the value of $v_1$. There is a cut-off point, $\widehat{v}_1$ such that if $v_1 > \widehat{v}_1$ the bidders $i$ with valuation $v_i \geq \underline{b}$ bid in the same interval as bidder 1 does. By contrast, if $v_1 \leq \widehat{v}_1$, high valuation bidders will bid above the support of bidder 1's random bid. We denote by $y(\overline{b})$ the valuation of a bidder $i$ who bids $\overline{b}$, that is, $y(\overline{b}) = b^{-1}(\overline{b})$. If $v_1 \leq \widehat{v}_1$, then $y(\overline{b}) < 1$. Figure 1 illustrates the mixed strategy equilibrium in this case.[8] In Proposition 2 we give the equilibrium bidding strategies. We first define the conditions that the points $\widehat{v}_1, \overline{b}$, and $y(\overline{b})$ must satisfy, and a condition that $\underline{b}$ must satisfy in *any* mixed strategy equilibrium in undominated strategies.[9] To do it, we make an additional, simplifying assumption.

**Assumption 1** Given $v_1$, the function $G(b) = F(b)^{n-1}(v_1 - b)$ has a unique maximum in $[0, v_1]$.

The next Lemma establishes that this maximum is the lower bound of bidder 1's random bid.

**Lemma 1** *The infimum of the support of bidder 1's random bid, $\underline{b}$, must satisfy the following condition:*[10]

$$v_1 = \underline{b} + \frac{F(\underline{b})}{(n-1)f(\underline{b})} \tag{6}$$

**Proof.** See Appendix. ■

Notice that in Lemma 1 we are not discussing the optimality (from the point of view of bidder 1) of bidding $\underline{b}$. What we do is to rule out any equilibrium where the lower bound of her random bid is other than $\underline{b}$.[11]

We now define some points which are necessary to characterize our equilibrium. We will see later on how these points have been arrived at.

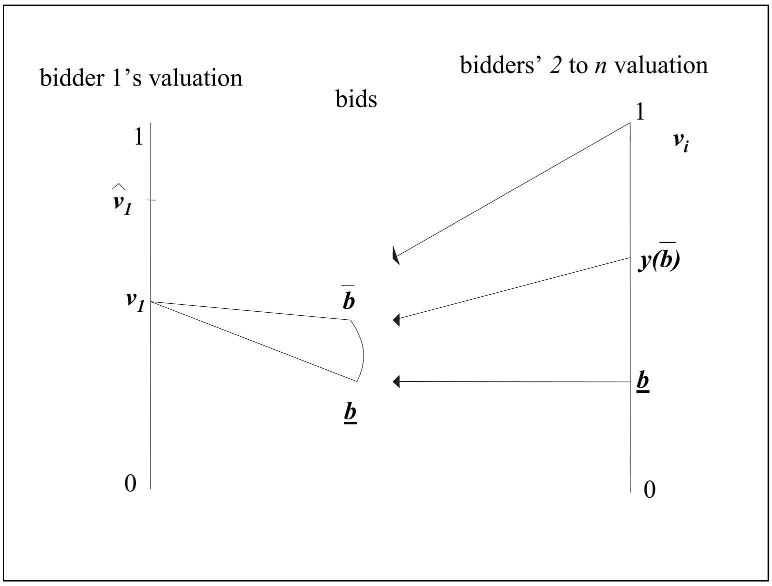
[8] Lebrun (2002) gives an example where a similar equilibrium arises: in a three-bidder asymmetric auction where one of the bidders' valuation is known, two different equilibria arise depending on how high that valuation is. The main difference with our model is that the other bidders' distributions of valuations are discrete.

[9] Notice that we have already ruled out any pure strategy equilibrium to this game (in undominated strategies), and, moreover, any equilibrium in which bidder 1 plays a pure strategy. Therefore, condition (6) given below is very general, and must be satisfied in any equilibrium of the game.

[10] In their pioneering analysis of asymmetric auctions, Maskin and Riley (1996) identified the lowest bid of the bidder whose minimal valuation is highest (which in our case is $v_1$), and the same has also been recently reported by Lebrun (2004). Lemma 1 is a particular case that can be derived from their results.

[11] Although Assumption 1 simplifies the proofs, it is not necessary to obtain our equilibrium, nor for Lemma 1 to hold. If Assumption 1 did not hold and several points satisfied equation (6), $\underline{b}$ would be the absolute maximum of $G(b)$. Had $G(b)$ several absolute maxima, our $\underline{b}$ would be the largest among them.

Figure 1: The mixed strategy equilibrium when $v_1 < \widehat{v}_1$: bidder 1 randomizes her bid in the interval $[\underline{b}, \overline{b}]$. Among her rivals, those with low valuations bid below $\underline{b}$, those with "intermediate" valuations bid in the same interval as bidder 1, and the rest bid above $\overline{b}$.




Let $\widehat{v}_1$ be the bidder 1's valuation at which

$$\widehat{v}_1 = \frac{(n-2) + \underline{b}F(\underline{b})^{n-1}}{(n-2) + F(\underline{b})^{n-1}}. \tag{7}$$

Notice that $\underline{b}$ depends on $v_1$, so that here we are defining a fixed point, where $\underline{b}$ must satisfy equation (6) with $v_1 = \widehat{v}_1$. As we said, $\widehat{v}_1$ is the maximal valuation of bidder 1 such that her rivals bid above the support of her random bid.

If $v_1 \leq \widehat{v}_1$, let $\overline{b}$ and $y(\overline{b})$ be the points satisfying the following conditions:[12]

$$F(y(\overline{b}))^{n-1}(v_1 - \overline{b}) = F(\underline{b})^{n-1}(v_1 - \underline{b}) \tag{8}$$
$$(n-2)\left(y(\overline{b}) - \overline{b}\right) = (n-1)(v_1 - \overline{b}) \tag{9}$$

where, as we said, $y(\overline{b})$ is the valuation of a bidder $i$ who submits a bid of $\overline{b}$.

If $v_1 > \widehat{v}_1$, let $\overline{b} = v_1 - (v_1 - \underline{b})F(\underline{b})^{n-1}$ and $y(\overline{b}) = 1$.

Further on (Lemmas 3 and 4), we will see that $\widehat{v}_1$, $\overline{b}$ and $y(\overline{b})$ as defined in equations (7), (8), and (9) exist and are unique.

[12] Notice that the points defined in these two equations depend again on $v_1$, not only directly, but also through $\underline{b}$.

As long as $v_1 > 0$, equations (6) to (9) imply $0 < \underline{b} < \overline{b} < v_1 < y(\overline{b}) \leq 1$.[13]

**Proposition 2** *There exists an equilibrium where bidder 1 randomizes her bid in* $[\underline{b}, \overline{b}]$ *with density* $h(x)$ *and distribution function* $H(x)$ *satisfying the differential equation*

$$h(x) = \left[ \frac{1}{F^{-1}\left[F(\underline{b})\sqrt[n-1]{\frac{v_1-\underline{b}}{v_1-x}}\right] - x} - \frac{(n-2)}{(n-1)(v_1-x)} \right] H(x) \qquad \forall x \in (\underline{b}, \overline{b}]$$

*which implies* $H(\underline{b}) = 0$*, and bidder* $i = 2, ..., n$ *bids according to the function* $b(v_i)$*, where*

$$b(v_i) = \begin{cases} v_i & v_i \leq \underline{b} \\ v_1 - \frac{(v_1-\underline{b})F(\underline{b})^{n-1}}{F(v_i)^{n-1}} & \underline{b} < v_i \leq y(\overline{b}) \\ v_i - \frac{\int_{y(\overline{b})}^{v_i} F(x)^{n-2}dx}{F(v_i)^{n-2}} - \frac{\left[y(\overline{b})-\overline{b}\right]F(y(\overline{b}))^{n-2}}{F(v_i)^{n-2}} & v_i > y(\overline{b}) \end{cases}$$

Notice that if $v_1 > \widehat{v}_1$ the bidding strategy of bidders 2 to $n$ consists only of the two first regions given above: $y(\overline{b}) = 1$, so that no bidders have valuation above $y(\overline{b})$. The equilibrium is then very similar to the one described by Vickrey (1961), and Kaplan and Zamir (2000) for the two-bidder case. By contrast, if $v_1 < \widehat{v}_1$ the bidders with valuations higher than $y(\overline{b})$ bid above the support of bidder 1's random bid, competing for the good among themselves, rather than with bidder 1.[14]

To prove Proposition 2 we do as follows: first, we characterize the bidding strategy of bidders $2, ..., n$ assuming they behave symmetrically, and given that bidder 1 randomizes her bid in the interval $[\underline{b}, \overline{b}]$. Then we prove that $h(x)$ is the density function of player 1's random bid and that $H$ is a distribution function with no mass point at $\underline{b}$ (Lemma 2); that $\overline{b}$ and $y(\overline{b})$ must satisfy equations (8) and (9) whenever $v_1 < \widehat{v}_1$, and that they exist and are unique (Lemma 3), that bidders 2 to $n$ may bid above bidder 1's random bid support if and only if $v_1 < \widehat{v}_1$, where $\widehat{v}_1$ exists and is unique (Lemma 4) and, to finish, that the bidding functions given above are optimal globally, i.e., that they constitute an equilibrium to the game (Lemma 5).

As we have seen, the bidding strategy of bidders 2 to $n$, $b(v_i)$, can be divided in two or three different regions of values $v_i$ depending on $v_1$:

[13] If $v_1 = 0$ we have $\underline{b} = \overline{b} = y(\overline{b}) = 0$.

[14] According to the results in Lebrun (2002) if the equilibrium described above were unique, infinitesimal changes in the distribution of valuations would lead to infinitesimal changes in the equilibrium of the game, that is, the Nash equilibrium would be continuous with respect to the distribution of valuations. This would imply that when the support of one bidder's valuation in an asymmetric auction shrinks from a nondegenerate interval to a point $\{v_1\}$ the limit of the pure strategy equilibria would converge to our mixed strategy equilibrium.

(1) We assume that the bidders with valuation $v_i \in [0, \underline{b}]$ bid their own valuation. They have no chance of winning, since player 1 bids at least $\underline{b}$. Therefore, to bid $v_i$ is a best response for them.

(2) For $v_i > \underline{b}$, bidder $i \neq 1$ should bid at least $\underline{b}$, since bidding below $\underline{b}$ yields an expected profit of zero. Assume that bidders 2 to $n$ bid symmetrically. Denote by $b_s(v_i)$ the bidding function of these players that would make bidder 1 indifferent about bidding any quantity in $[\underline{b}, \overline{b}]$. Its inverse, $b_s^{-1}$, must satisfy the following equation:

$$F[b_s^{-1}(x)]^{n-1}(v_1 - x) = F(\underline{b})^{n-1}(v_1 - \underline{b}) \quad \forall x \in [\underline{b}, \overline{b}]. \tag{10}$$

Doing the change of variable $b_s^{-1}(x) = v_i$, and rearranging we obtain[15]

$$b_s(v_i) = v_1 - \frac{(v_1 - \underline{b})F(\underline{b})^{n-1}}{F(v_i)^{n-1}}. \tag{11}$$

Of course, the optimal bidding strategy for bidders 2 to $n$ depends on the probability distribution of bidder 1's random bid. Therefore, we will choose that distribution function in order to make bidding according to $b_s(v_i)$ optimal for bidder $i$ with valuation $v_i \in [\underline{b}, y(\overline{b})]$.

(3) As we have said, if $v_1$ is low enough, bidders with a high valuation will bid above $\overline{b}$ (for example, it is easy to see that this will be the case if $v_1 = 0$). Now, provided $v_1$ is low enough (namely provided that $v_1 < \widehat{v}_1$) there will be a third region of valuations, $v_i \in [y(\overline{b}), 1]$, where player $i$ bids above $\overline{b}$, so that he knows that he will not be beaten by player 1 and that he is competing against only $n - 2$ bidders. A necessary condition for equilibrium in this region is:

$$v_i = \arg\max{}_z \; F(z)^{n-2}(v_i - b_t(z)). \tag{12}$$

And also, $b_t(y(\overline{b})) = \overline{b}$.

Then, we must have:

$$b_t(v_i) = v_i - \frac{\int_{y(\overline{b})}^{v_i} F(x)^{n-2}dx}{F(v_i)^{n-2}} - \frac{\left[y(\overline{b}) - \overline{b}\right] F(y(\overline{b}))^{n-2}}{F(v_i)^{n-2}}. \tag{13}$$

We now study how bidder 1 must randomize her bid to make $b_s$ optimal for her rivals.

[15] Note that $b_s$ corresponds to bidder $i$'s optimal bidding function in the second region of valuations (the subindex "s" stands for "second"). For expositional purposes, it is useful to refer to this function in particular (instead of the bidding function in Proposition 2). Also, since $b_s$ is the way bidders 2 to $n$ must bid in order to make bidder 1 indifferent, note that it can be defined in a wider interval of valuations, $[\underline{b}, 1]$ than the one in which it is the optimal bidding function. The same happens for the function $b_t$ below, which can be defined in a larger interval of valuations than the one where it is the optimal strategy.

**Lemma 2** *Denote the distribution function of bidder 1's random bid by H, assume it is differentiable, and denote its derivative by h. The function H must satisfy the following differential equation:*

$$h\left[b_s(v_i)\right] = \left[\frac{1}{(v_i - b_s(v_i))} - \frac{(n-2)}{(n-1)(v_1 - b_s(v_i))}\right] H\left[b_s(v_i)\right] \tag{14}$$

*given the initial condition* $H(\overline{b}) = 1$.[16]

**Proof.** See Appendix. ■

Doing the change of variable $b_s(v_i) = x$ and using equation (10) we obtain the expression for $h$ given in Proposition 2.

Equation (14) stems from the maximization problem of bidders 2 to $n$. They can only win the auction if they submit a bid above that of bidder 1. Instead of computing the optimal bidding strategy as usual, what we do is to compute the adequate $H$ (probability that bidder 1's random bid is below a certain point) in order to make our function $b_s$ optimal.

Note that we cannot use the initial condition $H(\underline{b}) = 0$ to find a particular solution of equation (14): given that $b_s(\underline{b}) = \underline{b}$, the right hand side of the equation is not continuous at that point, and one of the necessary conditions for existence of a solution is violated. Instead, as we have seen, we use the condition $H(\overline{b}) = 1$. As $h$ is not defined at $\underline{b}$, we let $h(\underline{b})$ be any positive constant. Note that this is consistent with the fact that $H(\underline{b}) = 0$, which implies that there is no mass point at $\underline{b}$.

Now we are ready to give a condition $y(\overline{b})$ must satisfy and to establish the existence and uniqueness of $\overline{b}$ and $y(\overline{b})$:

**Lemma 3** *When* $v_1 \leq \widehat{v}_1$,
*(i)* $y(\overline{b})$ *is the point at which* $b'_s(y(\overline{b})) = b'_t(y(\overline{b}))$.
*(ii)* $\overline{b}$ *and* $y(\overline{b})$ *satisfying equations (8) and (9) exist and are unique.*

**Proof.** See Appendix. ■

Loosely speaking, condition (i) in Lemma 3 requires "smoothness" of the bidding function of bidders 2 to n. Together with the condition $b_t(y(\overline{b})) = \overline{b}$ given before, which requires continuity, we obtain equations (8) and (9) which establish the points where the bidders "shift" their bidding functions from $b_s$ to $b_t$.

Lemma 3 leads us to the next result.

**Lemma 4** *Bidders 2, 3,..., n bid above the support of bidder 1's random bid if and only if* $v_1 < \widehat{v}_1$, *where* $\widehat{v}_1$, *as defined in equation (7) exists and is unique.*

**Proof.** See Appendix. ■

Imposing condition (i) in Lemma 3 we compute the value of $y(\overline{b})$ as a function of $v_1$. When $v_1 = \widehat{v}_1$, then $y(\overline{b}) = 1$ and so it is for higher values of $v_1$.

[16] The Cauchy-Peano theorem on differential equations, among others, guarantees existence of a solution to equation (14).

When $v_1 > \widehat{v}_1$ the equilibrium is very similar to the one described in Vickrey (1961) for two bidders. In this case $\overline{b}$ is the maximal bid for all players, and to compute it we just need to equate bidder 1's profits when she bids $\underline{b}$ and when she bids $\overline{b}$ (in which case she wins with probability one). When there are more than two bidders we cannot use this method because, unless $v_1$ is high enough, it will be optimal for the highest bidders to bid more than $\overline{b}$, and therefore, bidder 1 won't win with probability one when she submits her highest bid. Instead, to compute $\overline{b}$, we need to find first $y(\overline{b})$, the point satisfying condition (i) stated in Lemma 3.

It is clear that bidder 1 maximizes her expected profits by bidding on $[\underline{b}, \overline{b}]$.[17] However, we have constructed the equilibrium bidding strategy of the other bidders by imposing only the first order conditions for each interval of valuations. We have not proved that our solution is a global maximum, nor have we checked the second order conditions. This is done in the proof of Lemma 5, where we state that this bidding strategy is indeed optimal for them.

**Lemma 5** *For every $v_i$, bidding according to $b(v_i)$ in Proposition 2 is a global maximizer of bidder i's maximization problem.*

**Proof.** See Appendix. ■

# 3 A comparison with the standard auction

Now we can try to compare this auction with a standard one (without reserve price, and in which all valuations are private knowledge) from the point of view of the bidders and the seller.

First, notice that our equilibrium can lead to inefficient outcomes: as long as bidder 1 randomizes her bid, it is not guarantied anymore that the highest-valuation bidder will win the object. Second, while it is obvious that bidder 1´s expected profit is lower in this auction, it is not clear whether the rest of the bidders are worse or better off.[18] Indeed, on the one hand they have an advantage over bidder 1, which should allow them to perform better, but on the other hand, player 1 may bid more aggressively, due to the disadvantage she suffers.[19] Moreover, her bid has a similar effect on her rivals as a random reserve price, which can force them to bid more aggressively than they would in the standard auction. Which of these two effects is stronger is not obvious. Since the bidding is more or less aggressive depending on the valuation of bidder

[17] Note that $b_s(v_i)$, bidder $i$'s bidding strategy making bidder 1 indifferent about bidding on the interval $[\underline{b}, \overline{b}]$ can be defined in a larger interval, namely $[\underline{b}, 1]$, and it is easy to prove that $b_t(v_i) > b_s(v_i)\ \forall v_i > y(\overline{b})$ so that bidder 1 has not incentives to bid above $\overline{b}$.

[18] If bidder 1 bids $\underline{b}$ in the standard auction she wins with a higher probability than in the asymmetric auction, since her rivals never bid up to their valuation. Therefore, the expected profits to this bidder in the standard auction cannot be less than in the asymmetric one.

[19] Without further conditions on the distribution of the bidders' valuations it is not easy to compare bidder 1's bid in the asymmetric auction with that of the standard auction. However, we next show that if the bidders' valuations are uniformly distributed, bidder 1 submits a higher bid in the asymmetric auction with probability one.

1, it is not clear either whether the seller's expected revenue in this auction is higher or lower than that in the standard auction.

In this section we consider the case of three bidders whose valuations are drawn from the uniform distribution on [0,1]. With this example we illustrate the equilibrium described above, and compare the expected profits of the seller and the buyers in the auction we have analyzed with those in a standard auction. We obtain the following:

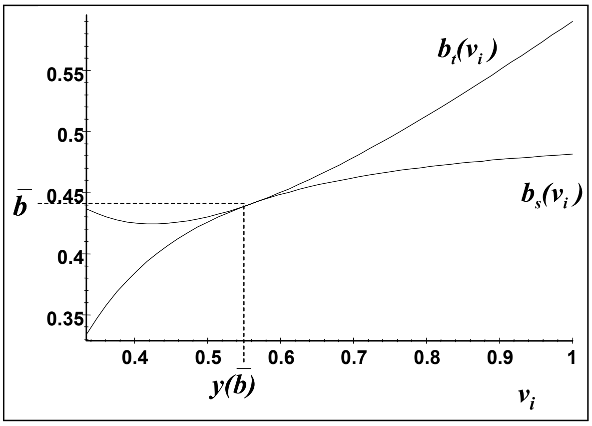
$$\begin{aligned} \underline{b} &= \frac{2v_1}{3} \\ \widehat{v}_1 &= 0.89411 \end{aligned}$$

$$\begin{array}{ll} \overline{b} = 0.88157 v_1, \ y(\overline{b}) = 1.1184 v_1 & \forall v_1 \leq \widehat{v}_1 \\ \overline{b} = v_1 - \frac{4v_1^3}{27}, \ y(\overline{b}) = 1 & \forall v_1 > \widehat{v}_1 \end{array}$$

$$\begin{aligned} b_s(v_i) &= v_1 - \frac{4v_1^3}{27v_i^2} \\ b_t(v_i) &= \frac{v_i}{2} + \frac{0.3605 v_1^2}{v_i} \end{aligned}$$

$$h(x) = \left[ \frac{1}{\frac{2v_1}{9(v_1 - x)} \sqrt{3\,(v_1 - x)\, v_1} - x} - \frac{1}{2(v_1 - x)} \right] H(x)$$

Figure 2: $b_s(v_i)$ and $b_t(v_i)$ where $v_1 = 0.5$.




An interesting property of the uniform distribution is that the lower bound of bidder 1's random bid, $\underline{b}$, is precisely what bidder 1 would bid in the standard auction, that is $\underline{b} = \frac{n-1}{n} v_1$. This implies that when the valuations are uniformly distributed, bidder 1 will always bid more aggressively than she would in a standard auction.

In Figure 2 we draw $b_s(v_i)$ and $b_t(v_i)$ as defined in the interval $[\underline{b}, 1]$, for the particular case where $v_1 = 0.5$. In that case we have $\overline{b} = 0.44079$ and $y(\overline{b}) = 0.5592$. Note that $b_s(v_i)$ and $b_t(v_i)$ satisfy the tangency condition stated in Lemma 3. Figures 3 and 4 show, respectively, $H(x)$ and $h(x)$ when $v_1 = 0.5$.

Figure 3: $H(b)$ where $v_1 = 0.5$.

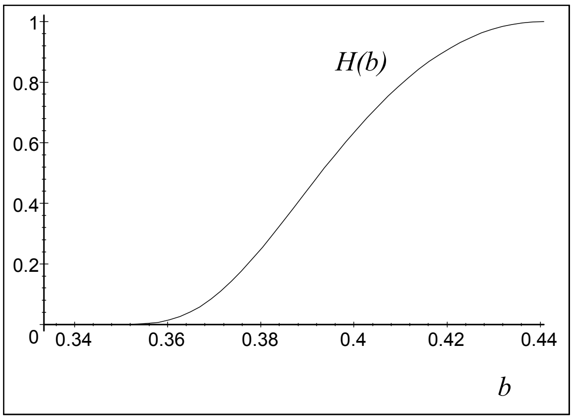
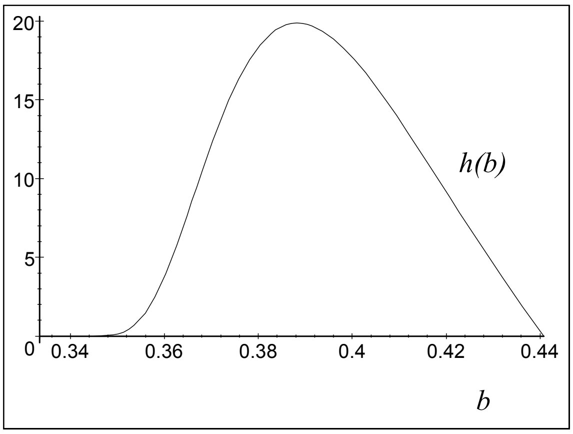



Figure 4: $h(b)$ where $v_1 = 0.5$.




Now we can compute the expected profits of the buyers and seller. We compute the expected profits to the participants in this kind of auction ex-ante, that is, before they learn any valuations, and we compare these results with those of a standard auction, in which all bidders' valuations are private information. To do it, we integrate numerically for all valuations of $v_1$ from 0 to 1, (and then for each of the other bidders). We also use the results in Vickrey (1961) to compute the same for the case of two bidders. The results are given in the Tables 1 and 2. In columns 2 and 3 we write, respectively, the ex ante expected profits of participating in our asymmetric auction and a standard auction. In the fourth column we write the difference between the second and the third one, that is, the gains to each agent of participating in our asymmetric auction instead of the standard one. In the fifth and sixth rows we write, respectively, the rents that the seller extracts from bidder 1 and from each of the other bidders.

Table 1: Expected profits in the case where $n = 2$

| Expected profit | Asymmetric Auction | Symmetric A. | Gains |
|---|---|---|---|
| Player 1 | 0.0833 | 0.1667 | -0.0833 |
| Player 2 | 0.2007 | 0.1667 | 0.034 |
| Seller | 0.3694 | 0.3333 | 0.0361 |
| Revenue from bidder 1 | 0.1875 | 0.1667 | 0.0208 |
| Revenue from bidder 2 | 0.1819 | 0.1667 | 0.0152 |
| Social Surplus | 0.6534 | 0.6667 | -0.0132 |

Table 2: Expected profits in the case where $n = 3$

| Expected profit | Asymmetric Auction | Symmetric A. | Gains |
|---|---|---|---|
| Player 1 | 0.03704 | 0.0833 | -0.0463 |
| Player $i = 2, 3$ | 0.0985 | 0.0833 | 0.0151 |
| Seller | 0.5101 | 0.5 | 0.0101 |
| Revenue from bidder 1 | 0.1426 | 0.1667 | -0.0241 |
| Revenue from bidders $2, 3$ | 0.1837 | 0.1667 | 0.0170 |
| Social Surplus | 0.7440 | 0.75 | -0.0060 |

As we see, with $n = 2$ the player whose valuation has not been revealed benefits from the revelation of information less than the seller does. When $n = 3$ the opposite happens. However, in both cases all the agents but bidder 1 have a higher expected profit in the asymmetric auction than in the standard one. As for the rents that the seller extracts from the agents, we see that, with $n = 2$ both rents are higher in the asymmetric auction than in the standard one, while with $n = 3$ the rents extracted from bidder 1 are lower than those in the standard auction. The loss of efficiency is not very large compared to the seller's increase in expected profits. Hence, an authority interested in both efficiency and maximal revenue may still prefer this kind of mechanism to the standard auction when he knows the valuation of one of the bidders. Comparing our results with those of KZ we observe that, in the particular case of the uniform distribution and two bidders, the seller's expected revenues are slightly higher when the seller reveals the ranking of the valuations than when he announces one of them (0.3696 instead of 0.3694). To finish, it may be interesting to point out that, conditional on bidder 1's valuation, the expected revenue to the seller will be higher in the asymmetric auction (compared to the standard one) if $v_1 > 0.43$ when $n = 2$ and if $v_1 > 0.47$ when $n = 3$.[20]

# 4 Conclusions

When one of the bidders' valuations is common knowledge no equilibrium in pure strategies exists. We have characterized a mixed strategy equilibrium in which the bidder whose valuation is common knowledge randomizes her bid, while the other players play pure strategies.

When the valuation which is revealed is low, the bidders with high valuations compete for the object among themselves rather than with bidder 1, whom they will beat for sure. As we have seen, when a bidder $i$ has the same valuation as bidder 1, he bids in the support of bidder 1's random bid. This implies that in this auction the inefficiencies go in both senses: player 1 can be beaten by a lower valuation rival but can also beat a higher valuation one.

As for the seller's expected revenue, revealing one of the bidders' valuations may result in more aggressive bidding from all the players if that valuation is high, thus enhancing the seller's expected profit. However, when the valuation which is revealed is low enough, the effect will be the opposite. Assuming that ex-ante the bidders' valuations are symmetrically distributed, and then one of them is revealed, we compute the seller's expected profit for the particular case of the uniform distribution and two and three bidders. We observe that in these cases it is in the interest of the seller to reveal one of the bidders' valuations if he has that information. The bidder whose valuation has been revealed is, as expected, worse off, while her rivals are better off.

Note that if a second-price auction took place under our assumptions, it would still be equivalent to a symmetric second-price auction: the fact that after the valuations have been drawn one of them is revealed does not change the bidders' strategic behavior, given that in both cases truthful revealing is optimal for the bidders.[21] Therefore, our comparison of the first-price asymmetric auction with a standard auction (where no valuation is revealed) applies too to the asymmetric second-price auction, i.e. although the effect on the seller's expected revenue is ambiguous, we know that at least in some cases (the uniform distribution with 2 or 3 bidders) the first-price auction yields more revenues than the second-price auction does.

As we said, when the bidder whose valuation is revealed is the winner of a previous auction or the incumbent in a market, it is likely that her valuation is high. In this case, our numerical computation is not useful, since we were

[20] Both numbers rounded to the second decimal. The first value was already given by Vickrey (1961).

[21] In case the common knowledge valuation were the highest, equivalence would hold only if the second highest bidder still submits his bid.

assuming ex-ante symmetric distributions of valuations. However, under this assumption, the policy of announcing the incumbent's valuation will indeed increase in the seller's expected profit.

Notice also that the random bid of bidder 1 operates, from the point of view of the other bidders, as a random (or secret) reserve price. Hence, our analysis can help to analyze such scenarios.

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# 5 Appendix

**Proof of Lemma 1**
Note that

$$\underline{b} = ArgMax_b\ F(b)^{n-1}(v_1 - b), \tag{15}$$

whose FOC yields (6). We prove Lemma 1 by contradiction.

Suppose that an equilibrium exists where the lower bound of bidder 1's random bid is $b^* \neq \underline{b}$. The optimal response for a bidder $i$ with valuation $v_i > b^*$ implies bidding at least $b^*$, since bidding less yields a zero expected profit. Therefore, bidder 1 will win the auction if and only if all her rivals have valuations less than or equal to $b^*$. Her expected profit is $F(b^*)^{n-1}(v_1 - b^*)$.

In undominated strategies bidders $2, ..., n$ do not bid above their valuation, so $b(v_i) \leq v_i$. By bidding $\underline{b}$, bidder 1 obtains an expected profit of $F(b^{-1}(\underline{b}))^{n-1}(v_1 - \underline{b}) \geq F(\underline{b})^{n-1}(v_1 - \underline{b}) > F(b^*)^{n-1}(v_1 - b^*)$. Hence, bidding $b^*$ cannot be optimal for bidder 1. ■

**Proof of Lemma 2**
In order to sustain the mixed strategy equilibrium, bidders 2 to $n$ with valuations in $[\underline{b}, y(\overline{b})]$ must bid according to the function $b_s(v_i)$. These players, who bid in the interval $[\underline{b}, \overline{b}]$ face the following maximization problem:

$$Max_z H(b_s(z))F(z)^{n-2}(v_i - b_s(z))\ \forall z \in [\underline{b}, y(\overline{b})] \tag{16}$$

where $b_s$ is the bidding function given in (11). In equilibrium the derivative with respect to $z$, evaluated at $v_i$ must be zero. Differentiating $b_s(v_i)$ we obtain $b_s'(v_i) = \frac{(n-1)f(v_i)(v_1 - b_s(v_i))}{F(v_i)}$. Simplifying and rearranging, we get equation (14).

We now show that $h$ is a density function, i.e., $h(x) \geq 0\ \forall x \in [\underline{b}, \overline{b}]$, and that $H(\underline{b}) = 0$.[22]

- $h$ **is a density function.**

We prove this by contradiction. Denote by $M(v_i)$ the term in brackets multiplying $H$ in equation (14). We show first that $M \geq 0\ \forall v_i \in [\underline{b}, y(\overline{b})]$. This holds if

$$(n-1)(v_1 - b_s(v_i)) \geq (n-2)(v_i - b_s(v_i))$$

rearranging and substituting $b_s$ by its value in (11) we obtain

$$(n-2)(v_1 - v_i) \geq -\frac{(v_1 - \underline{b})F(\underline{b})^{n-1}}{F(v_i)^{n-1}}. \tag{17}$$

The left (right) hand side is strictly decreasing (increasing) in $v_i$. Using equations (8) and (9) it is easy to see that condition (17) holds with equality when $v_i =$

[22] As we have said, $v_i \in [\underline{b}, y(\overline{b})]$ implies $b_s(v_i) \in [\underline{b}, \overline{b}]$. To simplify notation, in this proof we often denote the argument of $h$ or $H$ as $x \in [\underline{b}, \overline{b}]$ (instead of $b_s(v_i)$).

$y(\overline{b})$. Therefore, the inequality holds strictly when $v_i < y(\overline{b})$. Since $M > 0$ $\forall v_i < y(\overline{b})$ we must have that $sign(h) = sign(H)$ $\forall v_i < y(\overline{b})$.

Now, given that $H$ is the primitive of $h$ and that $H(\overline{b}) = 1$ we have

$$H(x) = 1 - \int_x^{\overline{b}} h(s)ds \quad \forall x \in [\underline{b}, \overline{b}].$$

Suppose $\exists z \in (\underline{b}, \overline{b})$ such that $h(z) < 0$. Then $H(z) < 0$ and, since $H$ is continuous,[23] $\exists x \in (z, \overline{b})$ such that $H(x) = 0$. Let $X = \min\{x > z \text{ such that } H(x) = 0\}$. Then $H(X) = 1 - \int_X^{\overline{b}} h(s)ds = 0$, so $\int_X^{\overline{b}} h(s)ds = 1$. Note that by continuity, both $h$ and $H$ must be negative in the interval $[z, X)$.[24]

On the other hand $H(z) = 1 - \int_z^{\overline{b}} h(s)ds = 1 - \left(\int_z^X h(s)ds + \int_X^{\overline{b}} h(s)ds\right) = 0 - \int_z^X h(s)ds$. But $\int_z^X h(s)ds < 0$, so $H(z) > 0$. Contradiction.

- $H(\underline{b}) = 0$.

Although we have arbitrarily set $h(\underline{b})$, we can prove that $\lim_{x \longrightarrow \underline{b}^+} H(x) = 0$. Note that it follows that $H(\underline{b}) = 0$, since $H$ must be non-decreasing. What we do is to prove that $\lim_{v_i \longrightarrow \underline{b}^+} h[b_s(v_i)]$ is bounded. Given that, we will have:

$$\lim_{v_i \longrightarrow \underline{b}^+} H[b_s(v_i)] = \lim_{v_i \longrightarrow \underline{b}^+} \frac{h[b_s(v_i)]}{M(v_i)} = \frac{C}{\infty} = 0.$$

Differentiating equation (14) with respect to $v_i$ we have $h' = M'H + Mh$; Substituting $h$ we get $h' = M'H + M(MH) = (M' + M^2)H$. We have $\lim_{v_i \longrightarrow \underline{b}^+} h'[b_s(v_i)] = \lim_{v_i \longrightarrow \underline{b}^+} [M' + M^2] H$. We show that this limit is positive (in fact $\lim_{v_i \longrightarrow \underline{b}^+} [M' + M^2] = \infty$). Therefore, $h'$ is positive. As $h$ is continuous and positive in $(\underline{b}, \overline{b}]$ and its slope is positive at $\underline{b}^+$, $\lim_{v_i \longrightarrow \underline{b}^+} h[b_s(v_i)]$ must be finite.

To compute this limit, take equation (14) and substitute $b_s(v_i)$ by its value in equation (11). We obtain $M$ as a function of $v_i$. Differentiating $M$ we obtain:

$$\begin{aligned} M' + M^2 &= F(v_i)^{n-2}\left[\frac{(n-1)f(v_i)K - F(v_i)^n}{[(v_i - v_1)F(v_i)^{n-1} + K]^2} - \frac{(n-2)f(v_i)}{K}\right] \\ &\quad + \left[\frac{F(v_i)^{n-1}}{F(v_i)^{n-1}(v_i - v_1) + K} - \frac{(n-2)F(v_i)^{n-1}}{(n-1)K}\right]^2 \end{aligned}$$

Where $K = (v_1 - \underline{b})F(\underline{b})^{n-1}$.

Doing some computations, taking into account that $(v_1 - \underline{b}) = \frac{F(\underline{b})}{(n-1)f(\underline{b})}$, and taking limits, we obtain:

$$\lim_{v_i \longrightarrow \underline{b}^+} [M' + M^2] = \frac{F(\underline{b})^{2n-2}}{[0]^2} - \frac{(n-2)f(\underline{b})^2}{F(\underline{b})^2} = \infty$$

[23] Since it is differentiable.

[24] Here we use the fact that $h$ is differentiable (and therefore continuous) in $(\underline{b}, \overline{b}]$, as it is the product of two functions ($M$ and $H$) that are so.

■

**Proof of Lemma 3**

$(i)$ Take the functions $b_s$ and $b_t$ as defined in all the interval $[\underline{b}, b_s(1)]$ and $[\underline{b}, b_t(1)]$ respectively. Suppose that $b_s'(y(\overline{b})) > b_t'(y(\overline{b}))$. Since $b_s(y(\overline{b})) = b_t(y(\overline{b}))$ and both functions are continuous, we must have $b_s(x) > b_t(x)$ in a neighborhood at the right of $y(\overline{b})$. Suppose that bidder 1 bids $b' > \overline{b}$. Her expected profit is $F[b_t^{-1}(b')]^{n-1}(v_1 - b')$, which is larger than $F[b_s^{-1}(b')]^{n-1}(v_1 - b')$, her expected profit of submitting any bid $b \in [\underline{b}, \overline{b}]$. Therefore, it is in the interest of player 1 to deviate bidding above $\overline{b}$. (As $b_s$ was defined as the way bidders 2 to $n$ should bid to make bidder 1 indifferent between all her possible bids, as long as these bidders bid below $b_s$ when their valuations are above $y(\overline{b})$ bidder 1 will be strictly better off bidding above $\overline{b}$.)

Suppose now that $b_s'(y(\overline{b})) < b_t'(y(\overline{b}))$. Deriving the first order conditions to (12) and (16) we show that $b_s'(y(\overline{b})) < b_t'(y(\overline{b}))$ would imply $h(y(\overline{b})) < 0$, which is impossible. First, from (12) we have that $b_t$ must satisfy

$$b_t'(v_i)F(v_i) = (n-2)f(v_i)(v_i - b_t(v_i))$$

From the first order condition of the maximization problem (16) we have:

$$b_s'(v_i)F(v_i) = \frac{(n-2)f(v_i)H(b_s(v_i))(v_i - b_s(v_i))}{H(b_s(v_i)) - h(b_s(v_i))(v_i - b_s(v_i))}$$

Now, evaluating both equations at $y(\overline{b})$, taking into account that $b_s(y(\overline{b})) = \overline{b}$, simplifying and rearranging we have that $b_s'(y(\overline{b})) < b_t'(y(\overline{b}))$ implies:

$$h(b_s[y(\overline{b})]) \left[y(\overline{b}) - \overline{b}\right] < 0$$

which cannot hold unless $h(\overline{b}) < 0$, since $y(\overline{b}) > \overline{b}$.

$(ii)$ Combining equations (8) and (9) we obtain:

$$F(y(\overline{b}))^{n-1}(y(\overline{b}) - v_1) = \frac{1}{n-2} F(\underline{b})^{n-1} (v_1 - \underline{b}) \tag{18}$$

Note that:

- Both sides of equation (18) are zero when $v_1 = y(\overline{b}) = 0$ (note that then $\underline{b} = 0$).
- Given $y(\overline{b})$, the left hand side of equation (18) is decreasing in $v_1$ while the right hand side is non-decreasing in it (because of the "incentive compatibility constraint", since it is $\frac{1}{n-2}$ times bidder 1's expected profit). Therefore, $y(\overline{b})$ must be strictly increasing in $v_1$, so that equation (18) holds as $v_1$ increases.
- Combining equations (7) and (18), it is easy to see that when $v_1 = \widehat{v}_1$, then $y(\overline{b}) = 1$. Therefore, for any $v_1 < \widehat{v}_1$, there exists a unique $y(\overline{b})$ satisfying equation (18).

- Equation (9) can be rearranged to obtain $\overline{b} = (n-1)\,v_1 - (n-2)\,y(\overline{b})$. Since $y(\overline{b})$ is unique, so is $\overline{b}$. ■

**Proof of Lemma 4**

Deriving the first order condition from (12) and differentiating equation (11) we obtain

$$b_t'(v_i) = \frac{(n-2)f(v_i)(v_i - b_t(v_i))}{F(v_i)} \tag{19}$$

$$b_s'(v_i) = \frac{(n-1)f(v_i)(v_1 - b_s(v_i))}{F(v_i)} \tag{20}$$

Equating the slopes of $b_s$ and $b_t$, and simplifying we obtain:

$$(n-1)\,(v_1 - b_s(v_i)) = (n-2)\,(v_i - b_t(v_i))$$

Imposing $b_t(v_i) = b_s(v_i)$, substituting $b_s(v_i)$ by its value from equation (11), simplifying, and rearranging we have:

$$v_i = v_1 + \frac{(v_1 - \underline{b})F(\underline{b})^{n-1}}{(n-2)F\,(v_i)^{n-1}}. \tag{21}$$

Note that the left hand side of equation (21) increases with $v_i$ while the right hand side decreases. It is clear that if $v_1$ is high (for example, if $v_1 = 1$) equation (21) does not have a solution on $[0,1]$. This will happen when evaluating equation (21) at $v_i = 1$ the right hand side is still greater than the left hand side, that is, when $1 < v_1 + \frac{(v_1-\underline{b})F(\underline{b})^{n-1}}{(n-2)}$, or, rearranging, when $v_1 > \widehat{v}_1$. On the other hand, we can rewrite equation (7) as:

$$F(\underline{b})^{n-1}(v_1 - \underline{b}) = (n-2)(1 - v_1) \tag{22}$$

The left hand side is the expected profit to bidder 1. It is zero for $v_1 = 0$ and non-decreasing in $v_1$.[25] The right hand side is positive if $v_1 = 0$ and strictly decreasing in $v_1$. Hence, a unique $v_1$, $\widehat{v}_1$, satisfies the above condition. ■

**Proof of Lemma 5**

We have described the optimal strategy of bidders $i = 2, .., n$ but we have not proved that our solution is a global maximum, and we have not checked the second order conditions. We proceed by regions:

1) Bidders with valuation $v_i \leq \underline{b}$ : A bidder has probability zero of winning and hence, an expected profit of zero. Bidding above $\underline{b}$ implies negative expected profit, since there is a positive probability of winning and paying a price higher than one's own valuation. Thus, there are no incentives to bid above $\underline{b}$, while a player is indifferent bidding any quantity below it.

2) Bidders with valuation $v_i \in (\underline{b}, y(\overline{b})]$ : bidder $i$ has a positive expected profit bidding according to $b_s(v_i)$. Thus he has not incentives to deviate by

[25] Because of the incentive-compatibility constraint.

bidding less than $\underline{b}$, which implies zero profits. We also need to show that bidding according to $b_t(z)$ with $z > y(\overline{b})$ is not optimal for bidders in this region. To see that, it is enough to show that $\frac{dE(\pi_{v_i}(z))}{dz} < 0 \ \forall z > y(\overline{b})$, that is, the expected profit of a player type $v_i < y(\overline{b})$ who acts as if she were of type $z \geq y(\overline{b})$ decreases with $z$. We have $E(\pi_{v_i}(z)|z > y(\overline{b})) = F(z)^{n-2}(v_i - b_t(z))$. Differentiating with respect to $z$, substituting $b_t(z)$ by its value in (13), and $b_t'(z)$ by its derivative, and simplifying we obtain $E'(\pi(z)) = v_i - z$, which is negative $\forall z > y(\overline{b}) > v_i$. On the other hand, if $v_1$ is high enough so that bidders $i$ to $n$ bid only in two regions, it is easy to see that bidding above $\overline{b}$ cannot be optimal: by bidding $\overline{b}$ a player obtains the good with probability one, while bidding above only implies paying a higher price for it.

It is also necessary to check that bidding according to $b_s(v_i)$ is indeed the best strategy for players with valuation $v_i \in [\underline{b}, y(\overline{b})]$, that is, that $b_s(v_i)$ is a maximum of the objective function. From the first order condition of problem (16) we have

$$\frac{dE(\pi_{v_i}(z))}{dz} = -b_s'(z)H(b_s(z))F(z)^{n-2}+$$
$$+\left[h(b_s(z))b_s'(z)F(z)^{n-2} + (n-2)F(z)^{n-3}f(z)H(b_s(z))\right](v_i - b_s(z)) \qquad (23)$$

which is zero evaluated at $z = v_i \ \forall v_i \in [\underline{b}, y(\overline{b})]$. Now, suppose a bidder type $v_i$ behaves as if he were of type $z < v_i$. Expression (23) would be zero if $v_i = z$. Since $v_i$ is greater than $z$, the second term, which is positive, will be greater than it was when $v_i = z$. Therefore, the expected profit increases with $z$ and, as long as $z < v_i$, it is in the interest of the bidder to increase his bid. Conversely, if the bidder acts as if he were of type $z > v_i$, the term $(v_i - b_s(z))$ decreases and so does the second term in (23), which implies that the derivative is negative, and therefore it is in the bidder's interest to decrease his bid. Hence, bidding $b_s(v_i)$ is optimal for a bidder of type $v_i \in [\underline{b}, y(\overline{b})]$.

3) Bidders with valuation $v_i > y(\overline{b})$ : Again, bidding less than $\underline{b}$ implies zero expected profit, while bidding above bidder $i$ can obtain a positive profit. Second, we need to show that bidding in the interval $[\underline{b}, \overline{b}]$ is not optimal for bidders in this region. As before, introducing in equation (23) the valuation of a bidder in this region we have that for all $z \in [\underline{b}, y(\overline{b})]$ the derivative of the expected profit with respect to $z$ is positive, that is, it is in the interest of player $i$ to increase his bid. Finally, we can use the same reasoning to show that our $b_t(v_i)$ is a maximizer of the problem (12) and not a minimum: differentiating the expression in (12) we get the derivative of the expected profit with respect to $v_i$. If $i$ behaves as if he were type $z < v_i$ this derivative is positive, and if he behaves as type $z > v_i$ it is negative. ■