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On Compactness in Locally Convex Spaces

B. Cascales and J. Orihuela

Departamento de Analisis Matematico, Facultad de Matematicas, Universidad de Murcia, E-30.001-Murcia-Spain

1. Introduction and Terminology

The purpose of this paper is to show that the behaviour of compact subsets in many of the locally convex spaces that usually appear in Functional Analysis is as good as the corresponding behaviour of compact subsets in Banach spaces. Our results can be intuitively formulated in the following terms: *Dealing with metrizable spaces or their strong duals, and carrying out any of the usual operations of countable type with them, we ever obtain spaces with their precompact subsets metrizable, and they even give good performance for the weak topology, indeed they are weakly angelic,* [14], *and their weakly compact subsets are metrizable if and only if they are separable.*

The first attempt to clarify the sequential behaviour of the weakly compact subsets in a Banach space was made by V.L. Šmulian [26] and R.S. Phillips [23]. Their results are based on an argument of metrizability of weakly compact subsets (see Floret [14], pp. 29–30). Šmulian showed in [26] that a relatively compact subset is relatively sequentially compact for the weak topology of a Banach space. He also proved that the concepts of relatively countably compact and relatively sequentially compact coincide if the weak-* dual is separable. The last result was extended by J. Dieudonné and L. Schwartz in [9] to submetrizable locally convex spaces. The converse to Šmulian's theorem was stated by W.F. Eberlein [10]. This result was extended by A. Grothendieck [15], to spaces of continuous functions on compact spaces endowed with the pointwise convergence topology. A combination of results by A. Grothendieck, D.H. Fremlin, J.D. Pryce and M. De Wilde allow K. Floret [14], p. 36, to give a proof of a general version for the Eberlein-Šmulian theorem. In spite of its powerful applications, the scope of the Eberlein-Šmulian theorem does not include some important classes of locally convex spaces and it gives no information about the metrizability of the compact subsets.

Dealing with compactness in a locally convex space E two questions appear to arise:

(1) Are the compact subsets of E metrizable?

(2) Is the space E weakly angelic?

Positive answers are known to (1), apart from the Šmulian theorem, for (DF)-spaces, H. Pfister [22], and dual metric spaces, M. Valdivia [28], p. 67 and [29]. Both problems for (LF)-spaces, posed by K. Floret in [13], has been recently solved by the authors. In [6], we give a positive answer to (1) for any countable inductive limit of metrizable spaces. In [21], the second author gives a positive answer to (2) in the same case as well as in dual metric spaces. The common structure that appears in the dual E' of a (LF)-space E suggests to us the introduction of a class $\mathfrak{G}$ of locally convex spaces for which (1) and (2) have positive answers.

The class $\mathfrak{G}$ contains the metrizable and dual metric spaces and it is stable taking subspaces, separated quotients, completions, countable direct sums and countable products. Further a compact space K is Talagrand-compact, [27] and [2], if and only if it is homeomorphic to a weakly compact subset of a locally convex space of the class $\mathfrak{G}$. Our results on metrizability are based upon a theorem of uniform spaces that we obtain in the second paragraph using K-analytic structures related with ordered families of compact subsets [5]. The results on weak angelicity are based upon a theorem about the angelic character of spaces of continuous functions with the pointwise convergence topology, given by the second author in [21], and lead us to an extended version of the Eberlein-Šmulian theorem.

We also give some applications to spaces of vector-valued continuous functions, to the general problem of retractivity in inductive limits and to the study of locally convex spaces with analytic duals.

All the topological spaces considered here will be Hausdorff. Whenever we work with spaces of continuous functions on a topological space it will be assumed to be completely regular. All the topological vector spaces (TVS) and all the locally convex spaces (LCS) will be defined over the field $\mathbb{K}$ of real or complex numbers.

We shall denote by $\mathbb{N}$ the set of positive integers endowed with the discrete topology and by $\mathbb{N}^{\mathbb{N}}$ the set of sequences of positive integers, $\alpha=(a_n)$, endowed with the product topology. In $\mathbb{N}^{\mathbb{N}}$ we consider the following relation of order $\leqq$, for $\alpha=(a_n)$ and $\beta=(b_n)$ in $\mathbb{N}^{\mathbb{N}}$ we say that $\alpha\leqq\beta$ if and only if $a_n\leqq b_n$ for every positive integer n. Standard references for notations and concepts are [11, 14] and [18].

2. A Result on Metrizability

This paragraph is devoted to a study of the metrizability of precompact subsets in some uniform spaces. As is well known, for a compact subset K of a topological space X the metrizability of K is equivalent to the separability of $C(K)$, the space of continuous functions on K endowed with the uniform convergence topology. This idea is the key to a proof of the following theorem, in which the separability of $C(K)$ is obtained using K-analytic structures. For the concept of K-analytic space we refer to [25].

Theorem 1. *Let $(X, \mathscr{U})$ be a uniform space and let us suppose that the uniformity $\mathscr{U}$ has a basis $\mathscr{B}=\{N_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ verifying the following condition:*

(a) *For any* α *and* β *in* $\mathbb{N}^{\mathbb{N}}$ *with* $\alpha \leqq \beta$ *we have that* $N_\beta \subset N_\alpha$.

Then the precompact subsets of $(X, \mathscr{U})$ *are metrizable in the induced uniformity.*

Proof. It will be enough to prove the result for the compact subsets of X because the corresponding result for precompact subsets is derived looking at the completion $(\tilde{K}, \tilde{\mathscr{U}})$, [18]. Let K be a compact subset of X and let us denote by $\| \ \|_\infty$ the supremum norm of $C(K)$. We are going to prove that $C(K)$ is K-analytic and therefore it is a Lindelöf and Banach space and so separable. Given $\alpha = (a_n)$ in $\mathbb{N}^{\mathbb{N}}$ we put $\alpha | n = (a_n, a_{n+1}, \ldots)$, $n = 1, 2, \ldots$, and we define the subset

$$A_\alpha = \{f \in C(K)\colon \ \|f\|_\infty \leqq a_1, \ |f(x) - f(y)| \leqq 1/n \text{ if } (x, y) \in (K \times K) \cap N_{\alpha|n}, \ n = 1, 2, \ldots\}$$

A_α is a uniformly bounded and uniformly equicontinuous subset of $C(K)$. The Ascoli theorem assures us that A_α is a compact subset of $C(K)$. On the other hand, the family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ has the following properties:

(i) $\bigcup \{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\} = C(K)$.

(ii) For any α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha \leqq \beta$ we have that $A_\alpha \subset A_\beta$.

Indeed, from the fact that every continuous function on K is bounded and uniformly continuous together with the order condition (a) on $\mathscr{B}$, the validity of condition (i) is obtained: Given $f \in C(K)$ there exist $M > 0$ and a sequence $(\alpha_k = (a_n^k))$ in $\mathbb{N}^{\mathbb{N}}$ such that $\|f\|_\infty \leqq M$ and $|f(x) - f(y)| \leqq 1/k$ for $(x, y) \in (K \times K) \cap N_{\alpha_k}$, $k = 1, 2, \ldots$. If we take $a_1 = \max\{a_1^1, M\}$ and $a_k = \max\{a_k^1, a_{k-1}^2, \ldots, a_1^k\}$, $k = 2, 3, \ldots$, for the sequence $\alpha = (a_k)$ we have that $f \in A_\alpha$. Property (ii) is a straightforward consequence of the order condition (a) required for $\mathscr{B}$. Using the family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ we are going to describe a K-analytic structure on $C(K)$.

For the positive integers $k, n_1, n_2, \ldots, n_k$ we write

$$C_{n_1 n_2 \ldots n_k} = \bigcup \{A_\alpha\colon \ \alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}, \ a_j = n_j, \ j = 1, 2, \ldots, k\}$$

and for every $\alpha = (n_k)$ in $\mathbb{N}^{\mathbb{N}}$ we put $B_\alpha = \bigcap_{k=1}^{\infty} C_{n_1 n_2 \ldots n_k}$. We obtain in this way a mapping B from $\mathbb{N}^{\mathbb{N}}$ into the family of all the parts of $C(K)$ that obviously satisfies $\bigcup \{B_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\} = C(K)$. The set-valued mapping B will be a K-analytic mapping in $C(K)$ if we can show that it is an upper semi-continuous compact (usco) set-valued mapping. To see this, it is enough to show that if (α_n) is a sequence in $\mathbb{N}^{\mathbb{N}}$ converging to α and f_n belongs to B_{α_n} for every positive integer n, then the sequence (f_n) has an adherent point in $C(K)$ belonging to B_α. This property is easily derived from Lemma A below and the proof is concluded. Q.E.D.

Lemma A. *Let* X *be a topological space with a family* $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *of compact subsets such that* $A_\alpha \subset A_\beta$ *whenever* $\alpha \leqq \beta$ *in* $\mathbb{N}^{\mathbb{N}}$. *Given positive integers* $k, a_1, a_2, \ldots, a_k$, *we write*

$$C_{a_1 a_2 \ldots a_k} = \bigcup \{A_\alpha\colon \ \alpha = (b_n) \in \mathbb{N}^{\mathbb{N}}, \ b_j = a_j, \ j = 1, 2, \ldots, k\}.$$

If $\alpha=(n_k)$ *belongs to* $\mathbb{N}^{\mathbb{N}}$ *and* $x_k \in C_{n_1 n_2 \dots n_k}$, $k=1,2,\dots$, *then the sequence* (x_k) *has an adherent point in X belonging to* $\bigcap_{k=1}^{\infty} C_{n_1 n_2 \dots n_k}$.

Proof. Given α and (x_k) as above there is a sequence $(\alpha_k=(a_n^k))$ in $\mathbb{N}^{\mathbb{N}}$ with $a_j^k=n_j$, $j=1,2,\dots,k$; $k=1,2,\dots$, such that $x_k \in A_{\alpha_k}$, $k=1,2,\dots$. For arbitrary positive integers m and n we put $b_n^m=\max\{a_n^k\colon k=m,m+1,\dots\}$, which is clearly finite, and $\beta_m=(b_n^m)$. We have that $b_j^m=n_j$, $j=1,2,\dots,m$ and $\beta_m \geqq \alpha_k$, $k=m,m+1,\dots$. It follows that $x_k \in A_{\beta_m}$, $k=m,m+1,\dots$, and so (x_k) has an adherent point in X belonging to $\bigcap_{m=1}^{\infty} A_{\beta_m}$ which is contained in $\bigcap_{k=1}^{\infty} C_{n_1 n_2 \dots n_k}$ and the proof is finished. Q.E.D.

Note. The former lemma has been used by the first author in a previous paper, [5], to prove, among other, the following result:

If X is an angelic topological space then X is K-analytic if and only if there is a family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *of compact subsets of X covering it such that* $A_\alpha \subset A_\beta$ *whenever* $\alpha \leqq \beta$ *in* $\mathbb{N}^{\mathbb{N}}$.

The usco mapping is defined by $\alpha=(a_n) \to \bigcap_{n=1}^{\infty} C_{a_1 a_2 \dots a_n}$.

At first glance the conditions required in Theorem 1 seem to be very technical, however they turn up *in many of the uniformities of topological vector spaces* as we shall see in the next paragraph. At the moment we give some applications for spaces of continuous functions. A good reference for strict topologies in spaces of bounded continuous functions is [31].

Corollary 1.1. *Let X be a topological space having a K-analytic and dense subspace. Then the following conditions are verified:*

(i) *The space of continuous functions on X endowed with the compact-open topology,* $C_c(X)$, *has its compact subsets metrizable.*

(ii) *The space of bounded continuous functions on X,* $C_b(X)$, *endowed with the compact-open topology has its compact subsets metrizable.*

(iii) *The space* $C_b(X)$ *endowed with whatever strict topology* β_p, β_t, β_τ, β_s *or* β_σ *has its compact subsets metrizable.*

Proof. (i) Let Y be a K-analytic and dense subspace of X. As Talagrand has shown, [27], Y can be written as the union of a family $\{K_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ where every K_α is compact and $K_\alpha \subset K_\beta$ whenever $\alpha \leqq \beta$ in $\mathbb{N}^{\mathbb{N}}$. Therefore $X=\overline{\bigcup\{K_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}}$ and the topology $\mathfrak{T}$, on $C(X)$, of uniform convergence on every K_α is a Hausdorff topology coarser than the compact-open topology on $C(X)$. The uniformity that describes the topology $\mathfrak{T}$ verifies the conditions of Theorem 1, and so its precompact subsets are metrizable and "a fortiori" the compact subsets of $C_c(X)$ are metrizable.

(ii) is a consequence of (i).

(iii) For the strict topologies on $C_b(X)$ we have the following relations

$$\beta_t \leqq \beta_\tau \leqq \beta_s \leqq \beta_\sigma \quad \text{and} \quad \beta_t \leqq \beta_p$$

β_t being finer than the compact open-topology, [31]. It is now clear that (iii) follows from (ii). Q.E.D.

3. Metrizability of Precompact Subsets in Locally Convex Spaces

We shall now use Theorem 1 from the previous section to provide large classes of spaces with metrizable precompact subsets in the framework of LCS.

The uniform structure of a LCS E is related to the filter basis of neighbourhoods of the origin in E. The neighbourhoods of the origin in E are related by polarity to the equicontinuous subsets of E'. So in order to get a LCS $E[\mathfrak{T}]$ that fulfills the hypothesis of Theorem 1 it seems to be reasonable to ask for the following structure in the dual E':

There is a family $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ of equicontinuous subsets of E' such that the following conditions are satisfied:

(a) $\bigcup\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}=E'$.

(b) *For any α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha\leqq\beta$ we have that $A_\alpha\subset A_\beta$.*

For such a space $E[\mathfrak{T}]$ its precompact subsets are metrizable.

Indeed, if we denote by $\mathfrak{T}'$ the topology on E of uniform convergence on every A_α we have that $\sigma(E,E')\leqq\mathfrak{T}'\leqq\mathfrak{T}$ and $E[\mathfrak{T}']$ fulfills the conditions of Theorem 1. So the $\mathfrak{T}'$-precompact subsets of E are metrizable and "a fortiori" the $\mathfrak{T}$-precompact subsets are also metrizable because $\mathfrak{T}$ and $\mathfrak{T}'$ agree on the $\mathfrak{T}$-precompact subsets of E, [18] § 28.5.(2).

There is a large class of LCS $E[\mathfrak{T}]$ which have the former structure, nevertheless in order to obtain a general result that includes many of the previous results on metrizability of compact subsets [6, 22, 29,...]/, we are going to enlarge this structure looking at the countable equicontinuous only:

Theorem 2. *Let $E[\mathfrak{T}]$ be a LCS with a family $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ of subsets of E' verifying the following conditions:*

(a) $\bigcup\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}=E'$.

(b) *For any α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha\leqq\beta$ we have that $A_\alpha\subset A_\beta$.*

(c) *For any α in $\mathbb{N}^{\mathbb{N}}$ the countable subsets of A_α are equicontinuous.*

Then the precompact subsets of $E[\mathfrak{T}]$ are metrizable.

Proof. Let $\mathfrak{T}'$ be the topology on E of uniform convergence on every A_α and $\mathfrak{T}^s$ the topology of uniform convergence on all the sequences contained in some A_α, $\alpha\in\mathbb{N}^{\mathbb{N}}$. We have that $\sigma(E,E')\leqq\mathfrak{T}^s\leqq\mathfrak{T}$ and $\sigma(E,E')\leqq\mathfrak{T}^s\leqq\mathfrak{T}'$, being the precompact subsets of $E[\mathfrak{T}']$ metrizable after Theorem 1. The proof will be finished if we can show that a precompact subset A in $E[\mathfrak{T}]$ is also precompact in $E[\mathfrak{T}']$ because in this case the topologies $\mathfrak{T}$, $\mathfrak{T}'$ and $\mathfrak{T}^s$ coincide on A, [18] § 28.5.(2). We have that A is precompact in $E[\mathfrak{T}^s]$. To see that A is precompact in $E[\mathfrak{T}']$ it will be enough to show that every sequence contained in A has a Cauchy subnet in $E[\mathfrak{T}']$, [18] § 5.6.(3). Because of the precompactness of A in $E[\mathfrak{T}^s]$, given a sequence $\{x_n\colon n\in\mathbb{N}\}$ in A there is a subnet of it, $\{x_\ell\colon \ell\in L, \geqq\}$, which is a $\mathfrak{T}^s$-Cauchy net. It is not difficult to prove that $\mathfrak{T}^s$ and $\mathfrak{T}'$ coincide on the countable subsets of E. Particularly, $\mathfrak{T}^s$ and $\mathfrak{T}'$ coincide on $M=\{x_\ell-x_{\ell'}\colon \ell,\ell'\in L\}$ and therefore $\{x_\ell\colon \ell\in L,\geqq\}$ is a $\mathfrak{T}'$-Cauchy net and the proof is concluded. Q.E.D.

We can give a picture of the possible applications of Theorem 2 with some examples:

Examples 1.2. *The following LCS $E[\mathfrak{T}]$ fulfill the conditions of Theorem 2, and so they have their precompact subsets metrizable:*

A) *The inductive limits, $E[\mathfrak{T}]=\varinjlim E_n[\mathfrak{T}_n]$, of increasing sequences of metrizable LCS.*

If $U_1^n \supset U_2^n \supset \ldots \supset U_j^n \supset \ldots$ is a fundamental system of neighbourhoods of the origin in $E_n[\mathfrak{T}_n]$ and for every $\alpha=(a_n)\in\mathbb{N}^{\mathbb{N}}$ we put $A_\alpha=\bigcap_{n=1}^{\infty}(U_{a_n})^0$, then the family $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ satisfies the conditions of Theorem 2. This structure has been used by the authors in a previous paper, [6].

B) *The generalized inductive limits, $E[\mathfrak{T}]=\varinjlim(E_n[\mathfrak{T}_n], A_n)$, of sequences of pairs $\{(E_n[\mathfrak{T}_n], A_n)\colon n=1,2,\ldots\}$ where every A_n is $\mathfrak{T}_n$-metrizable.*

It is not difficult to extend the former construction to this more general setting dealing with the system of neighbourhoods of the origin for generalized inductive limit topologies, [28] Chap. 1, 9. The first author has made use of this structure in a previous paper [4], to describe some properties of generalized inductive limit topologies.

C) *The (DF)-spaces $E[\mathfrak{T}]$*

If $B_1\subset B_2\subset\ldots\subset B_n\subset\ldots$ is a fundamental system of bounded subsets in $E[\mathfrak{T}]$ and for every $\alpha=(a_n)\in\mathbb{N}^{\mathbb{N}}$ we put $A_\alpha=\bigcap_{n=1}^{\infty}a_nB_n^0$, then the family $\{A_\alpha\colon \alpha\in \mathbb{N}^{\mathbb{N}}\}$ satisfies the conditions of Theorem 2. The metrizability of precompact subsets in (DF)-spaces has been proved by H. Pfister in [22] with different techniques.

D) *The dual metric spaces $E[\mathfrak{T}]$*

Let us recall that a dual metric space is a quasi-ℓ_∞-barrelled LCS, [17], with a fundamental sequence of bounded sets. The same construction of (DF)-spaces works for dual metric spaces. M. Valdivia proved in [29] that the precompact subsets of dual metric spaces are metrizable.

Note. The proof that Valdivia does to obtain that the precompact subsets of a dual metric space are metrizable is based on the following result ([28], p. 67):

Let $E[\mathfrak{T}]$ be a LCS such that its dual E' endowed with the topology of uniform convergence on the $\mathfrak{T}$-compact subsets of E, $\rho(E',E)$, is quasi-Suslin. Then the compact subsets of $E[\mathfrak{T}]$ are metrizable.

We can give a proof of this result of Valdivia using Theorem 1.

A topological space X is said to be quasi-Suslin, [28], if there is a mapping T from $\mathbb{N}^{\mathbb{N}}$ into the family of all the parts of X, $\mathscr{P}(X)$, such that:

(a) $\bigcup\{T_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}=X$.

(b) If α_n is a sequence in $\mathbb{N}^{\mathbb{N}}$ that converges to α and x_n belongs to T_{α_n} for every positive integer n, then the sequence (x_n) has an adherent point in X belonging to T_α.

If $T\colon \mathbb{N}^{\mathbb{N}}\to\mathscr{P}(E')$ is a quasi-Suslin mapping in $E'[\rho(E',E)]$, for every $\alpha\in\mathbb{N}^{\mathbb{N}}$ the set $A_\alpha=\bigcup\{T_\beta\colon \beta\in\mathbb{N}^{\mathbb{N}},\ \beta\leqq\alpha\}$ is $\rho(E',E)$-countably compact in E', [5] and

[21]. The family $\{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ covers E' and $A_\alpha \subset A_\beta$ whenever $\alpha \leqq \beta$ in $\mathbb{N}^{\mathbb{N}}$. Let $\mathfrak{T}'$ be the topology on E of uniform convergence on every A_α and $\mathfrak{T}^s$ the topology of uniform convergence on all the sequences contained in some A_α, $\alpha \in \mathbb{N}^{\mathbb{N}}$. We have that $\sigma(E, E') \leqq \mathfrak{T}^s \leqq \mathfrak{T}'$ and that for a compact subset A of $E[\mathfrak{T}]$ the topology induced by $\mathfrak{T}^s$ on A is coarser than the induced by $\mathfrak{T}$. - Let us observe that every sequence contained in some A_α is $\mathfrak{T}$-equicontinuous on A after Ascoli's theorem. Now, we can conclude that A is metrizable as we have done in Theorem 2: $\mathfrak{T}$, $\mathfrak{T}'$ and $\mathfrak{T}^s$ coincide on A and A is $\mathfrak{T}'$-metrizable ($\mathfrak{T}$-metrizable) after Theorem 1.

Other consequences can be derived from Theorems 1 and 2. M. Valdivia has described in [30] the class of quasi-LB spaces. A LCS $E[\mathfrak{T}]$ is said to be a quasi-LB space if there is a family $\{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of Banach discs in $E[\mathfrak{T}]$ such that (a) $\{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\} = E$ and (b) for any α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha \leqq \beta$ we have that $A_\alpha \subset A_\beta$.

Corollary 2.2. *The following LCS have their precompact subsets metrizable:*

(i) *The quasi-barrelled LCS, $E[\mathfrak{T}]$, with strong dual $E'[\beta(E', E)]$ quasi-LB space.*

(ii) *The quasi-ℓ_∞-barrelled LCS, $E[\mathfrak{T}]$, with strong dual $E'[\beta(E', E)]$ quasi-LB space.*

(iii) *The strong duals $E'[\beta(E', E)]$ of quasi-LB spaces $E[\mathfrak{T}]$.*

Proof. (i) and (ii) are straightforward consequences of Theorem 2.

(iii) Let us consider a quasi-LB structure $\{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of $E[\mathfrak{T}]$ and let $\mathfrak{T}'$ be the topology on E' of uniform convergence on every A_α. Using Theorem 1, the precompact subsets of $E'[\mathfrak{T}']$ are metrizable. On the other hand, $\beta(E', E)$ is finer than $\mathfrak{T}'$ and they agree on the $\beta(E', E)$-precompact subsets of E', [18] §28.5.(2). It is clear now that the precompact subsets of $E'[\beta(E', E)]$ are metrizable. Q.E.D.

Note. In [30], Valdivia has shown that a locally complete webbed LCS, see [8], is a quasi-LB space and that in a webbed LCS $E[\mathfrak{T}]$ there is a family of bounded subsets $\{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ such that $\bigcup\{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\} = E$ and $A_\alpha \subset A_\beta$ whenever $\alpha \leqq \beta$ in $\mathbb{N}^{\mathbb{N}}$. It clearly follows that the former corollary is also valid if we put webbed spaces instead of quasi-LB spaces.

The class of quasi-LB spaces is stable by closed subspaces, separated quotients, countable topological direct sums and countable topological products, [30]. If we denote the family of LCS verifying the conditions required in (i) (resp. (ii)) of Corollary 2.2 by $\mathscr{F}$ (resp. $\mathscr{F}'$), the stability properties of $\mathscr{F}$ and $\mathscr{F}'$ are good enough to enlarge some of the results gathered in the former examples and corollary: $\mathscr{F} \subset \mathscr{F}'$, the metrizable spaces belong to $\mathscr{F}$, the dual metric spaces belong to $\mathscr{F}'$, and both of the classes, $\mathscr{F}$ and $\mathscr{F}'$, are stable by closed subspaces of finite codimension, separated quotients, countable topological direct sums, countable topological products and completions. These stability properties are based upon the stability properties of quasi-barrelled, quasi ℓ_∞-barrelled, [17], and quasi-LB spaces, [30].

We give no proof of these properties because we are going to study a better class containing the former ones:

Definition 3. *Let $\mathfrak{G}$ be the class of LCS E that fulfill the conditions of Theorem 2. A family $\{A_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ in E′ verifying the conditions* (a), (b) *and* (c) *of Theorem 2 shall be called a $\mathfrak{G}$-representation of E in E′.*

$\mathfrak{G}$ is a large class that is made up of LCS where the precompact subsets are metrizable. *Further*, looking at the proof of Theorem 2 again *it can be asserted that for an space $E[\mathfrak{T}]\in\mathfrak{G}$, if $\mathfrak{T}^s$ is the topology on E of uniform convergence on all the equicontinuous sequences of E′, then the precompact subsets of $E[\mathfrak{T}^s]$ are metrizable.* – Let us observe that $\mathfrak{T}^s$ is strictly coarser than $\mathfrak{T}$ in general.

We are now going to deal with the stability properties of the class $\mathfrak{G}$ to reinforce the former results on metrizability of precompact subsets. Our proofs are inspired by the stability properties of quasi-LB spaces of M. Valdivia [30].

Proposition 4. *Let $\{E_n[\mathfrak{T}_n]: n=1,2,\ldots\}$ be a sequence of spaces of the class $\mathfrak{G}$ and $E[\mathfrak{T}]=\bigoplus\{E_n[\mathfrak{T}_n]: n=1,2,\ldots\}$ its locally convex direct sum. Then $E[\mathfrak{T}]$ belongs to $\mathfrak{G}$.*

Proof. For every positive integer n let $\{A^n_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of $E_n[\mathfrak{T}_n]$ in E'_n. Let φ be a one to one mapping from $\mathbb{N}$ onto $\mathbb{N}\times\mathbb{N}$. If $\alpha_j=(a_{j,n})\in\mathbb{N}^{\mathbb{N}}$, $j=1,2,\ldots$, we write $b_n=a_{\varphi(n)}$, $n=1,2,\ldots$ and $\psi\{(\alpha_j: j=1,2,\ldots)\}=(b_n)$. Then ψ is a one to one mapping from $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ onto $\mathbb{N}^{\mathbb{N}}$. If $\alpha\in\mathbb{N}^{\mathbb{N}}$ and $\{\alpha_j: j=1,2,\ldots\}=\psi^{-1}(\alpha)$ we set $A_\alpha=\prod\{A^j_{\alpha_j}: j=1,2,\ldots\}$. The family $\{A_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of E in $E'\simeq\prod\{E'_n: n=1,2,\ldots\}$, [18] § 22.5.(2) and § 22.5.(4). Q.E.D.

Proposition 5. *Let $\{E_n[\mathfrak{T}_n]: n=1,2,\ldots\}$ be a sequence of spaces of the class $\mathfrak{G}$ and $E[\mathfrak{T}]=\prod\{E_n[\mathfrak{T}_n]: n=1,2,\ldots\}$ its topological product. Then $E[\mathfrak{T}]$ belongs to the class $\mathfrak{G}$.*

Proof. For every positive integer n let $\{A^n_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of $E_n[\mathfrak{T}_n]$ in E'_n. Given $\alpha=(a_n)\in\mathbb{N}^{\mathbb{N}}$ we put $A_\alpha=A^1_\alpha\oplus A^2_\alpha\oplus\ldots\oplus A^{a_1}_\alpha$ and we obtain a family $\{A_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ in $E'\simeq\bigoplus\{E'_n: n=1,2,\ldots\}$, [18] § 22.5.(2), which is a $\mathfrak{G}$-representation of E in E', [18] § 22.5.(3). Q.E.D.

Proposition 6. *Let E be a space of the class $\mathfrak{G}$ and F a closed subspace of E. Then E/F belongs to the class $\mathfrak{G}$.*

Proof. The dual space $(E/F)'$ is isomorphic to $F^\perp$, the orthogonal subspace to F *in* E', and the equicontinuous subsets of $(E/F)'$ are identified with the equicontinuous subsets of E' which are contained in $F^\perp$, [18] § 22.1.(2). If $\{A_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of E in E', then $\{A_\alpha\cap F^\perp: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of (E/F) in $(E/F)'$. Q.E.D.

Proposition 7. *Let E be a space of the class $\mathfrak{G}$ and F any subspace of E. Then F belongs to the class $\mathfrak{G}$.*

Proof. If $\{A_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of E in E', then the restrictions to F $\{A_\alpha|_F: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation of F in F'. Q.E.D.

Proposition 8. *Let $E[\mathfrak{T}]$ be a space of the class $\mathfrak{G}$. Then the completion $\hat{E}[\hat{\mathfrak{T}}]$ belongs to the class $\mathfrak{G}$.*

Proof. $E[\mathfrak{T}]$ and its completion $\hat{E}[\hat{\mathfrak{T}}]$ have the same dual E' and the same equicontinuous subsets, [18] §21.4.(5). Therefore, a $\mathfrak{G}$-representation of E in E' is also a $\mathfrak{G}$-representation of $\hat{E}$ in E'. Q.E.D.

Usual corollaries for countable projective or inductive limits follow directly from the former propositions. As a application we give the following:

Proposition 9. *Let* $E[\mathfrak{T}]=\varinjlim E_n[\mathfrak{T}_n]$ *be the inductive limit of a sequence of spaces* $\{E_n[\mathfrak{T}_n]: n=1,2,\ldots\}$ *such that* $E'_n[\beta(E'_n,E_n)]\in\mathfrak{G}$, $n=1,2,\ldots$. *Then* $E'[\beta(E',E)]$ *has its precompact subsets metrizable.*

Proof. E' can be identified with a subspace of the product $\prod\{E'_n: n=1,2,\ldots\}$. The topological product $\prod E'_n[\beta(E'_n,E_n)]$ induces on E' a topology $\mathfrak{T}'$ such that $E'[\mathfrak{T}']\in\mathfrak{G}$. It follows that $\mathfrak{T}'\leqq\beta(E',E)$ and that $\beta(E',E)$ has a system of neighbourhoods of the origin which are $\mathfrak{T}'$-closed subsets. The result follows from Theorem 2 and [18] §28.5.(2). Q.E.D.

Examples 10. *There are LCS E such that* $E'=\bigcup\{A_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ *with* A_α *equicontinuous and having compact subsets which are not metrizable.*

Let us consider the Banach space ℓ^∞ and the dense subspace ℓ_0^∞ generated by the characteristic functions of subsets of $\mathbb{N}$. $\ell_0^\infty=\bigcup\{A_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ with A_α bounded and finite dimensional because the cardinal number of $\mathbb{N}^{\mathbb{N}}$ coincides with the cardinal number of the set of all finite subsets of the set of subsets of $\mathbb{N}$. However, there are $\sigma(\ell^{\infty\prime},\ell_0^\infty)$-compact subsets which are not $\sigma(\ell^{\infty\prime},\ell_0^\infty)$-metrizable: the unit closed ball of ℓ^∞ provides us with a set in such a situation, [14] p. 8.

Another example can be constructed using Hilbert spaces: if we take $\ell^2(\mathbb{N}^{\mathbb{N}})$ it is not difficult to show that the dense subspace F generated by a complete orthonormal system, $\{e_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$, is of the form $F=\bigcup\{A_\alpha: \alpha\in\mathbb{N}^{\mathbb{N}}\}$ with A_α bounded and finite dimensional. As before, there are $\sigma(\ell^2(\mathbb{N}^{\mathbb{N}}),F)$-compact subsets in $\ell^2(\mathbb{N}^{\mathbb{N}})$ which are not $\sigma(\ell^2(\mathbb{N}^{\mathbb{N}}),F)$-metrizable because $\ell^2(\mathbb{N}^{\mathbb{N}})$ is a non-separable Hilbert space.

4. On Angelic Spaces

A Hausdorff topological space S is angelic [14], if the closure of every relatively countably compact subset A of S is compact and consists precisely of the limits of sequences from A. If the closure of every relatively countably compact subset of a space S is metrizable, then S is an angelic space. Therefore, all the uniform and locally convex spaces studied in the previous sections are angelic spaces. *Particularly, our class* $\mathfrak{G}$ *is a wide and very stable class of angelic LCS.*

The strongest results about the angelic character of a given space S are those that give the angelicity for coarser topologies than the original one. Working with a common structure in the dual E' of a LCS E, more general than the $\mathfrak{G}$-representations, we shall simultaneously derive the metrizability of the compact subsets of E and the angelic character of $E[\sigma(E,E')]$:

Theorem 11. *Let $E[\mathfrak{T}]$ be a LCS and $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ a family of subsets of E' verifying the following conditions:*

(a) $\bigcup\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ *is total in* $E'[\sigma(E',E)]$.

(b) *For α and β in $\mathbb{N}^{\mathbb{N}}$ such that $\alpha\leqq\beta$ we have that $A_\alpha\subset A_\beta$.*

(c) *For any α in $\mathbb{N}^{\mathbb{N}}$ the countable subsets of A_α are equicontinuous.*

Then we have that:

(i) *The closure of every relatively countably compact subset of $E[\mathfrak{T}]$ is metrizable.*

(ii) *$E[\sigma(E,E')]$ is an angelic space.*

Proof. (i) Proceeding as we have done in Theorem 2 we obtain that the precompact subsets of $E[\mathfrak{T}^s]$ are metrizable where we are denoting by $\mathfrak{T}^s$ the topology on E of uniform convergence on all the sequences contained in some A_α, $\alpha\in\mathbb{N}^{\mathbb{N}}$. The Hausdorff topology $\mathfrak{T}^s$ is clearly coarser than $\mathfrak{T}$. Let A be a relatively countably compact subset of $E[\mathfrak{T}]$ and $\Phi\colon E[\mathfrak{T}]\to E[\mathfrak{T}^s]$ the identity mapping. $\Phi(A)$ is a precompact subset of $E[\mathfrak{T}^s]$ and thus its closure $\overline{\Phi(A)}$ is metrizable. Φ is continuous and injective, so we can apply "the angelic lemma", [14] p. 28, and we conclude that $\Phi(\bar{A})$ is closed and $\Phi|_{\bar{A}}$ is a homeomorphism. The conclusion follows from this fact.

(ii) We consider $X=\overline{\bigcup\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}}$ endowed with the topology induced by $\sigma(E',E)$. Every A_α is relatively countably compact in X. Let $C_P(X)$ be the space of continuous functions on X endowed with the pointwise convergence topology. The second author has shown in [21] that for such a space X the space $C_P(X)$ is angelic. Now, the restriction mapping $E[\sigma(E,E')]\to C_P(X)$ is continuous and injective, and an application of "the angelic lemma" gives us the conclusion. Q.E.D.

As a consequence of Theorem 11, every space of the class $\mathfrak{G}$ is weakly angelic. The good stability properties of the class $\mathfrak{G}$ reinforce the recent answers given by the second author in [21] to the problems of weak angelic character of (LF)-spaces, dual metric spaces, ...

Note. The angelic character of spaces $C_P(X)$ is proved in [21] for a class of topological spaces X that we call web-compact spaces. A space X is web-compact if and only if there is a metrizable and separable space P together with a mapping T from P into the set of all the parts of X such that: $\bigcup\{Tx\colon x\in P\}$ is dense in X and $\bigcup\{Tx_n\colon n=1,2,\ldots\}$ is relatively countably compact in X whenever (x_n) is a convergent sequence in X. Particularly, K-analytic, countable determined [27], quasi-Suslin and every space X with a dense and ordered family $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ of relatively countably compact subsets are web-compact spaces. The proof is based upon an argument of accessibility by sequences that extends previous results by J.D. Pryce [24], M. De Wilde [7] and K. Floret [14] which only work for spaces X with a dense σ-relatively countably compact subset.

We continue dealing with spaces of continuous functions, but now, we are going to study vector-valued continuous functions. Given a topological space X and a LCS E we denote by $C(X,E)$ (resp. $C_b(X,E)$) all continuous (resp.

bounded continuous) functions from X into E. We shall use the strict topologies in the vectorial case, β_0, β_1, β'_1, β, β' and β'_∞ as Khurana does in [19]. It should be noted that if E is the field $\mathbb{K}$ then we have that $\beta_0=\beta_t$, $\beta=\beta'=\beta_\tau$, $\beta'_\infty=\beta_\infty$ and $\beta_1=\beta'_1=\beta_\sigma$ with the notations of Wheeler [31] that we have used in Corollary 1.1.

Corollary 1.11. *Let X be a topological space with a dense K-analytic subspace and E a LCS of the class $\mathfrak{G}$. Then the following statements are verified:*

(i) *$C(X,E)$ endowed with the compact-open topology has metrizable compact subsets and with the weak topology is an angelic space.*

(ii) *$C_b(X,E)$ endowed with the compact-open topology has metrizable compact subsets and with the weak topology is an angelic space.*

(iii) *$C_b(X,E)$ endowed with any strict topology β_0, β, β', β'_∞, β_1 or β'_1 has metrizable compact subsets and endowed with their corresponding weak topology is an angelic space.*

Proof. (i) Let $\{K_\alpha: \alpha\in\mathbb{N}^\mathbb{N}\}$ be a family of compact subsets of X such that $K_\alpha \subset K_\beta$ whenever $\alpha\leqq\beta$ in $\mathbb{N}^\mathbb{N}$ and $X=\overline{\bigcup\{K_\alpha: \alpha\in\mathbb{N}^\mathbb{N}\}}$. We abridge $G=C(X,E)$ endowed with the compact-open topology, $\mathfrak{K}$, and we put G' for its dual space. For every $s\in X$ and $x'\in E'$ the mapping $\delta_{s,x'}$: $C(X,E)\to\mathbb{K}$ given by $\delta_{s,x'}(g)=x'g(s)$, $g\in G$, belongs to G'. For every $\alpha\in\mathbb{N}^\mathbb{N}$ we define $C_\alpha=\{\delta_{s,x'}: s\in K_\alpha, x'\in A_\alpha\}$. It is clear that $C_\alpha\subset C_\beta$ for every $\alpha\leqq\beta$ in $\mathbb{N}^\mathbb{N}$. It is not difficult to show that the countable subsets of every C_α are $\mathfrak{K}$-equicontinuous. On the other hand, for g belonging to G such that $\delta_{s,x'}(g)=0$ for every $\delta_{s,x'}\in\bigcup\{C_\alpha: \alpha\in\mathbb{N}^\mathbb{N}\}$ we have that $g=0$, and so $\bigcup\{C_\alpha: \alpha\in\mathbb{N}^\mathbb{N}\}$ is total in $G'[\sigma(G',G)]$. Now, the conclusion follows from Theorem 11.

(ii) is a consequence of (i) because the metrizability of compact subsets and the angelic character is inherated by subspaces.

(iii) For the strict topologies on $C_b(X,E)$ we have that $\beta\geqq\beta'\geqq\beta_0$, $\beta_1\geqq\beta'_1\geqq\beta_0$, $\beta'_1\geqq\beta'_\infty\geqq\beta'$, β_0 being finer than the compact-open topology on $C_b(X,E)$, [19]. Now, (iii) follows from (ii) if we take into account that the metrizability of compact subsets and the angelic character are preserved by refining the topologies. Q.E.D.

5. Talagrand-Compact spaces and the Class $\mathfrak{G}$

As we have seen in the previous sections the spaces of the class $\mathfrak{G}$ are good spaces from the viewpoint of compactness: they have metrizable precompact subsets and they are weakly angelic. In this paragraph we are going to connect the weakly compact subsets of spaces of the class $\mathfrak{G}$ with the class of Talagrand-compact spaces. A compact space K is said to be a Talagrand-compact (or a compact of type $\mathscr{E}_1$) if the space $C(K)$ is a weakly K-analytic Banach space [27]. Every compact space such that $C(K)$ is weakly compactly generated (WCG) is Talagrand-compact [27]. The fact that for a compact space K the space $C(K)$ is WCG if and only if K is Eberlein compact, that is, if and only if K is homeomorphic to a weakly compact subset of a Banach space, was shown by Amir and Lindestrauss [1]. We can characterize the Talagrand-compact spaces as the weakly compact subsets of LCS of the class $\mathfrak{G}$:

Theorem 12. *For a compact space K they are equivalent:*

(i) *K is Talagrand-compact.*

(ii) *K is homeomorphic to a weakly compact subset of a LCS $E[\mathfrak{T}]$ of the class $\mathfrak{G}$.*

Proof. (ii)$\Rightarrow$(i) Let $E[\mathfrak{T}]$ be a LCS belonging to $\mathfrak{G}$ and K a compact subset of $E[\sigma(E,E')]$. To see that K is Talagrand-compact we proceed in the following way: Let $\{A_\alpha: \alpha\in\mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-representation of E in E' and we put $B_\alpha=\{x'|_K: x'\in A_\alpha\}$ for every $\alpha\in\mathbb{N}^\mathbb{N}$. It is clear enough that B_α is contained in $C(K)$ and $B_\alpha \subset B_\beta$ for every $\alpha\leqq\beta$ in $\mathbb{N}^\mathbb{N}$. On the other hand every B_α is a relatively countably compact subset of $C_P(K)$ and so a relatively compact subset because of Grothendieck's theorem [15]. If we consider the closure $\overline{B_\alpha}$ of B_α in $C_P(K)$, then the union of the ordered family of compact subsets $\{\overline{B_\alpha}: \alpha\in\mathbb{N}^\mathbb{N}\}$ separates the points of K and we obtain that K is Talagrand-compact applying Proposition 6.13 of [27].

(i)$\Rightarrow$(ii) If K is a Talagrand-compact, then $C(K)$ is weakly K-analytic. Particularly, according to [27] there is a family $\{A_\alpha: \alpha\in\mathbb{N}^\mathbb{N}\}$ of weakly compact subsets of $C(K)$ such that $C(K)=\bigcup\{A_\alpha: \alpha\in\mathbb{N}^\mathbb{N}\}$ and $A_\alpha\subset A_\beta$ whenever $\alpha\leqq\beta$ in $\mathbb{N}^\mathbb{N}$. Using Krein's theorem, [18] § 28.5.(4'), we can assume that every A_α is absolutely convex and weakly compact in $C(K)$. If $M(K)$ is the space of Radon measures on K, the dual space of $C(K)$, and $\mathfrak{T}$ is the topology on $M(K)$ of uniform convergence on every A_α, then $M(K)[\mathfrak{T}]$ belongs to $\mathfrak{G}$ and for its dual space we have that $M(K)[\mathfrak{T}]'=C(K)$. It now follows that K is a weakly compact subset of $M(K)[\mathfrak{T}]$ and the proof is finished. Q.E.D.

The weakly compact subsets of LCS of the class $\mathfrak{G}$ have the good properties of the Talagrand-compact spaces.

Corollary 1.12. *Let $E[\mathfrak{T}]$ be a LCS of the class $\mathfrak{G}$. For every weakly compact subset K of E the weight of K is equal to the density character of K. Particularly, K is metrizable if and only if K is separable.*

Proof. It is a consequence of Theorem 12 and Théorème 6.2 of [27]. Q.E.D.

Talagrand's proof of Théorème 6.2, [27], is based upon his Théorème 6.1, [27], that assures for a weakly countably determined Banach space E the equality of its density character with that of its weak-* dual $E'[\sigma(E',E)]$. Dealing with the class $\mathfrak{G}$ we have the following:

Theorem 13. *Let $E[\mathfrak{T}]$ be a LCS of the class $\mathfrak{G}$ which is weakly countably determined. Then the density character of $E[\mathfrak{T}]$ is equal to the density character of $E'[\sigma(E',E)]$.*

Proof. For a set B we denote by $|B|$ its cardinal number.

Let B be a dense subset in $E[\sigma(E,E')]$. The second author has proved that the space $E'[\sigma(E',E)]$ is angelic (Theorem 6 [21]). If we bear in mind the note that follows Lemma A we obtain that $E'[\sigma(E',E)]$ is a K-analytic space using the $\mathfrak{G}$-representation of E in E'. Let $\sigma(E',B)$ the topology on E' of pointwise convergence on B. Applying the Théorème 2.4 of [27] we can be sure that $E'[\sigma(E',E)]$ has a dense subset with cardinality less than or equal to $|B|$.

Conversely, if B is a dense subset of $E'[\sigma(E',E)]$, using the topology $\sigma(E,B)$ on E of pointwise convergence on B and the fact that $E[\sigma(E,E')]$ is countably determined another application of Théorème 2.4 of [27] gives us a dense subset of $E[\sigma(E,E')]$ with cardinality less than or equal to $|B|$ and so the proof is finished. Q.E.D.

In the separable case the former ideas lead us to the following:

Theorem 14. *Let $E[\mathfrak{T}]$ be a LCS of the class $\mathfrak{G}$. If $E[\mathfrak{T}]$ is separable, then $E'[\sigma(E',E)]$ is separable.*

Proof. Let N be a countable and dense subset of $E[\mathfrak{T}]$. The topology $\sigma(E',N)$ on E' is metrizable and coarser than $\sigma(E',E)$ and so $E'[\sigma(E',E)]$ is angelic. The note that follows Lemma A assures us that $E'[\sigma(E',E)]$ is a K-analytic space. The Théorème 2.4 of [27] assures us that $E'[\sigma(E',E)]$ is separable. Q.E.D.

A LCS E with $E'[\sigma(E',E)]$ separable is realcompact for its weak topology [28] p. 137, *and so every separable LCS E of the class $\mathfrak{G}$ is weakly realcompact.*

Corollary 1.14. *Let $E[\mathfrak{T}]$ be a LCS of the class $\mathfrak{G}$ and K a weakly compact subset of E. If K is contained in a separable subspace of E then K is metrizable.*

Proof. Let F be a separable subspace of E containing K. F belongs to $\mathfrak{G}$ after Proposition 7 and so $F'[\sigma(F',F)]$ is separable by Theorem 14. Šmulian's theorem gives us the result. Q.E.D.

We can go further and say that for a separable space $E[\mathfrak{T}]$ of the class $\mathfrak{G}$ its weak-* dual $E'[\sigma(E',E)]$ is analytic. We need the following previous:

Theorem 15. *Let X be a submetrizable topological space. The following statements are equivalent:*

(i) *X is analytic.*

(ii) *There is a family $\{K_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ of compact subsets of X covering it and such that $K_\alpha\subset K_\beta$ whenever $\alpha\leqq\beta$ in $\mathbb{N}^{\mathbb{N}}$.*

Proof. (i)$\Rightarrow$(ii) Every analytic space is a continuous image of $\mathbb{N}^{\mathbb{N}}$. Given $\alpha\in\mathbb{N}^{\mathbb{N}}$ the subset $\{\beta\in\mathbb{N}^{\mathbb{N}}\colon \beta\leqq\alpha\}$ is compact in $\mathbb{N}^{\mathbb{N}}$ and its image is the compact K_α required in (i).

(ii)$\Rightarrow$(i) Let d be a metric on X such that its associated topology is coarser than the original one on X. Taking into account the note that follows Lemma A, the space X is K-analytic and so (i) is satisfied after Theorem 5.5.1 of [25]. Q.E.D.

Let us remark that an analytic space is submetrizable whenever it is regular and so the condition of submetrizability on X can not be omitted in Theorem 15, [25] 5.5.1. For a LCS $E[\mathfrak{T}]$ we denote by $\mathfrak{T}_{pc}(E',E)$ the topology of uniform convergence on all the precompact subsets of $E[\mathfrak{T}]$.

Corollary 1.15. *Let $E[\mathfrak{T}]$ be a separable LCS of the class $\mathfrak{G}$. Then $E'[\mathfrak{T}_{pc}(E',E)]$ is analytic and in particular $E'[\sigma(E',E)]$ is analytic.*

Proof. The separability of E implies that $E'[\mathfrak{T}_{pc}(E',E)]$ is submetrizable. The $\mathfrak{G}$-representation of E in E' gives us a family $\{A_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ of countable equicontinuous subsets of E' such that $A_\alpha\subset A_\beta$ whenever $\alpha\leqq\beta$ in $\mathbb{N}^{\mathbb{N}}$. Every A_α is relatively countably compact in $E'[\mathfrak{T}_{pc}(E',E)]$. The angelic character of $E'[\mathfrak{T}_{pc}(E',E)]$ provides us with the situation of Theorem 15, hence the result. Q.E.D.

The former corollary contains as particular cases some of the results proved by M. De Wilde in [8], VII.2, and of course new cases of applications because of the good stability properties of the class $\mathfrak{G}$.

6. Regularity Conditions in Inductive Limits

The regularity and retractivity conditions have been studied by different authors: K. Floret introduces the notion of sequentially retractive inductive limit in [12], Bierstedt and Meise introduce the compact-regular inductive limits in [3] and H. Neus gives the definition of sequentially compact-regular inductive limit in [20] as well as deep results on the equivalence between the different notions of regularity and retractivity for inductive limits of sequences of normed spaces. In [28], M. Valdivia has studied the former properties and other related ones for the weak topology of inductive limits of normed spaces. Recently, the authors have proved in [6] that for an inductive limit of an increasing sequence of metrizable LCS the conditions of sequentially retractive, sequentially compact-regular, compact-regular and precompactly retractive [6], are equivalent. Other results in this context for generalized inductive limits have been obtained by the first author in [4].

The purpose of this short paragraph is to obtain the former equivalences in the general case of inductive limits of spaces of the class $\mathfrak{G}$, and thus – in some way – give a certain answer to K. Floret, who expounds in [13] that it is desirable to study this kind of results in general cases.

Theorem 16. *If $E[\mathfrak{T}]=\varinjlim E_n[\mathfrak{T}_n]$ is an inductive limit of an increasing sequence of subspaces $E_n[\mathfrak{T}_n]$ belonging to the class $\mathfrak{G}$, then the following statements are equivalent:*

(i) *$E[\mathfrak{T}]$ is sequentially retractive.*
(ii) *$E[\mathfrak{T}]$ is sequentially compact-regular.*
(iii) *$E[\mathfrak{T}]$ is compact-regular.*
(iv) *$E[\mathfrak{T}]$ is precompactly retractive.*

If every $E_n[\mathfrak{T}_n]$ is complete, the former conditions are also equivalent to the following:

(v) *For every precompact subset A of $E[\mathfrak{T}]$ there is a positive integer n such that A is contained in $E_n[\mathfrak{T}_n]$ and it is precompact in this space.*

(vi) $E'[\mathfrak{T}_{pc}(E',E)]$ *is a closed subspace of the topological product* $\prod_{n=1}^{\infty} E'_n[\mathfrak{T}_{pc}(E'_n, E_n)]$.

Proof. The stability properties of the class $\mathfrak{G}$ give us that the space $E[\mathfrak{T}]$ belongs to $\mathfrak{G}$, and therefore its precompact subsets are metrizable. Based upon the metrizability of precompact subsets, the same proof that we have given in Theorem 3 and Corollary 1.3 of [6] works in this case, and so we obtain that the conditions (i), (ii), (iii), (iv) and (v) are equivalent.

(v)$\Rightarrow$(vi) The space E' is identified with a closed subspace of $\prod_{n=1}^{\infty} E'_n[\mathfrak{T}_{pc}(E'_n, E_n)]$ through the restriction mapping $u \to (u|_{E_1}, u|_{E_2}, \ldots, u|_{E_n}, \ldots)$. The topology induced by $\prod_{n=1}^{\infty} E'_n[\mathfrak{T}_{pc}(E'_n, E_n)]$ on E' is the topology of uniform convergence on the located precompact and so (vi) follows from (v).

(vi)$\Rightarrow$(v) If we assume that (vi) is satisfied, then for every precompact subset A of $E[\mathfrak{T}]$ there is a precompact subset B in some $E_n[\mathfrak{T}_n]$ such that $B^0 \subset A^0$, where the polars have been taken in $\langle E, E' \rangle$. Using the bipolar theorem, A is contained in the closed absolutely convex hull of B which is again a precompact subset of $E_n[\mathfrak{T}_n]$ because this space is complete, hence the result. Q.E.D.

Note. Using the filters lemma of Grothendieck, [16] p. 107 Lemme 7, the following can be shown:

Let $E[\mathfrak{T}]$ be a LCS of the class $\mathfrak{G}$ and let $\{E_n[\mathfrak{T}_n]: n=1,2,\ldots\}$ be an increasing sequence of subspaces of E covering it such that the topologies induced by $\mathfrak{T}$ and $\mathfrak{T}_{n+1}$ on E_n are coarser than $\mathfrak{T}_n$, $n=1,2,\ldots$. Then they are equivalent:

(i) *For every sequence (x_m) in $E[\mathfrak{T}]$ which is convergent to the origin, there is a positive integer n such that (x_m) is contained in E_n and converges to the origin in $E_n[\mathfrak{T}_n]$.*

(ii) *For every precompact subset A of $E[\mathfrak{T}]$ there is a positive integer n such that A is contained in $E_n[\mathfrak{T}_n]$ and the topologies $\mathfrak{T}$ and $\mathfrak{T}_n$ coincide on A.*

This viewpoint unifies regularity and retractivity properties for inductive limits (as Theorem 16) with the Mackey convergence properties. For instance, for a dual metric space $E[\mathfrak{T}]$, that belongs to $\mathfrak{G}$, the equivalence between (i) and (ii) tells us that the Mackey convergence property in $E[\mathfrak{T}]$ and the strict Mackey convergence property for precompacts are equivalent (for the definitions see [16]). These properties and other have been studied by the first author [4], in the more general case of generalized inductive systems.

For the weak topologies, the same proof that we have given in [6] for an inductive limit of metrizable subspaces is also valid to obtain the following:

Theorem 17. *Let $E[\mathfrak{T}] = \varinjlim E_n[\mathfrak{T}_n]$ be an inductive limit of an increasing sequence of subspaces $E_n[\mathfrak{T}_n]$ belonging to the class $\mathfrak{G}$. The following two conditions are equivalent:*

(i) *For every sequence (x_m) in $E[\mathfrak{T}]$ which is convergent to the origin, there is a positive integer n such that (x_m) is contained in E_n and converges to the origin in $\sigma(E_n, E'_n)$.*

(ii) *For every precompact subset A of $E[\mathfrak{T}]$ there is a positive integer n such that A is contained in E_n and $\sigma(E, E')$ coincides with $\sigma(E_n, E'_n)$ on A.*

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Note Added in Proof

After the preparation of this paper we have shown the following: If X is a web-compact topological space where the relatively countably compact subsets are also relatively compact, then X contains a dense and countably determined subspace. As a consequence, if X is a web compact space, E is a LCS of the class $\mathfrak{G}$, then the compact subsets of $C_p(X, E)$ are always Gul'ko compact spaces and that space is also weakly angelic. These results will be appear elsewhere.