A coercive James's weak compactness theorem and nonlinear variational problems

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Abstract

We discuss a perturbed version of James's sup theorem for weak compactness that not only properly generalizes that classical statement, but also some recent extensions of this central result: the sublevel sets of an extended real–valued and coercive function whose subdifferential is surjective are relatively weakly compact. Furthermore, we apply it to generalize and unify some facts in mathematical finance and to prove that the unique possible framework in the development of an existence theory for a wide class of nonlinear variational problems is the reflexive one.

Keywords: James's theorem, weak compactness, coercive functions, subdifferential, nonlinear variational problems, mathematical finance.

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1 Introduction

The two most important results about weak compactness in a Banach space are Eberlein–Smuliam's and James's sup theorems. This latter asserts that a weakly closed subset A of a real Banach space E is weakly compact provided that each continuous and linear functional on E attains its supremum on A. In the last years, a few generalizations of James's sup theorem have appeared, some motivated by its use in mathematical finance, in which the linear optimization condition is replaced by another one of a perturbed nature; that is, for a fixed and adequate extended real–valued function f, $x^* - f$ attains its supremum, where x^* is any continuous and linear functional on E.

The first of these results deals with a specific subset of the space, its closed unit ball. Inspired by the fact that the set of norm attaining functionals in a real Banach space is not more than the range of the duality mapping, which in turn is the range of the subdifferential of a certain coercive, convex and lower semicontinuous function, B. Calvert and S. Fitzpatrick announced in [6, 10] that a real Banach space is reflexive whenever its dual space coincides with the range of an extended real–valued coercive, convex and lower semicontinuous function whose effective domain has nonempty norm–interior. However, the erratum [6] makes [10] more difficult to follow, since the main addendum requires correcting non–written proofs of some statements in [10] which are adapted from [15].

Subsequently, and for arbitrary subsets, in [24] it was proven that a closed and convex subset A of a real Banach space E is weakly compact each time there exists a bounded function $f : A \longrightarrow \mathbb{R}$ such that for all continuous and linear functional x^* on E, the function $x^*_{|A} - f$ attains its supremum on A. Finally, [18] contains another James's type result, but for a concrete class of Banach spaces and also under a certain boundedness assumption: if E is a separable real Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and lower semicontinuous function whose effective domain is bounded, and such that for all continuous and linear functional x^* on E, the function $x^* - f$ attains its supremum, then its sublevel sets are weakly compact. This same statement has been shown by F. Delbaen in [9] for a concrete nonseparable space of integrable functions.

In this paper we introduce a new version of James's theorem that generalizes all these results: if E is a real Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a coercive function with subdifferential onto, then its sublevel sets are relatively weakly compact. As a consequence, we extend some well–known results in mathematical finance and prove that if for a real Banach space a suitable abstract optimization problem –which includes lots of nonlinear or nonsmooth variational ones that arise in connection with numerous applied problems, in the theory of partial differential equations and many other areas of pure and applied mathematics– admits a solution, then it is reflexive.

The organization of the paper is as follows. Section 2 is concerned with introducing some basic notions in extended real–valued functions –in particular that of coercivity, which appears with diverse meanings in the literature–, establishing the mentioned version of James's theorem for coercive functions whose subdifferential is surjective, showing a topological property of the epigraph of such functions, and deducing the known results. Section 3 deals with applying our results to mathematical finance. Finally, in Section 4 we prove that reflexivity is the natural context where a variety of nonlinear variational results can be developed.

2 James's theorem for coercive functions

First of all, we give a brief review of elementary notions related to extended real–valued functions. Let E be a real Banach space. We denote its topological dual space by E^* and its closed unit ball by B_E. If $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a function, let us write $\mathrm{dom}(f)$ for its *effective domain*, that is,

$$\mathrm{dom}(f) := \{x \in E : f(x) < +\infty\}.$$

Given $x_0 \in E$, $\partial f(x_0)$ stands for the *subdifferential* of f at x_0, i.e., the subset of E^*

$$\{x^* \in E^* : \text{for all } x \in E, \quad x^*(x) - f(x) \leq x^*(x_0) - f(x_0)\}$$

if $x_0 \in \mathrm{dom}(f)$, while for $x_0 \notin \mathrm{dom}(f)$ it is simply $\emptyset$. Under additional assumptions of convexity and lower semicontinuity, some results as the Brønsted–Rockafellar theorem ([20, Theorem 3.17]) or [4, Theorem 4.2.8] guarantee that for a certain $x \in \mathrm{dom}(f)$, $\partial f(x) \neq \emptyset$. The *range of the subdifferential* of f,

$$\{x^* \in E^* : \text{there exists } x \in E \text{ with } x^* \in \partial f(x)\},$$

is denoted by $\partial f(E)$, and for a subset B of E^* we write

$$(\partial f)^{-1}(B) := \{x \in E : \text{ there exists } x^* \in B \text{ such that } x^* \in \partial f(x)\}.$$

Finally, the function f is said to be *proper* when its effective domain is nonempty, and *coercive* provided that

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

Obviously, such is the case when $\mathrm{dom}(f)$ is a bounded subset of E. It follows from the Brønsted–Rockafellar theorem that if f is coercive, then its subdifferential is large: $\partial f(E)$ is norm–dense in E^* (see [22, Theorem 2.3]). In our statements a stronger condition is assumed, that $\partial f(E) = E^*$, which is a perturbed optimization condition, since it is equivalent to the assertion

$$\text{for all } x^* \in E^*, \quad x^* - f \text{ attains its supremum on } E.$$

Theorem 1 below focuses the main efforts that we must make in order to prove Theorem 2, which is our most important result. Before stating it, we present this easy result:

Lemma 1. *If E is a real Banach space, $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a function such that*

$$\textit{for all } x^* \in E^*, \quad x^* - f \textit{ is bounded above}$$

and A is a nonempty subset of E with $f(A)$ bounded above, then A is bounded.

Proof. To prove that A is bounded it suffices to check, in view of the uniform boundedness principle, that for all $x^* \in E^*$ the set $\{x^*(a) : a \in A\}$ is bounded above. Hence, let us consider a functional $x_0^* \in E^*$ for which, by hypothesis, there exists $\alpha \in \mathbb{R}$ such that

$$\text{for all } x \in E, \quad x_0^*(x) - f(x) \leq \alpha.$$

In particular,

$$\text{for all } a \in A, \quad x_0^*(a) \leq f(a) + \alpha,$$

and since $f(A)$ is bounded above, then the set $\{x_0^*(a) : a \in A\}$ is also bounded above. □

The following notation will be used in the proof of Theorem 1: given a bounded sequence $\{x_n\}_{n\geq 1}$ in a real Banach space E, we write $\mathrm{co}_\sigma\{x_n : n \geq 1\}$ for the bounded subset of E

$$\left\{ \sum_{n=1}^{+\infty} \lambda_n x_n : \text{ for all } n \geq 1, \ \lambda_n \geq 0 \text{ and } \sum_{n=1}^{+\infty} \lambda_n = 1 \right\}.$$

Theorem 1. *Let E be a real Banach space, let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function and suppose that*

$$\partial f(E) = E^*$$

and that

$$\text{for all } \rho > 0, \quad (\partial f)^{-1}(\rho B_{E^*}) \text{ is bounded.}$$

Then

$$\text{for all } c \in \mathbb{R}, \quad f^{-1}((-\infty, c]) \text{ is relatively weakly compact.}$$

Proof. Firstly, since f is a bounded below function, equivalently $0 \in \partial f(E)$, we assume without loss of generality that $f > 0$ on E. We proceed by *reductio ad absurdum*, so let us suppose that for some $c > 0$ the sublevel set $f^{-1}((-\infty, c])$ is not relatively weakly compact. Taking into account that $f^{-1}((-\infty, c])$ is bounded, in view of Lemma 1, then Eberlein–Smulian's and Alaoglu's theorems guarantee the existence of x_0^{**} in $\overline{f^{-1}((-\infty, c])}^{w(E^{**}, E^*)} \backslash E$ and a sequence $\{x_n\}_{n\geq 1}$ in $f^{-1}((-\infty, c])$ in such a way that x_0^{**} is a $w(E^{**}, E^*)-$cluster point of $\{x_n\}_{n\geq 1}$. Let

$$\alpha := \mathrm{dist}(x_0^{**}, E) > 0$$

and $\lambda > 0$ such that

$$\lambda c < \frac{\alpha}{4}. \tag{1}$$

Let us define

$$\rho := \frac{2}{\lambda} \tag{2}$$

and, making use of the hypothesis, let $M > 0$ with

$$(\partial f)^{-1}(\rho B_{E^*}) \subset M B_E. \tag{3}$$

We have that $\partial f(E) \neq \emptyset$, which in particular says that f is proper. So, let $x_0 \in \mathrm{dom}(f)$, let

$$c_0 := \max\{c, f(x_0) + \rho(M + \|x_0\|)\} \tag{4}$$

and let X be the set

$$X := f^{-1}((-\infty, c_0]),$$

which is bounded thanks to Lemma 1. Moreover, we know by the Hahn–Banach theorem that there exists $x^{***} \in B_{E^{***}}$ such that

$$x^{***}(x_0^{**}) = \alpha$$

and

$$\text{for all } x \in E, \quad x^{***}(x) = 0.$$

Now we derive from Goldstine's theorem the existence of a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} satisfying

$$\text{for all } p \geq 1, \quad \lim_n x_n^*(x_p) = x^{***}(x_p) = 0 \tag{5}$$

and

$$\lim_n x_0^{**}(x_n^*) = \alpha.$$

We can clearly assume that

$$\text{for all } n \geq 1, \quad x_0^{**}(x_n^*) > \frac{\alpha}{2}. \tag{6}$$

Note that, given a $w(E^*, E)$–cluster point x_0^* of the sequence $\{x_n^*\}_{n\geq 1}$, we have that

$$x_0^{**}(x_0^*) = 0, \tag{7}$$

because x_0^{**} is a $w(E^{**}, E^*)$–cluster point of the sequence $\{x_n\}_{n\geq 1}$ and for all $p \geq 1$ (5) holds. Let us apply [24, Lemma 9(c)], obtaining a subsequence $\left\{x_{n_k}^*\right\}_{k\geq 1}$ of $\{x_n^*\}_{n\geq 1}$ such that

$$\text{for all } h_0 \in \mathrm{co}_\sigma\{x_{n_k}^* : k \geq 1\},$$

$$\sup_X \left(h_0 - \limsup_k x_{n_k}^* - \lambda f \right) = \sup_X \left(h_0 - \liminf_k x_{n_k}^* - \lambda f \right). \tag{8}$$

Let us now observe that for x_0^{**} we have by (6) that

$$\text{for all } h_0 \in \mathrm{co}_\sigma \left\{x_{n_k}^* : k \geq 1\right\}, \quad x_0^{**}(h_0) > \frac{\alpha}{2}. \tag{9}$$

In addition, for every $w(E^*, E)$–cluster point y_0^* of $\left\{x_{n_k}^* : k \geq 1\right\}$, condition (7) above implies $x_0^{**}(y_0^*) = 0$. Let us fix such a cluster point y_0^*. It follows that for all $x \in X$

$$\begin{aligned} h_0(x) - \liminf_k x_{n_k}^*(x) - \lambda f(x) &\geq (h_0 - y_0^* - \lambda f)(x) \\ &\geq h_0(x) - \limsup_k x_{n_k}^*(x) - \lambda f(x). \end{aligned}$$

Therefore, we derive from (8) that

$$\text{for all } h_0 \in \mathrm{co}_\sigma \left\{x_{n_k}^* : k \geq 1\right\},$$

$$\begin{aligned}\sup_X \left(h_0 - \limsup_k x^*_{n_k} - \lambda f\right) &= \sup_X \left(h_0 - \liminf_k x^*_{n_k} - \lambda f\right) \\ &= \sup_X \left(h_0 - y_0^* - \lambda f\right).\end{aligned}$$

But given $h_0 \in \mathrm{co}_\sigma \left\{x^*_{n_k} : k \geq 1\right\}$, since $f^{-1}((-\infty, c]) \subset X$, (1) and (9) are valid and $x_0^{**}(y_0^*) = 0$, we find

$$\begin{aligned}\sup_X (h_0 - y_0^* - \lambda f) &\geq \sup_{f^{-1}((-\infty,c])} (h_0 - y_0^* - \lambda f) \\ &\geq \sup_{f^{-1}((-\infty,c])} (h_0 - y_0^*) - \lambda c \\ &= \sup_{\overline{f^{-1}((-\infty,c])}^{w(E^{**},E^*)}} (h_0 - y_0^*) - \lambda c \\ &\geq x_0^{**}(h_0) - x_0^{**}(y_0^*) - \lambda c \\ &> \frac{\alpha}{4}.\end{aligned}$$

Now [24, Corollary 8] applies to obtain $g_0 \in B_{E^*}$ and a sequence $\{g_k\}_{k\geq 1}$ in B_{E^*} satisfying that for all bounded function $\tilde{g} : X \longrightarrow \mathbb{R}$ with

$$\liminf_k g_k \leq \tilde{g} \leq \limsup_k g_k \text{ on } X$$

we have that

$$g_0 - \tilde{g} - \lambda f \text{ does not attain its supremum on } X.$$

Let $x^* \in B_{E^*}$ be a $w(E^*, E)$–cluster point of the sequence $\{g_k\}_{k\geq 1}$. Then, writing $z^* := g_0 - x^*$,

$$z^* - \lambda f \text{ does not attain its supremum on } X. \tag{10}$$

But, in view of (2)

$$\left\| \frac{z^*}{\lambda} \right\| \leq \rho,$$

hence, by (3) there exists $z \in MB_E$ such that $z^*/\lambda \in \partial f(z)$. In particular,

$$\frac{z^*}{\lambda}(x_0) - f(x_0) \leq \frac{z^*}{\lambda}(z) - f(z)$$

and so, by (4)

$$\begin{aligned}f(z) &\leq f(x_0) + \frac{z^*}{\lambda}(z - x_0) \\ &\leq f(x_0) + \rho(M + \|x_0\|) \\ &\leq c_0.\end{aligned}$$

Thus, $z \in X$, which contradicts the fact that $z^*/\lambda \in \partial f(z)$ and (10). □

The condition that the subdifferential maps bounded sets to bounded sets coincides with a familiar notion:

Lemma 2. *Assume that E is a real Banach space and that $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a function with*

$$\partial f(E) = E^*.$$

Then

$$f \text{ is coercive}$$

if, and only if,

$$\text{for all } \rho > 0, \quad (\partial f)^{-1}(\rho B_{E^*}) \text{ is bounded.}$$

Proof. We start with the necessity. This statement so far has only been stated for proper, coercive, convex and lower semicontinuous functions in [22, Proposition 2.4]. The proof given now is completely analogous to that one, but we include it here for the sake of completeness.

Let $\rho > 0$ and let $x_0 \in \mathrm{dom}(f)$. Since f is coercive, chose $\delta > 1$ such that

$$\text{for all } x \in E \text{ with } \|x\| > \delta, \quad \rho + \rho\|x_0\| + |f(x_0)| < \frac{f(x)}{\|x\|}. \tag{11}$$

Let us finish by showing that $(\partial f)^{-1}(\rho B_{E^*}) \subset \delta B_E$. Indeed, given $x \in E$ with $\|x\| > \delta$ and $x^* \in \rho B_{E^*}$, it follows from (11) that

$$\begin{aligned} \frac{x^*(x) - f(x)}{\|x\|} &\leq \rho - \frac{f(x)}{\|x\|} \\ &< -\rho\|x_0\| - |f(x_0)| \end{aligned}$$

and so,

$$\begin{aligned} x^*(x) - f(x) &< \|x\|(-\rho\|x_0\| - |f(x_0)|) \\ &\leq \delta(-\rho\|x_0\| - |f(x_0)|) \\ &\leq -\rho\|x_0\| - |f(x_0)| \\ &\leq x^*(x_0) - f(x_0). \end{aligned}$$

Altogether then, for all $x \in E$ such that $\|x\| > \delta$ and for all $x^* \in \rho B_{E^*}$ we have that

$$x^*(x) - f(x) < x^*(x_0) - f(x_0),$$

which clearly yields

$$(\partial f)^{-1}(\rho B_{E^*}) \subset \delta B_E.$$

To see the sufficiency, we first show that if f^* is the Fenchel–Legendre conjugate of f, that is, for each $x^* \in E^*$

$$f^*(x^*) = \sup_E (x^* - f),$$

then

$$\text{for all } \rho > 0, \quad f^*(\rho B_{E^*}) \text{ is bounded above.}$$

Let $\rho > 0$ and chose $\gamma > 0$ with

$$(\partial f)^{-1}(\rho B_{E^*}) \subset \gamma B_E.$$

Then $f^*(\rho B_{E^*})$ is bounded above by $\rho\gamma - \inf_E f$ ($\inf_E f$ is finite because $0 \in \partial f(E)$), since given $x^* \in \rho B_{E^*}$, taking $x \in \gamma B_E$ with $x^* \in \partial f(x)$ yields

$$\begin{aligned} f^*(x^*) &\leq f^*(x^*) + f(x) - \inf_E f \\ &= x^*(x) - \inf_E f \\ &\leq \rho\gamma - \inf_E f. \end{aligned}$$

Now we conclude that f is coercive. Let $\rho > 0$ and let $\omega \in \mathbb{R}$ such that $f^*(\rho B_{E^*}) \subset [-\infty, \omega]$. Then, following the ideas in [2, Lemma 3.2],

$$\text{for all } x^* \in \rho B_{E^*}, \quad f^*(x^*) \leq \omega,$$

equivalently

$$\text{for all } x^* \in \rho B_{E^*} \text{ and for all } y \in E, \quad x^*(y) - f(y) \leq \omega,$$

that is,

$$\text{for all } y \in E, \quad f(y) \geq \rho\|y\| - \omega$$

and so

$$\liminf_{\|x\| \to +\infty} \frac{f(y)}{\|y\|} \geq \rho.$$

The arbitrariness of $\rho > 0$ gives the coercivity of f. $\square$

As an immediate consequence of Theorem 1 and Lemma 2 we arrive at our main statement:

Theorem 2. *If E is a real Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a coercive function such that*

for all $x^* \in E^*$ the function $x^* - f$ attains its supremum on E,

then

for all $c \in \mathbb{R}$ the sublevel set $f^{-1}((-\infty, c])$ is relatively weakly compact.

As mentioned in the Introduction, the main result in [6, 10] is a particular case of Theorem 2 (this will be shown in Theorem 7 below) and its suggested proof is very intricate, since it involves the correction of non–written lemmas in [6], which are modified versions of others in [15]. Instead, our approach is based on an adequate use of results in [24]. Indeed, we have arrived to our proof after a careful analysis of the bounded and separable case presented as Theorem A.1 in [18] together with the perturbed version of James's Theorem studied in [24].

In view of the hypotheses of Theorem 1, one could conjecture that it suffices to assume in Theorem 2 that $\partial f(E)$ has nonempty norm–interior, instead of $\partial f(E) = E^*$. However, this is not the case. Indeed, given a nontrivial real Banach space E there exists a norm $\|\cdot\|$ on E , which is equivalent to the original one, in such a way that the range of the corresponding *duality mapping*, that is, the subdifferential of the continuous function

$$f(x) = \frac{1}{2}\|x\|^2, \qquad (x \in E),$$

has nonempty norm–interior (see [1, Corollary 2]). But its sublevel sets are not weakly compact, unless E is reflexive.

A sort of converse of Theorem 2 can be derived even when no coercivity is assumed:

Theorem 3. *Suppose that E is a real Banach space and that $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a weakly lower semicontinuous function such that*

$$\text{for all } c \in \mathbb{R}, \quad f^{-1}((-\infty, c]) \text{ is weakly compact}$$

and

$$\text{for all } x^* \in E^* \text{ the function } x^* - f \text{ is bounded above.}$$

Then

$$\partial f(E) = E^*.$$

Proof. Let $x_0^* \in E^*$ and let us prove that $x_0^* \in \partial f(E)$. Since $f - x_0^*$ is bounded from below, then it is proper, so let $c \in \mathbb{R}$ such that $(f - x_0^*)^{-1}((-\infty, c]) \neq \emptyset$. It follows from Lemma 1 that this subset is bounded, so let $\alpha \in \mathbb{R}$ with $(f - x_0^*)^{-1}((-\infty, c]) \subset \alpha B_E$. Then, for all $x \in (f - x_0^*)^{-1}((-\infty, c])$,

$$f(x) \leq x_0^*(x) + c \leq \alpha\|x_0^*\| + c,$$

that is,

$$(f - x_0^*)^{-1}((-\infty, c]) \subset f^{-1}((-\infty, \alpha\|x_0^*\| + c]),$$

which implies, $f^{-1}((-\infty, \alpha\|x_0^*\| + c])$ being weakly compact, that $(f - x_0^*)^{-1}((-\infty, c])$ is also weakly compact. Finally, we deduce from the weak lower semicontinuity of $f - x_0^*$ and the weak compactness of $(f - x_0^*)^{-1}((-\infty, c])$ that $f - x_0^*$ attains its infimum on E, or in other words, $x_0^* \in \partial f(E)$. □

Topological results along the lines of Theorems 2 and 3 appeared in Theorems 2.4 and 2.1 in [3], respectively.

There is a close connection between Theorem 3 and an open problem posed in [6]. More specifically, given a real Banach space and a convex and lower semicontinuous function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ whose effective domain $\mathrm{dom}(f)$ has nonempty norm–interior, the question is whether or not the following facts are equivalent:

(i) $\partial f(E) = E^*$.

(ii) E is reflexive and for all $x^* \in E^*$, $x^* - f$ is bounded above.

Let us notice that Theorem 3 and Lemma 1 assert, as a particular case, that the implication (ii) $\Rightarrow$ (i) holds. Regarding the reversed one, (i) $\Rightarrow$ (ii), we will see in Theorem 7 that it is true if f is assumed to be coercive. Moreover, we have been looking for a general statement with the use of the epigraph of the function f, instead of the approach followed here, but it requires an additional hypothesis on the dual unit ball to work, as seen in [19]. Anyway, the implication (i) $\Rightarrow$ (ii) is valid (and not only it but also the converse of Theorem 3) for a wide class of Banach spaces. To be more precise, let us recall that given a real Banach space E, a sequence $\{y_n\}_{n\geq 1}$ in E is a *convex block sequence* of another sequence $\{x_n\}_{n\geq 1}$ if there are a sequence of finite subsets of integers $\{F_n\}_{n\geq 1}$ such that

$$\max F_1 < \min F_2 \leq \max F_2 < \min F_3 \cdots < \max F_n < \min F_{n+1} < \cdots$$

together with sets of positive numbers $\{t_i^n : i \in F_n\} \subset (0, 1]$ satisfying

$$\sum_{i\in F_n} t_i^n = 1 \text{ and } y_n = \sum_{i\in F_n} t_i^n x_i.$$

It is not difficult to check that if E is separable then B_{E^*} is w^**–convex block compact*, that is, each sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} has a convex block w^*–convergent sequence. Indeed, a subsequence of a given sequence is a convex block sequence of it, thus every sequentially compact set is convex block compact. In particular, the dual unit ball of E is metrizable and w^*–compact, thus it also is w^*–sequentially compact. Moreover, J. Bourgain proved in [5] that if the Banach space E doest not contain a copy of $l^1(\mathbb{N})$ then its dual ball is w^*–convex block compact. This result was extended for spaces not containing a copy of $l^1(\mathbb{R})$ under Martin's axiom and the negation of the Continuum Hipothesis in [14].

The mentioned result, [19, Theorem 4], is stated as follows:

Theorem 4. *Let E be a real Banach space whose dual unit ball is w^*–convex block compact and let $f : E \longrightarrow \mathbb{R} \cup \{\infty\}$ be a proper map such that*

$$\partial f(E) = E^*.$$

Then for all $c \in \mathbb{R}$ the corresponding sublevel set $f^{-1}((-\infty, c])$ is relatively weakly compact.

The fact that we conclude (i) $\Rightarrow$ (ii) from Theorem 4, for those Banach spaces whose dual unit ball is w^*–convex block compact, is shown in Theorem 8.

Thus, it is natural to ask whether the converse of Theorem 3 is valid in general, and equivalently, if is it possible to avoid the coercivity assumption in Theorem 2 or the w^*–convex block compactness for the dual unit ball in Theorem 4:

Problem 1. *Let E be a real Banach space and let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that $\partial f(E) = E^*$. Does it imply that*

$$\text{for all } c \in \mathbb{R} \text{ the sublevel set } f^{-1}((-\infty, c]) \text{ is relatively weakly compact?}$$

If one tries to deal with this question, in view of Theorem 2 a first possibility is to know if the coercivity of f follows from the surjectivity of its subdifferential. But this is not the case:

Proposition 1. *For each real infinite–dimensional reflexive Banach space E there exists a proper, convex and lower semicontinuous function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that*

$$\partial f(E) = E^*$$

but f is not coercive.

Proof. Thanks to [2, Theorem 3.6], if E is a real infinite–dimensional reflexive Banach space, then there exists a proper, convex and lower semicontinuous function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$\text{for all } x^* \in E^*, \quad x^* - f \text{ is bounded above,}$$

but f is not coercive. Now apply Theorem 3, Lemma 1 and the reflexivity of E to conclude that

$$\partial f(E) = E^*.$$

□

As a consequence of Theorem 3, and under the additional hypothesis that f is weakly lower semicontinuous, the reverse of Theorem 2 is also true, which leads to this characterization of the weak compactness for the sublevel sets:

Theorem 5. *Let E be a real Banach space and let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, coercive and weakly lower semicontinuous function. Then*

$$\partial f(E) = E^*$$

if, and only if,

$$\text{for all } c \in \mathbb{R} \text{ the sublevel set } f^{-1}((-\infty, c]) \text{ is weakly compact.}$$

Proof. The "only if" part is Theorem 2 and the weak lower semicontinuity of f. For the proof of the "if" part we make use of Theorem 3. We first show that f is bounded from below, because otherwise there exists a sequence $\{x_n\}_{n\geq 1}$ in E such that

$$\lim_n f(x_n) = -\infty, \tag{12}$$

so for a certain $\nu \geq 1$ we have

$$\{x_n : n \geq \nu\} \subset f^{-1}((-\infty, 0]),$$

which together with the weak compactness of $f^{-1}((-\infty, 0])$ and the weak lower semicontinuity of f contradicts (12). Therefore, according to Theorem 3, we just need to prove that any function of the form $x^* - f$, with $x^* \in E^*$, is bounded above. Thus let $x_0^* \in E^*$. As

$$\lim_{\|x\| \to +\infty} \frac{f(x) - x_0^*(x)}{\|x\|} = +\infty,$$

then there exists $\delta > 0$ such that

$$\|x\| \geq \delta \Rightarrow \frac{f(x) - x_0^*(x)}{\|x\|} \geq 1,$$

so for all $x \in E$ with $\|x\| \geq \delta$ we find

$$x_0^*(x) - f(x) \leq -\delta,$$

while $x_0^* - f$ is also bounded above in δB_E, because f is bounded from below, and so we conclude the proof. □

Let us note that in Theorem 5 the coercivity of f is essential: it suffices to consider any real reflexive Banach space E and the proper, noncoercive and weakly lower semicontinuous function $f = \|\cdot\|$. Then, for all $c \in \mathbb{R}$ the sublevel set $f^{-1}((-\infty, c])$ is weakly compact, though $\partial f(E) = B_{E^*}$.

Theorem 2 includes, as a particular case, the classical James's theorem [17, Theorem 5]. Furthermore, all the generalizations of this result previously established also follow from Theorem 2. In fact, we improve some of such results, as [24, Theorem 16] in the context of Banach spaces. In the original statement, the subset A below is in addition assumed to be convex.

Corollary 1. *Let A be a weakly closed subset of a real Banach space and let $\psi : A \longrightarrow \mathbb{R}$ be a bounded function such that*

for all $x^* \in E^*$ the function $x^* - \psi$, when restricted to A, attains its supremum.

Then A is weakly compact.

Proof. Apply Theorem 2 to the coercive function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined for $x \in E$ as

$$f(x) = \begin{cases} \psi(x), & \text{if } x \in A \\ +\infty, & \text{otherwise} \end{cases}$$

and take into account that $A = f^{-1}((-\infty, \sup_A \psi])$. □

Another consequence of Theorem 2, in the framework of mathematical finance, is described in Section 3. A further one is the main result in [6, 10], which we shall comment on Section 4.

Now we derive a topological consequence related to the epigraph of a proper function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$, i.e.,

$$\mathrm{epi}(f) := \{(x,t) \in E \times \mathbb{R} : f(x) \leq t\}.$$

First we show a Krein–Smulian's type result, useful for our purposes. Given $c \in \mathbb{R}$, $\mathrm{epi}(f,c)$ denotes the *truncated epigraph*

$$\mathrm{epi}(f,c) := \{(x,t) \in E \times \mathbb{R} : f(x) \leq t \leq c\}.$$

Lemma 3. *Let E be a real Banach space and let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a weakly lower semicontinuous function with $\partial f(E) = E^*$. Then*

$$\textit{for all } c \in \mathbb{R}, \quad \mathrm{epi}(f,c) \textit{ is } w(E \times \mathbb{R}, E^* \times \mathbb{R})\textit{–compact}$$

if, and only if,

$$\mathrm{epi}(f) = \overline{\mathrm{epi}(f)}^{w(E^{**}\times\mathbb{R},E^*\times\mathbb{R})}.$$

Proof. Let us first assume that $\mathrm{epi}(f)$ is a $w(E^{**} \times \mathbb{R}, E^* \times \mathbb{R})-$closed subset of $E^{**} \times \mathbb{R}$. Let $c \in \mathbb{R}$ such that $\mathrm{epi}(f,c) \neq \emptyset$ and let us show that $\mathrm{epi}(f,c)$ is $w(E \times \mathbb{R}, E^* \times \mathbb{R})-$compact. The fact that f is weakly lower semicontinuous implies that $\mathrm{epi}(f,c)$ is $w(E \times \mathbb{R}, E^* \times \mathbb{R})-$closed, and the fact that $\partial f(E) = E^*$ and Lemma 1 that it is bounded. Therefore, in order to prove that $\mathrm{epi}(f,c)$ is $w(E \times \mathbb{R}, E^* \times \mathbb{R})-$compact, it suffices to check that

$$\mathrm{epi}(f,c) = \overline{\mathrm{epi}(f,c)}^{w(E^{**}\times\mathbb{R},E^*\times\mathbb{R})}.$$

Thus, let us fix $(x_0^{**}, t_0) \in \overline{\mathrm{epi}(f,c)}^{w(E^{**}\times\mathbb{R},E^*\times\mathbb{R})}$ and let us show that $(x_0^{**}, t_0) \in \mathrm{epi}(f,c)$. Let $\{(x_\lambda, t_\lambda)\}_{\lambda\in\Lambda}$ be a net in $\mathrm{epi}(f,c)$ such that

$$(x_0^{**}, t_0) = w(E^{**} \times \mathbb{R}, E^* \times \mathbb{R}) - \lim_\lambda \{(x_\lambda, t_\lambda)\}.$$

But $\overline{\mathrm{epi}(f,c)}^{w(E^{**}\times\mathbb{R},E^*\times\mathbb{R})} \subset \overline{\mathrm{epi}(f)}^{w(E^{**}\times\mathbb{R},E^*\times\mathbb{R})} = \mathrm{epi}(f)$, so, on the one hand, $x_0 := x_0^{**} \in E$ and, on the other hand, the limit above is nothing more than

$$(x_0, t_0) = w(E \times \mathbb{R}, E^* \times \mathbb{R}) - \lim_\lambda \{(x_\lambda, t_\lambda)\},$$

and $(x_0^{**}, t_0) = (x_0, t_0) \in \mathrm{epi}(f,c)$, since $\mathrm{epi}(f,c)$ is $w(E \times \mathbb{R}, E^* \times \mathbb{R})-$closed.

And conversely, let us choose $(x_0^{**}, t_0) \in \overline{\mathrm{epi}(f)}^{w(E^{**}\times\mathbb{R},E^*\times\mathbb{R})}$ and we shall prove that $(x_0^{**}, t_0) \in \mathrm{epi}(f)$. For any net $\{(x_\lambda, t_\lambda)\}_{\lambda\in\Lambda}$ in $\mathrm{epi}(f)$ with

$$(x_0^{**}, t_0) = w(E^{**} \times \mathbb{R}, E^* \times \mathbb{R}) - \lim_\lambda \{(x_\lambda, t_\lambda)\}$$

we can assume without loss of generality that the scalar net $\{t_\lambda\}_{\lambda\in\Lambda}$ is bounded, so there exists $c \in \mathbb{R}$ in such a way that for all $\lambda \in \Lambda$, $(x_\lambda, t_\lambda) \in \mathrm{epi}(f, c)$. Finally, the fact that $(x_0^{**}, t_0) \in \mathrm{epi}(f, c) \subset \mathrm{epi}(f)$ follows from the weak compactness of $\mathrm{epi}(f, c)$. $\square$

The announced topological result is the next one, and it is derived from Theorem 2 and the immediate fact that the weak compactness of the sublevel set $f^{-1}((-\infty, c])$ is equivalent to that of the truncated epigraph $\mathrm{epi}(f, c)$:

Proposition 2. *Suppose that f is an extended real–valued, coercive and weakly lower semicontinuous function defined on a real Banach space E such that*

$$\text{for all } x^* \in E^*, \quad x^* - f \text{ attains its supremum on } E.$$

Then $\mathrm{epi}(f)$ *is* $w(E^{**} \times \mathbb{R}, E^* \times \mathbb{R})$*–closed in* $E^{**} \times \mathbb{R}$.

Obviously, in view of Theorem 4 we can replace the coercivity of f in the proposition above with the weak–star convex block compactness of B_{E^*}.

We conclude this section by emphasizing that the completeness assumption cannot be dropped in any of the preceding results, since in [16] R.C. James gave an example of a noncomplete normed space for which each continuous and linear functional attains its norm.

3 Consequences for mathematical finance

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with $\mathcal{X}$, a linear space of functions in $\mathbb{R}^\Omega$ that contains the constant functions. We assume here that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, although in practice this is not a restriction, since the property of being atomless is equivalent to the fact that we can define a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ that has a continuous distribution function. The space $\mathcal{X}$ is going to describe all possible financial positions $X : \Omega \longrightarrow \mathbb{R}$, where $X(\omega)$ is the discounted net worth of the position at the end of the trading period if the scenario $\omega \in \Omega$ is realized. The problem of quantifying the risk of a financial position $X \in \mathcal{X}$ is modeled with functions $\rho : \mathcal{X} \longrightarrow \mathbb{R}$ that satisfy:

- Monotonicity : if $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Cash invariance: if $m \in \mathbb{R}$ then $\rho(X + m) = \rho(X) - m$.

Such a function ρ is called a monetary measure of risk (see Chapter 4 in [11]). When it is also a convex function, then ρ is called a convex measure of risk. In many occasions we have

$\mathcal{X} = \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, and it is important to have results for representing the risk measure as

$$\rho(X) = \sup_{Y \in \mathbb{L}^1(\Omega,\mathcal{F},\mathbb{P})} \{-\mathbb{E}[Y \cdot X] - \rho^*(Y)\}. \tag{13}$$

Here ρ^* is the Fenchel–Legendre conjugate of ρ, that is, for all $Y \in (\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}))^*$,

$$\rho^*(Y) = \sup_{X \in \mathbb{L}^\infty(\Omega,\mathcal{F},\mathbb{P})} \{\langle Y, X\rangle - \rho(X)\}.$$

Having this representation is equivalent to the so–called Fatou property, i.e., for any bounded sequence $\{X_n\}_{n\geq 1}$ that converges pointwise a.s. to some X,

$$\rho(X) \leq \liminf_n \rho(X_n)$$

(see [11, Theorem 4.31]). A natural question is whether the supremum above is attained. In general the answer is no, as shown by the essential supremum map on $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, as seen in [11, Example 4.36]. The representation formula (13) with a maximum instead of a supremum has been studied by F. Delbaen; see Theorems 8 and 9 in [9], or [11, Corollary 4.35] in the case of coherent risk measures, i.e., the convex ones that also satisfy $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \geq 0$. The fact that the order continuity of ρ is equivalent to turning the supremum into a maximum, that is, for all $X \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$,

$$\rho(X) = \max_{Y \in \mathbb{L}^1(\Omega,\mathcal{F},\mathbb{P})} \{-\mathbb{E}[Y \cdot X] - \rho^*(Y)\}$$

for an arbitrary convex risk measure, is the statement of the so–called Jouini–Schachermayer–Touzi theorem in [9]. We refer to [18, Theorem 5.2] and [9, Theorem 2]. Let us remark that the order sequential continuity for a map ρ in $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is equivalent to having

$$\lim_n \rho(X_n) = \rho(X),$$

whenever $\{X_n\}_{n\geq 1}$ is a bounded sequence in $\mathbb{L}^\infty$ pointwise almost surely convergent to X. For this reason it is said that a map $\rho : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R} \cup \{+\infty\}$ verifies the Lebesgue property provided that it is sequentially order continuous. The precise statement is the following:

Theorem 6 (Jouini, Schachermayer and Touzi). *Let $\rho : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R}$ be a convex risk measure with the Fatou property and let $\rho^* : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \longrightarrow [0, +\infty]$ be its Fenchel–Legendre conjugate The following are equivalent:*

(i) For all $c \in \mathbb{R}$, $\{Y \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}) : \rho^*(Y) \leq c\}$ is a weakly compact subset of $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) For every $X \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, the supremum in the equality

$$\rho(X) = \sup_{Y \in \mathbb{L}^1(\Omega,\mathcal{F},\mathbb{P})} \{-\mathbb{E}[XY] - \rho^*(Y)\}$$

is attained.

(iii) *For each bounded sequence* $\{X_n\}_{n\geq 1}$ *in* $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ *tending a.s. to* $X \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$*, we have*

$$\lim_n \rho(X_n) = \rho(X).$$

As the first author observed in the Appendix of [18], the proof requires compactness arguments of the perturbed James's type. Indeed, in [9] this result is presented as a generalization of James's theorem. In that case the map ρ^* has a bounded domain, so it is going to be coercive; thus we can apply Theorem 2 for $f = \rho^*$ and we obtain the proof for the main implication (ii) $\Rightarrow$ (i) above.

The proof for (ii) $\Rightarrow$ (i) presented in [18] is based on a particular case of Corollary 2 given below. It was done for a convex map f and a separable real Banach space E, as seen in [18, Theorem A.1].

Corollary 2. *Let* E *be a real Banach space and let* $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a proper and weakly lower semicontinuous function such that* $\operatorname{dom}(f)$ *is a bounded subset of* E*. Suppose that there exists* $c \in \mathbb{R}$ *such that the sublevel set* $f^{-1}((-\infty, c])$ *fails to be weakly compact. Then there exists an* $x^* \in E^*$ *for which the function* $x^* - f$ *does not attain its supremum on* E*.*

F. Delbaen gave a different approach for Theorem 6. His proof is valid for non–separable $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ spaces, but it is based in a homogeneization trick to reduce the matter to a direct application of the classical James's theorem, as well as the Dunford–Pettis theorem characterizing weakly compact sets in $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. In any case, the arguments that F. Delbaen uses are only valid in $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

Our contribution here, summarized in Corollary 2, provides the proof for the implication (ii) $\Rightarrow$ (i) in Theorem 6 for arbitrary $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ spaces, thus including Delbaen's non–separable case as well.

4 Application to nonlinear variational problems

The particular version of the Weierstrass theorem asserting that if a proper and weakly lower semicontinuous extended real–valued function h defined on a real reflexive Banach space and such that $h(x) \to +\infty$ as $\|x\| \to +\infty$, then it attains its infimum, represents a powerful tool that has been intensively used in different fields. For instance, it proves the existence of a solution to nonlinear variational equations, derived from the weak formulation of a wide range of boundary value problems. See [8, Examples 4.2.2] for some illustrative applications. Indeed, a usual way in which this result is applied is as follows: for a real reflexive Banach space E and a proper, coercive, convex and lower semicontinuous function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$, given any $x^* \in E^*$, the optimization problem

$$\text{find } x_0 \in E \text{ such that } f(x_0) - x^*(x_0) = \inf_{x\in E} (f(x) - x^*(x))$$

has a solution. Next we prove that, if in a real Banach space one of these nonlinear optimization problems admits a solution, then the space is reflexive. Moreover we assume no convexity or weak lower semicontinuity on f. The specific case corresponding to convex and weakly lower semicontinuous functions was announced in [6, 10], although as we commented above, the proof outlined in that work is not entirely clear.

Since applied nonlinear variational problems are usually stated in terms of infima instead of suprema, we adopt this terminology in this section.

Now we derive the following result from Theorem 2:

Theorem 7. *Let E be a real Banach space and let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a coercive function such that* $\mathrm{dom}(f)$ *has nonempty norm–interior and for all $x^* \in E^*$ there exists $x_0 \in E$ with*

$$f(x_0) - x^*(x_0) = \inf_{x\in E} (f(x) - x^*(x)).$$

Then E is reflexive.

Proof. Let B be a nonempty open ball contained in $\mathrm{dom}(f)$. Then we have that

$$B = \bigcup_{p=1}^{+\infty} B \cap \overline{f^{-1}((-\infty, p])}^{w(E,E^*)}.$$

We can apply the Baire Category theorem to the open set B to get an integer $p \geq 1$ such that $B \cap \overline{f^{-1}((-\infty, p])}^{w(E,E^*)}$ has an interior point relative to B, so that there is an open set G in E such that $\emptyset \neq B \cap G \subset B \cap \overline{f^{-1}((-\infty, p])}^{w(E,E^*)}$ and thus $\emptyset \neq G \cap B \subset \overline{f^{-1}((-\infty, p])}^{w(E,E^*)}$. But $\overline{f^{-1}((-\infty, p])}^{w(E,E^*)}$ is weakly compact by Theorem 2, therefore, we have a closed ball of positive radius which is weakly compact and the space must be reflexive. □

It is clear that we can also arrive to the next result, but applying Theorem 4 instead of Theorem 2:

Theorem 8. *A real Banach space E whose dual unit ball is w^*–convex block compact is reflexive provided that there exists a function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that* $\mathrm{dom}(f)$ *has nonempty norm–interior and*

$$\partial f(E) = E^*.$$

We emphasize that for Theorems 7 and 8 to be valid, f is not required to fulfil any kind of (semi)continuity.

With the aim to apply these results to nonlinear variational problems in mind, let us first recall that the weak formulation of lots of differential problems leads us to consider a variational equation which, in the presence of symmetry, turns into a variational problem for which the key is to establish the existence of a minimum. Next we prove that these results, always stated in the reflexive context, only make sense for this kind of Banach space. In order to motivate it, let us consider the variational formulation of a coercive, linear and elliptic boundary value problem, or in a more general way, the variational equation

$$\text{find } x_0 \in E \text{ such that for all } x \in E, \quad a(x_0, x) = x_0^*(x),$$

where E is a real reflexive Banach space, $x_0^* \in E^*$ and $a : E \times E \longrightarrow \mathbb{R}$ is a coercive, continuous and bilinear form. The classical Lax–Milgram theorem [13, Theorem 12] guarantees the existence of one and only one solution to this equation. In particular, when a is symmetric, the functional

$$x \in E \mapsto \frac{1}{2}a(x, x) - x_0^*(x) \in \mathbb{R}$$

attains its infimum at x_0. Another type of variational equation, now a nonlinear one, arises if we consider certain nonlinear boundary value problems, for which the existence of a solution follows from well–known results as the next one, more general than the Lax–Milgram theorem. Before evoking it, we introduce some common notations. Given a real Banach space E, an operator $\Phi : E \longrightarrow E^*$ is said to be *monotone* provided that

$$\inf_{x,y \in E} (\Phi(x) - \Phi(y))(x - y) \geq 0,$$

strongly monotone as soon as

$$\inf_{\substack{x,y \in E \\ x \neq y}} \frac{(\Phi(x) - \Phi(y))(x - y)}{\|x - y\|^2} > 0,$$

hemicontinuous if for all $x, y, z \in E$, the function

$$t \in [0, 1] \mapsto (\Phi(x + ty))(z) \in \mathbb{R}$$

is continuous, *bounded* when the image under Φ of a bounded set is also bounded, and *coercive* whenever the function

$$x \in E \mapsto (\Phi(x))(x) \in \mathbb{R}$$

is coercive. The mentioned result appears in [7, Corollary 2.101]:

Proposition 3. *If E is a real reflexive Banach space and $\Phi : E \longrightarrow E^*$ is a monotone, hemicontinuous, bounded and coercive operator, then Φ is surjective.*

It is a fruitful result with applications to nonlinear variational analysis, including one of its most popular particular cases: in a real reflexive Banach space E, given $x_0^* \in E^*$, the equation

$$\text{find } x_0 \in E \text{ such that } \Phi(x_0) = x_0^*$$

admits a unique solution, whenever $\Phi : E \longrightarrow E^*$ is a Lipschitz continuous and strongly monotone operator. We refer to [12, Example 3.51] for a standard application.

In the following result we prove that the equation $\Phi(x_0) = x_0^*$ leads to a nonlinear optimization problem (in general nonsmooth, not even continuous) when Φ is symmetric, i.e.,

$$\text{for all } x, y \in E, \quad (\Phi(x))(y) = (\Phi(y))(x),$$

and, as a consequence of Theorem 7, the adequate and the only framework where Proposition 3 can be developed is the reflexive one:

Corollary 3. *A real Banach space E is reflexive, provided there exists a monotone, coercive, symmetric and surjective operator $\Phi : E \longrightarrow E^*$.*

Proof. It suffices to define the function $f : E \longrightarrow \mathbb{R}$ as

$$f(x) := \frac{1}{2}(\Phi(x))(x), \qquad (x \in E)$$

and to show that it fulfills the hypotheses in Theorem 7. It is obvious that f is coercive with the effective domain all E. In order to conclude the proof, let us check that the subdifferential of f is surjective. Thus, let $x_0^* \in E^*$. As Φ is surjective, let $x_0 \in E$ with $\Phi(x_0) = x_0^*$. Let us show that $x_0^* \in \partial f(x_0)$. Indeed, for each $x \in E$ we have, because of the monotonicity of Φ, that

$$(\Phi(x) - \Phi(x_0))(x - x_0) \geq 0,$$

which is equivalent, since Φ is symmetric, to

$$\frac{1}{2}(\Phi(x))(x) + \frac{1}{2}(\Phi(x_0))(x_0) - (\Phi(x_0))(x) \geq 0,$$

and thus, taking into account that $\Phi(x_0) = x_0^*$,

$$\begin{aligned} f(x_0) - x_0^*(x_0) &= -\frac{1}{2}(\Phi(x_0))(x_0) \\ &\leq \frac{1}{2}(\Phi(x))(x) - (\Phi(x_0))(x) \\ &= f(x) - x_0^*(x). \end{aligned}$$

The arbitrariness of $x \in E$ implies $x_0^* \in \partial f(x_0)$ and E is reflexive. □

Note that it suffices to suppose that Φ is defined on a subset of E with nonempty norm–interior, and in addition we do not need to assume any hypothesis of (semi)continuity on Φ.

A result along these lines, in the linear case, can be found in [23, Corollary 1.4].

In a similar way, Theorem 8 also provides us with a sufficient condition for the reflexivity of a space:

Corollary 4. *If E is a real Banach space such that B_{E^*} is w^*–convex block compact and there exists a monotone, symmetric and surjective operator $\Phi : E \longrightarrow E^*$, then E is reflexive.*

Moreover, now we show how Theorem 8 allows us to deal with multivalued operators defined on a real Banach space with a w^*–convex block compact dual unit ball . This assumption on the space is not a restriction in applications, since most of spaces used in concrete examples are separable, and so their dual unit balls are w^*–convex block compact.

Let us recall that given a real Banach space E and a multivalued operator $\Phi : E \longrightarrow 2^{E^*}$, the *domain* of Φ is the subset of E

$$\mathrm{D}(\Phi) := \{x \in E : \Phi(x) \text{ is nonempty}\},$$

and its *range* is the subset of E^*

$$\Phi(E) := \{x^* \in E^* : \text{there exists } x \in E \text{ with } x^* \in \Phi(x)\}.$$

In addition, Φ is said to be *monotone* if

$$\inf_{\substack{x,y \in \mathrm{D}(\Phi) \\ x^* \in \Phi(x),\ y^* \in \Phi(y)}} (x^* - y^*)(x - y) \geq 0$$

and *cyclically monotone* when the inequality

$$\sum_{j=1}^{n} x_j^*(x_j - x_{j-1}) \geq 0$$

holds, whenever $n \geq 2$, $x_0, x_1, \ldots, x_n \in \mathrm{D}(\Phi)$ with $x_0 = x_n$ and for $j = 1, \ldots, n$, $x_j^* \in \Phi(x_j)$.

We derive another sufficient condition for reflexivity in terms of cyclically monotone operators:

Corollary 5. *Let E be a real Banach space whose dual unit ball is w^*–convex block compact, and let $\Phi : E \longrightarrow 2^{E^*}$ be a cyclically monotone operator such that $\mathrm{D}(\Phi)$ has nonempty norm–interior and*

$$\Phi(E) = E^*.$$

Then E is reflexive.

Proof. Since Φ is cyclically monotone, [21, Theorem 1] guarantees that there exists a proper and convex function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that for all $x \in E$,

$$\Phi(x) \subset \partial f(x).$$

As $\mathrm{D}(\Phi) \subset \mathrm{dom}(f)$, Theorem 8 applies. $\square$

However, the following question, posed in [10, §3], seems to remain open, including the case of E having a w^*–convex block compact dual unit ball: let E be a real Banach space and let $\Phi : E \longrightarrow 2^{E^*}$ be a monotone operator whose domain has nonempty norm–interior and $\Phi(E) = E^*$. Is E reflexive?

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