# NETWORK CHARACTERIZATION OF GUL'KO COMPACT SPACES AND THEIR RELATIVES

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ABSTRACT. In this paper we characterize the classes of Gul'ko and Talagrand compact spaces through a network condition leading us to answer two questions posed by G. Gruenhage, [23], on covering properties.

Dedicated to Professor John Horvath on the occasion of his $80^{th}$ birthday

## 1. INTRODUCTION

A compact space $K$ is called **Eberlein compact** if it is homeomorphic to a weakly compact subset of a Banach space and it has a strong influence on both the geometry and topology of the Banach space it generates. Since the seminal paper by Amir and Lindenstrauss [1], where they showed the interplay between topological and geometrical properties of the so-called weakly compactly generated Banach spaces, a lot of research has been done on this class of Banach spaces and their relatives such as weakly K-analytic, weakly countably determined and weakly Lindelöf determined Banach spaces [42, 45, 26, 3, 38, 44, 39, 8, 16, 34, 35].

For a compact space $K$ we have that $K$ is Eberlein compact if, and only if, $C(K)$ is weakly compactly generated [1]; $K$ is said to be **Talagrand compact** when $(C(K), \tau_p)$ is $K$-analytic [42], i.e. there exists an onto usco map $\varphi : \mathbb{N}^{\mathbb{N}} \to 2^{(C(K),\tau_p)}$; and $K$ is said to be **Gul'ko compact** if $(C(K), \tau_p)$ is $K$-countably determined [42, 45], i.e. there exists an onto usco map $\varphi : \Sigma \subset \mathbb{N}^{\mathbb{N}} \to 2^{(C(K),\tau_p)}$. Main results are the fact that $K$ embeds in a $\Sigma$-product of real lines whenever $K$ is Gul'ko compact [26] and that $K$ embeds in $(c_0(\Gamma), \tau_p)$ whenever $K$ is an Eberlein compact space [1]. Compact spaces lying in $\Sigma$-products of real lines are called **Corson compact** spaces [10, 25, 5, 20]. We denote by $\tau_p$ the pointwise convergence topology on spaces of functions.

For an up-to-date account of these classes of compact spaces, as well as their interplay in Functional Analysis, we recommend the books [6, 15, 17]

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2000 *Mathematics Subject Classification.* 54D20, 54D30, 46B20, 46B50.

*Key words and phrases.* Gul'ko ompact, network, covering properties.

Research partially supported by Fundación Séneca PI-55/00872/FS/01 (CARM, Spain); by DGES, Project PBFM2002-01719 (Spain).

together with the survey papers [33, 30, 19], as well as some very recent papers [13, 2, 16, 4]. We have the following implications:

$$\text{Eberlein compact} \Rightarrow \text{Talagrand compact} \Rightarrow \text{Gul'ko compact}$$
$$\Rightarrow \text{Corson compact}$$

and no arrow can be reversed, [15, 42, 5, 43].

Given a set $A$ we shall denote by $\#A$ its cardinality and for a given family of subsets $\mathcal{A}$ of a set $X$, given $x \in X$ we shall denote by $\operatorname{ord}(x, \mathcal{A}) = \#\{A \in \mathcal{A} : x \in A\}$. We say that the family $\mathcal{A}$ is *point finite* (resp. *point countable*) if for every $x \in X$ we have $\operatorname{ord}(x, \mathcal{A}) < \omega$ (resp. $\operatorname{ord}(x, \mathcal{A}) = \omega$), where $\omega$ is the cardinality of the set of the positive integers, $\mathbb{N}$. $\mathcal{A}$ is said *$\sigma$-point finite* if $\mathcal{A} = \bigcup\{\mathcal{A}_n : n \in \mathbb{N}\}$ such that each family $\mathcal{A}_n$ is point finite.

Let us recall that a topological space $(X, \tau)$ is *metalindelöf* (resp. *$\sigma$-metacompact*) if every open cover of $X$ has a point countable (resp. $\sigma$-point finite) open refinement. A cover $\mathcal{U}$ of $X$ is a *weak $\theta$-cover* if $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}$ such that if $x \in X$, then $0 < \operatorname{ord}(x, \mathcal{U}_n) < \omega$ for some $n \in \omega$. $X$ is *weakly submetacompact* if every open cover of $X$ has an open refinement which is a weak $\theta$-cover (also called *weakly $\theta$-refinable* spaces [7] and *$\sigma$-relatively metacompact* [12]).

Gruenhage [20] characterized Corson (resp. Eberlein) compacta as those compact spaces $K$ such that $K^2$ is hereditarily metalindelöf (resp. $\sigma$-metacompact), or equivalently, such that $K^2 \setminus \Delta$ is metalindelöf (resp. $\sigma$-metacompact), where $\Delta = \{(x, x) \in K^2 : x \in K\}$ is the diagonal. There are Corson compact spaces which are not hereditarily weakly submetacompact [22]. Nevertheless every Gul'ko compact space is hereditarily weakly submetacompact, even more they are weakly $\sigma$-metacompact according to [23], where the following definition is introduced. In order to stress the difference between these concepts we refer to [7, 23]

**Definition 1.** *A topological space $(X, \tau)$ is weakly $\sigma$-metacompact if for every open cover $\mathcal{U}$ in $X$ we have an open refinement $\mathcal{V}$ such that $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \omega\}$ and for every $x \in X$ we have*

$$\mathcal{V} = \bigcup\{\mathcal{V}_n : \operatorname{ord}(x, \mathcal{V}_n) < \omega\}.$$

The paper of Gruenhage [23] had a strong influence in Functional Analysis since it was the inspiration for proving fragmentability properties of Gul'ko compact spaces and consequently that weakly countably determined Banach spaces are weak Asplund spaces [15].

In view of the results mentioned above it is natural the conjecture posed by Gruenhage that the condition $K$ is compact and $K^2$ is hereditarily weakly $\sigma$-metacompact would characterize Gul'ko compact spaces (see [23, remark

2]). Our main results in this paper provide a positive answer for this conjecture (see Theorema 9). In the course of the proof we shall present a characterization of Gul'ko compact spaces in terms of networks, providing more information on the relationship between Gul'ko compact spaces and compact spaces with the *linking separability property*, as it is presented by Dow, Junnila and Pelant [13]. In particular, the network obtained in any Gul'ko compact space, yields its hereditarily weakly $\sigma$-metacompactness. Let us recall that fragmentability together with hereditarily weakly submetacompactness imply to be descriptive in the sense of Hansell, [27], a property satisfied by all Gul'ko compact spaces, [36], which has become very important in the study of LUR renorming, [31].

Recall that a family $\mathcal{N}$ of sets in a topological space $(X, \tau)$ is said to be a *network* for the topology if for every open set $U \subset X$ and any point $x \in U$ there is $N \in \mathcal{N}$ such that $x \in N \subset U$.

Gruenhage also asked (see [23, Remark 2]) if the weaker condition that $K$ Corson compact and $K^2$ hereditarily weakly submetacompact characterizes Gul'ko compact spaces. In this case the answer is negative due to a previous example of Argyros and Mercourakis [3] which we have discussed in [36], (see Remark 5). An example of a compact space $K$ such that $K^2$ is hereditarily weakly submetacompact and not Corson compact, and so $K^2$ is not hereditarily metalindelöf, was already given in [23, Remark 2].

All our topological spaces are assumed to be Hausdorff and we refer the reader to [14, 15] for general background and for definitions of terms and concepts used below without any explanation.

## 2. ON WEAKLY $\sigma$-POINT FINITE FAMILIES

The combinatorial decomposition for weakly $\sigma$-metacompactness can be presented with the following definition, which has been used by Sokolov in [41] in order to give characterizations of Gul'ko compact spaces in the spirit of Rosenthal's theorem for Eberlein compact spaces [40]:

**Definition 2.** *A collection $\mathcal{U}$ of subsets of a given set $X$ is said to be weakly $\sigma$-point finite if $\mathcal{U} = \cup\{\mathcal{U}_n : n = 1, 2, \dots\}$ so that, for each $x \in X$ we have*

$$\mathcal{U} = \bigcup\{\mathcal{U}_s : \operatorname{ord}(x, \mathcal{U}_s) < \omega\}.$$

In our approach to prove Gruenhage's conjecture we shall need to find handy conditions characterizing weakly $\sigma$-point finite families in a given set $X$. It is our intention to present in this section some characterizations based on the lattice structure of the set $\mathcal{K}(M) := \{K \subset M : K \text{ is compact}\}$, where $M$ is a separable metric space. Let us begin with the following notion:

**Definition 3.** *Given a separable metric space $M$ and a family $\mathcal{W}$ of subsets of a given set $X$, we say that $\mathcal{W}$ is $M$-point finite if for every compact subset $K \in \mathcal{K}(M)$ we have a subfamily $\mathcal{W}_K$ of $\mathcal{W}$ such that*

(i) $\mathcal{W} = \bigcup\{\mathcal{W}_K : K \in \mathcal{K}(M)\}$*;*
(ii) $\mathcal{W}_{K_1} \subset \mathcal{W}_{K_2}$ *whenever* $K_1 \subset K_2$ *in* $\mathcal{K}(M)$*;*
(iii) $\mathcal{W}_K$ *is a point finite family in* $X$ *for every* $K \in \mathcal{K}(M)$*.*

**Remark 1.** *It is enough to ask (i), (ii) and (iii) in Definition 3 for compact sets $K$ in a fundamental system of compact subsets of $M$ only. Indeed, let $\mathcal{S} \subset \mathcal{K}(M)$ such that each compact set in $M$ is included into an element of $\mathcal{S}$. Let $\mathcal{W}_S$ be defined for every $S \in \mathcal{S}$ and let (i)-(iii) above hold if $\mathcal{K}(M)$ is replaced by $\mathcal{S}$. For $K \in \mathcal{K}(M)$ put $\mathcal{W}_K := \bigcap\{\mathcal{W}_S : S \supset K, S \in \mathcal{S}\}$. Then (i)-(iii) are satisfied.*

Another way of describing weakly $\sigma$-point finite families in a given set $X$ is with the concept of *web* [37], which allows us to see the combinatorial structure of weakly $\sigma$-point finite families in a way similar to a Souslin scheme [28].

For $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and $\mathcal{W}$ a family of subsets of $X$ we assume it is possible to assign to each $\alpha \in \Sigma$ a subfamily $\mathcal{W}_\alpha \subset \mathcal{W}$ such that $\mathcal{W} = \bigcup\{\mathcal{W}_\alpha : \alpha \in \Sigma\}$. For $\beta = (b_s) \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$ we denote by $\beta|n$ the finite sequence $(b_1, b_2, \dots, b_n)$. If $(a_1, a_2, \dots, a_n)$ is a finite sequence of positive integers, then we write

$$\mathcal{W}_{a_1,a_2,\dots,a_n} := \bigcup\{\mathcal{W}_\beta : \beta \in \Sigma, \beta|n \equiv (a_1, a_2, \dots, a_n)\}$$

(it could be empty when there is no $\beta$ in $\Sigma$ with $\beta|n = (a_1, a_2, \dots, a_n)$) and we have a 'web of subfamilies': i.e.

$$\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n; \dots ; \mathcal{W}_{n_1,n_2,\dots,n_k} = \bigcup_{m=1}^{\infty} \mathcal{W}_{n_1,n_2,\dots,n_k,m}$$

for every $n_1, n_2, \dots, n_k \in \mathbb{N}$ and $k \in \mathbb{N}$.

**Definition 4.** *We say that a family $\mathcal{W}$ of subsets of $X$ is web-point finite if there is $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and a web of subfamilies as above, so that for every $\alpha \in \Sigma$ and for every $x \in X$ there is an integer $n_0 := n(\alpha, x)$ such that*

$$\operatorname{ord}(x, \mathcal{W}_{\alpha|n_0}) < \omega$$

The following results collects the characterizations we are looking for:

**Theorem 1.** *For a nonempty set $X$ and a family $\mathcal{W}$ of subsets on it, the following are equivalent:*

(i) $\mathcal{W}$ *is weakly* $\sigma$*-point finite,*
(ii) $\mathcal{W}$ *is* $M$*-point finite for a suitable separable metric space* $(M, d)$*,*
(iii) $\mathcal{W}$ *is* $\Sigma$*-point finite for a suitable* $\Sigma \subset \{0, 1\}^{\mathbb{N}}$*,*

(iv) $\mathcal{W}$ *is* $\Sigma$*-point finite for a suitable* $\Sigma \subset \mathbb{N}^{\mathbb{N}}$.
(v) $\mathcal{W}$ *is web-point finite.*

*Proof.* (ii) $\Rightarrow$ (i) Let us consider $d_H$ the Hausdorff distance on $\mathcal{K}(M)$, i.e.

$$d_H(A,B) := \sup\{d(a,B), d(A,b) : a \in A, b \in B\}$$

and we have $(\mathcal{K}(M), d_H)$ is a separable metric space too. Then, we claim that for every $K \in \mathcal{K}(M)$ and every $x \in X$ there exists a neighbourhood $V$ of $K$ in $(\mathcal{K}(M), d_H)$ such that

$$\operatorname{ord}(x, \bigcup\{\mathcal{W}_S : S \in V\}) < \omega.$$

Indeed, if this is not the case for some $x \in X$ and $K \in \mathcal{K}(M)$, we could find $W_1$ such that

$$x \in W_1 \in \bigcup\{\mathcal{W}_S : S \in B_{d_H}(K,1)\}.$$

Now, assume the sets $W_1, \ldots, W_n$ are already defined for some $n \in \mathbb{N}$. We can find $W_{n+1}$ such that

$$x \in W_{n+1} \in \bigcup\{\mathcal{W}_S : S \in B_{d_H}(K, \tfrac{1}{n+1})\} \setminus \{W_1, W_2, \ldots, W_n\}.$$

Now, for $n \in \mathbb{N}$ find $K_n \in B_{d_H}(K, \frac{1}{n})$ so that $W_n \in \mathcal{W}_{K_n}$ and put $K_\infty := K \cup K_1 \cup K_2 \cup \ldots$ This is an element of $\mathcal{K}(M)$ and so $\operatorname{ord}(x, \mathcal{W}_{K_\infty})$ is finite, which is a contradiction, since $x \in W_n \in \mathcal{W}_{K_\infty}$ for every $n \in \mathbb{N}$.

Let us now fix a countable basis $\mathcal{B}$ for the space $(\mathcal{K}(M), d_H)$ and define

$$\mathcal{W}(B) := \bigcup\{\mathcal{W}_K : K \in \mathcal{B}\}$$

for every $B \in \mathcal{B}$. We will have $\mathcal{W} = \bigcup\{\mathcal{W}(B) : B \in \mathcal{B}\}$ and for every $x \in X$, $\mathcal{W} = \bigcup\{\mathcal{W}(B) : \operatorname{ord}(x, \mathcal{W}(B)) < \omega, B \in \mathcal{B}\}$. Indeed, for every $K \in \mathcal{K}(M)$ our claim above provides us with an element $V \in \mathcal{B}$ such that $K \in V$ and $\operatorname{ord}(x, \mathcal{W}(V)) < \omega$.

(i) $\Rightarrow$ (iii) Since $\mathcal{W}$ is weakly $\sigma$-point finite there are countably many subfamilies of $\mathcal{W}$ such that $\mathcal{W} = \bigcup\{\mathcal{W}_n : n = 1, 2, \ldots\}$ with the property that for every $x \in X$ the following holds

$$\mathcal{W} = \bigcup\{\mathcal{W}_s : \operatorname{ord}(x, \mathcal{W}_s) < \omega\} \qquad (*)$$

For every $V \in \mathcal{W}$ let us consider the element $P(V) \in \{0,1\}^{\mathbb{N}}$ defined by

$$P(V)(n) = \begin{cases} 0 \text{ if } V \notin \mathcal{W}_n \\ 1 \text{ if } V \in \mathcal{W}_n \end{cases}$$

and let us call $\Sigma := \{P \in \{0,1\}^{\mathbb{N}} : P = P(V) \text{ for some } V \in \mathcal{W}\}$. Let us note that for every $P \in \Sigma$ the family $\mathcal{W}_P := \{V \in \mathcal{W} : P(V) = P\}$ is a point finite family in $X$. Indeed, given $P \in \Sigma$ and $x \in X$ suppose $\#\{V \in \mathcal{W}_P : x \in V\} = \omega$. Enumerate these sets as $\{V_n\}_{n=1}^\infty$ and let $\{s_m\}_{m=1}^\infty$ be a sequence of positive integers such that $\operatorname{ord}(x, \mathcal{W}_q) < \omega$ if and only if $q \in \{s_m\}$. Hence, for every fixed $i \in \mathbb{N}$ we have $V_n \notin \mathcal{W}_{s_i}$ for all large $n \in \mathbb{N}$, and so $P(s_i) = P(V_n)(s_i) = 0$. Thus $P(s_i) = 0$ for all $i \in \mathbb{N}$, and therefore $P(V_n)(s_i) = 0$, i.e., $V_n \notin \mathcal{W}_{s_i}$ for all $n, i \in \mathbb{N}$. However, as $\mathcal{W} = \bigcup\{\mathcal{W}_{s_i} : i \in \mathbb{N}\}$ by $(*)$, we have a contradiction.

In fact, this argument can be extended to show that for every compact $K \subset \Sigma \subset \{0,1\}^{\mathbb{N}}$ the family $\mathcal{W}_K := \{V \in \mathcal{W} : P(V) \in K\}$ is point finite in $X$. Indeed, given $x \in X$ and $K \subset \Sigma$ compact, let $\{s_1, s_2, \ldots, s_n, \ldots\} = \{s \in \mathbb{N} : \operatorname{ord}(x, \mathcal{W}_s) < \omega\}$. If $\#\{V \in \mathcal{W}_K : x \in V\}$ were infinite, put them into a sequence $\{V_n\}$. Since $K$ is compact we may assume $P(V_n)$ converges to some $P(V) \in K$, with $V \in \mathcal{W}$. Now for every $j \in \mathbb{N}$, only finitely many members of $\{V_n : n = 1, 2, \ldots\}$ can be in $\mathcal{W}_{s_j}$, so $P(V_n)(s_j) = 0$ for $n$ large enough. Thus $P(V)(s_j) = 0$ for all $j \in \mathbb{N}$, and this means $V \notin \mathcal{W} = \bigcup\{\mathcal{W}_{s_j} : j = 1, 2, \ldots\}$ which is a contradiction.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) are obvious.

(ii) $\Rightarrow$ (v) Since $(\mathcal{K}(M), d_H)$ is a separable metric space, there is a subset $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and a continuous onto map $\varphi : \Sigma \to (\mathcal{K}(M), d_H)$. If we simply define

$$\mathcal{W}_\alpha := \mathcal{W}_{\varphi(\alpha)}$$

we obtain the web-point finite decomposition of Definition 4 because of our claim in the proof (ii) $\Rightarrow$ (i) above, together with the continuity of $\varphi$.

(v) $\Rightarrow$ (i) The web $\{\mathcal{W}_{n_1, n_2, \ldots, n_k} : n_1, n_2, \ldots, n_k \in \mathbb{N}\}$ is a countable family of subfamilies of $\mathcal{W}$ which satisfies Definition 2, since for every $x \in X$ and $\alpha \in \Sigma$ there is $n_0$ such that $\operatorname{ord}(x, \mathcal{W}_{\alpha|n_0}) < \omega$. □

For the corresponding covering property of Gruenhage we have

**Corollary 2.** *A topological space $(X, \tau)$ is weakly $\sigma$-metacompact if, and only if, every open cover has an $M$-point finite open refinement for some separable metric space $M$.*

It is a simple consequence of Definition 2 that every weakly $\sigma$-point finite family of subsets of a given set $X$ is point countable. For this reason, the theorem by Mercourakis [29, Theorem 3.3] giving a Rosenthal's type characterization for Gul'ko compact spaces reads now as follows:

**Theorem 3.** *For a compact space $K$ the following are equivalent:*

(i) *$K$ is a Gul'ko compact space,*
(ii) *there is a separable metric space $M$ together with an $M$-point finite family $\mathcal{F}$ of open $\mathcal{F}_\sigma$-sets in $K$ which is $T_0$-separating, i.e. such that for every $x$ and $y$ in $K$, $x \neq y$, there is $A \in \mathcal{F}$ such that $\#A \cap \{x, y\} = 1$.*

*Proof.* After Theorem 1 it is reduced to Mercourakis' Theorem 3.3 in [29]. □

**Remark 2.** *Sokolov's characterization* [41] *says that $K$ is a Gul'ko compact space if, and only if, $K$ has a weakly $\sigma$-point finite $T_0$-separating family of open $\mathcal{F}_\sigma$-sets. Our Theorem 1 shows that both Sokolov's and Mercourakis' Theorems are, in fact, the same result. (See footnote 1 in Gruenhage's paper* [23]*). The notion of $\Sigma$-point finite family appears for the first time in Mercourakis' Theorem 3.3 in* [29]*. The study of $K$-analytic and $K$-countably determined spaces using the lattice structure of $\mathcal{K}(M)$ began with M. Talagrand* [42]*, see also* [9].

**Remark 3** (Index-$\Sigma$-point finite families)**.** *Given an indexed family of subsets of a given set $X$, $\mathcal{A} = \{A_i : i \in I\}$, and $x \in X$ we may consider the "index-order" of the point in the family, i.e. $\#\{i \in I : x \in A_i\}$ instead of $\#\{A \in \mathcal{A} : x \in A\}$.*

*For instance, we shall say that the indexed family $\mathcal{A} = \{A_i : i \in I\}$ is index-weakly $\sigma$-point finite if $I = \bigcup\{I_n : n = 1, 2, \ldots\}$ in such a way that for each $x \in X$ we have*

$$I = \bigcup\{I_s : \#\{i \in I_s : x \in A_i\} < \omega\}$$

*For two families $\mathcal{A} = \{A_i : i \in I\}$ and $\mathcal{B} = \{B_j : j \in J\}$ we say that $\mathcal{A}$ is an indexed subfamily of $\mathcal{B}$ if there is a one-to-one map $\xi : I \to J$ such that $A_i = B_{\xi(i)}$ for all $i \in I$.*

*Given a separable metric space $M$ and an indexed family $\mathcal{A} = \{A_i : i \in I\}$ of a given set $X$, we shall say that $\mathcal{A}$ is index-M-point finite if for every compact subset $K \in \mathcal{K}(M)$ we have a subset $I_K \subset I$ such that if we denote by $\mathcal{A}_K := \{A_i : i \in I_K\}$*

(i) $I = \bigcup\{I_K : K \in \mathcal{K}(M)\}$,
(ii) $\mathcal{A}_{K_1}$ *is an indexed subfamily of* $\mathcal{A}_{K_2}$ *whenever* $K_1 \subset K_2$ *in* $\mathcal{K}(M)$,
(iii) *For every* $x \in X$ *and* $K \in \mathcal{K}(M)$ $\#\{i \in I_K : x \in A_i\} < \omega$.

*Of course, Theorem 1 remains true for these notions. In particular, a family $\mathcal{A} = \{A_i : i \in I\}$ is index-weakly $\sigma$-point finite if, and only if, $\mathcal{A}$ is index-M-point finite for a suitable separable metric space $M$. We shall use later these facts. A proof follows the same arguments used in Theorem 1 with a bit of extra care. For instance, we need the following:*

**Lemma 1.** *Let $\mathcal{A} = \{A_i : i \in I\}$ be an index-M-point finite family of subsets of a given set $X$. Then for every $x \in X$ and $K \in \mathcal{K}(M)$ there is a neighbourhood $V$ of $K$ in $(\mathcal{K}(M), d_H)$ such that*

$$\#\{i \in \bigcup\{I_S : S \in V\} : x \in A_i\} < \omega$$

*Proof.* If this is not the case, we choose, for every positive integer $n$,

$$\{i_1^n, \ldots, i_n^n\} \subset \bigcup\{I_S : d_H(S, K) < \frac{1}{2^n}\},$$

with $x \in A_{i_j^n}$ for $j = 1, 2, \ldots, n$ and $i_j^n \neq i_k^n$ if $j \neq k$. If $i_j^n \in S_j^n$ with $d_H(S_j^n, K) < \frac{1}{2^n}$, $j = 1, 2, \ldots, n$ we shall consider the sequence

$$\{S_1^1, S_1^2, S_2^2, \ldots, S_1^n, S_2^n, \ldots, S_n^n, \ldots\} \text{ in } \mathcal{K}(M)$$

which converges to $K$, so

$$K_\infty := S_1^1 \cup S_1^2 \cup S_2^2 \cup \ldots \cup S_1^n \cup \ldots \cup S_n^n \cup \ldots \cup K$$

is a compact subset of $M$ with $K_\infty \supset S_j^n$ for $n = 1, 2, \ldots$, $j = 1, 2, \ldots, n$, and $\mathcal{A}_{s_j^n}$ is an indexed subfamily of $\mathcal{A}_{K_\infty}$ for $n = 1, 2, \ldots$, $j = 1, \ldots, n$.

Thus $\{i_1^n, i_2^n, \ldots, i_n^n\} \subset I_{S_j^n}$ corresponds with a set of $n$ different points $\{i_1^{\infty,n}, i_2^{\infty,n}, \ldots, i_n^{\infty,n}\}$ in the index set $I_{K_\infty}$ with $x \in A_{i_j^{\infty,n}}$, $j = 1, 2, \ldots, n$, for every $n \in \mathbb{N}$, which is a contradiction with the fact that

$$\#\{i \in I_{K_\infty} : x \in A_i\} < \omega$$

$\square$

*Once we have this Lemma, the proof of Theorem 1 for indexed families follows the same pattern. Let us show, for example, that an index-weakly $\sigma$-point finite family $\mathcal{A} = \{A_i : i \in I\}$ must be index-$\Sigma$-point finite for a suitable $\Sigma \subset \{0,1\}^{\mathbb{N}}$. By assumption we have $I = \bigcup\{I_n : n \in \mathbb{N}\}$ so that, for each $x \in X$, we have $I = \bigcup\{I_s : \#\{i \in I_s : x \in A_i\} < \omega\}$. For every $i \in I$ we consider $P(i) \in \{0,1\}^{\mathbb{N}}$ defined by*

$$P(i)(n) = \begin{cases} 0 \text{ if } i \notin I_n \\ 1 \text{ if } i \in I_n \end{cases}$$

*and $\Sigma := \{P \in \{0,1\}^{\mathbb{N}} : P = P(i)$ for some $i \in I\}$. Then for $K$ compact subset of $\Sigma$, we set $I_K := \{i \in I : P(i) \in K\}$ and we have:*

*(i) $I = \bigcup\{I_K : K \in \mathcal{K}(\Sigma)\}$ since, for every $i \in I$, $P(i) \in \Sigma$.*

*(ii) $I_{K_1} \subset I_{K_2}$ whenever $K_1 \subset K_2$ are compact subsets of $\Sigma$.*

*(iii) For every $K \in \mathcal{K}(\Sigma)$ and $x \in X$ we have $\#\{i \in I_K : x \in A_i\} < \omega$. If not, we would have a sequence $\{i_n\}$ with $P(i_n) \in K$ and $x \in A_{i_n}$ for $n = 1, 2, \ldots$ Since $K$ is compact we may and do assume that $\{P(i_n) : n = 1, 2, \ldots\}$ converges to $P(i) \in K$ for some $i \in I$. Since $x \in A_{i_n}$, $n = 1, 2, \ldots$ we have $i \notin I_s$ for any $s$ such that $\#\{i \in I_s : x \in A_i\} < \omega$. But this contradicts $I = \bigcup\{I_s : \#\{i \in I_s : x \in A_i\} < \omega\}$ and the proof is over.*

## 3. NETWORKS FOR $c_1(X)$

Following Mercourakis [29] we shall work with the space

$$c_1(X) := \{f \in \ell^\infty(X) : \forall \varepsilon > 0 \text{ the set } \{t \in X : |f(t)| \geq \varepsilon\} \text{ is closed and discrete in } X\},$$

for a given topological space $X$, and we shall consider it as a Banach space endowed with the supremum norm, i.e. a closed subspace of $\ell^\infty(X)$. For every $f \in c_1(X)$ and every compact subset $K$ of $X$ we have $f_{|K} \in c_0(K)$ because a closed and discrete subset of a compact space must be finite. So when $X$ is a compact space, we have $c_1(X) \equiv c_0(X)$. The important case for us is when $X$ is $K$-countably determined. Indeed, a main result of Mercourakis [29] is the fact that a compact space $X$ is a Gul'ko compact if, and only if, $X$ embeds in a space $c_1(Y)$, with the pointwise convergence topology, for some $K$-countably determined space $Y$. Our main objective here is to show the existence of suitable networks in spaces $(c_1(Y), \tau_p)$, for $K$-countably determined spaces $Y$, which will characterize Gul'ko compact spaces in section 4.

Networks were introduced by Arkangel'skii in 1958 and they have been very useful since then. They have become a prominent tool in renorming theory after the seminal paper of Hansell, [27], who showed that different kind of networks in Banach spaces are related to fragmentability properties, [31]. The *linking separability property* (LSP, for short), is another tool we have used to connect networks for different metric spaces [34, 35, 32]. Dow, Junnila and Pelant have characterized quite recently, [13] the LSP in terms of a network condition too. For compact spaces, this condition lies strictly between being Gul'ko compact and Corson compact, [13]. It is a natural question in this context to look for a suitable network characterization of Eberlein, Talagrand, Gul'ko and Corson compact spaces. Eberlein compacta are characterized in [13] too. We will present here characterizations for Talagrand and Gul'ko compacta. In order to deal with the LSP and the Eberlein compact case, the following notion becomes essential, as it is shown in [13].

**Definition 5.** *A family $\mathcal{L}$ of subsets of a topological space $(X, \tau)$ is said to be point-finitely (resp. point-countably) expandable if there exists a family of open sets $\{G_L : L \in \mathcal{L}\}$ such that $L \subset G_L$ for every $L \in \mathcal{L}$ and, for every $x \in X$, the family $\{L \in \mathcal{L} : x \in G_L\}$ is finite (resp. countable). The family $\mathcal{L}$ is said to be $\sigma$-point-finitely expandable if we can write $\mathcal{L} = \bigcup\{\mathcal{L}_n : n \in \mathbb{N}\}$ so that each family $\mathcal{L}_n$ is point-finitely expandable.*

Dow, Junnila and Pelant characterize the LSP in a topological space by the existence of a $\sigma$-disjoint and point-countably expandable network. They show that a compact space is an Eberlein compact if, and only if, it has a $\sigma$-point-finitely expandable network. This fact, together with Gruenhage's characterization of Eberlein compact spaces as those compacta whose complement of the diagonal is a $\sigma$-metacompact space, [20], gives the proof. Indeed, if one has a $\sigma$-point-finitely expandable network in a topological space $X$, it follows that $X$ is hereditarily $\sigma$-metacompact, [13]. Our aim is to follow the same ideas for Gul'ko compact spaces. We construct the appropiate expandable network in $(c_1(Y), \tau_p)$ which will give us the hereditarily weakly $\sigma$-metacompactness property thanks to our results from section 2.

After our study of $\Sigma$-point finite families in section 2 we introduce now the following definition (see Remark 3).

**Definition 6.** *Let $\mathcal{A}$ be a family of sets in a topological space $(X, \tau)$. $\mathcal{A}$ is said to be $\Sigma$-point-finitely expandable if $\mathcal{A}$ can be indexed as $\mathcal{A} = \{A_i : i \in I\}$ and for every $i \in I$ there exists an open set $G_i \supset A_i$ in $X$ such that the indexed family $\{G_i : i \in I\}$ is index-$\Sigma$-point finite; i.e. for a suitable separable metric space $M$ we have, for every $K \in \mathcal{K}(M)$, subsets $I_K \subset I$ such that:*

(i) $I = \bigcup\{I_K : K \in \mathcal{K}(M)\}$,
(ii) $\{G_i : i \in I_{K_1}\}$ *is an indexed subfamily of* $\{G_i : i \in I_{K_2}\}$ *whenever* $K_1 \subset K_2$ *in* $\mathcal{K}(M)$,
(iii) *For every $x \in X$ and $K \in \mathcal{K}(M)$ we have*

$$\#\{i \in I_K : x \in G_i\} < \omega$$

Now we can formulate our main result in this section:

**Theorem 4.** *Let $(X, \tau)$ be a $K$-countably determined topological space. Then $(c_1(X), \tau_p)$ has a $\Sigma$-point-finitely expandable network.*

*Proof.* Let us begin with the particular case of $(X, \tau)$ being a compact space. Then $c_1(X) \equiv c_0(X)$ and we follow Hansell's construction in [27, Theorem 7.5]. Let us be precise with all the details since we shall need all of them for the proof of the non-compact case. A close construction is the one presented in [13].

Let us fix $\mathbb{I} = \{I_n; n = 1, 2, \dots\}$ a countable basis for the topology of $\mathbb{R} \setminus \{0\}$ made of bounded open intervals such that for each $n$ there is an $\varepsilon > 0$ such that either $I_n \subset (-\infty, -\varepsilon)$ or $I_n \subset (\varepsilon, +\infty)$. Let us fix an integer $n \in \mathbb{N}$ and the first $n$ elements from $\mathbb{I}$; i. e. $\mathbb{I}_n := \{I_1, I_2, \dots, I_n\}$. We shall consider maps $\varphi : \Lambda \longrightarrow \mathbb{I}_n$ where $\Lambda \subset X$, i. e. we choose "doors" from $\mathbb{I}_n$ for every point $x \in \Lambda$, and we need only finite sets, i. e. $\#\Lambda < +\infty$, to describe the topology $\tau_p$. So let us consider, for fixed $n \in \mathbb{N}$,

$$\mathcal{M}_n := \{(\Lambda, \varphi); \Lambda \subset X, \#\Lambda \leq n \text{ and } \varphi : \Lambda \longrightarrow \mathbb{I}_n\}$$

and define for $(\Lambda, \varphi) \in \mathcal{M}_n$ the $\tau_p-$open set

$$R(\Lambda, \varphi) := c_0(X) \cap \prod_{x \in X} R_x \text{ where } R_x = \begin{cases} \varphi(x) & \text{if } x \in \Lambda, \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

Moreover, for $m \in \mathbb{N}$ let us define

$$R_m(\Lambda, \varphi) := c_0(X) \cap \prod_{x \in X} R_x \text{ where } R_x = \begin{cases} \varphi(x) & \text{if } x \in \Lambda, \\ (-1/m, 1/m) & \text{otherwise} \end{cases}$$

and we have

$$R_m(\Lambda, \varphi) \subset R(\Lambda, \varphi)$$

and the family

$$\mathcal{R}_n := \{R(\Lambda, \varphi); (\Lambda, \varphi) \in \mathcal{M}_n\}$$

consists of $\tau_p-$open subsets of $c_0(X)$ and it is a point finite family in $c_0(X)$ for every fixed $n \in \mathbb{N}$. Indeed, given $f \in c_0(X)$ such that, $f \in R(\Lambda_m, \varphi_m)$ with $\{(\Lambda_m, \varphi_m) : m \in \mathbb{N}\}$ an infinite set in $\mathcal{M}_n$, then $\bigcup\{\Lambda_m : m \in \mathbb{N}\}$ must be infinite too, since each $\varphi_m$ takes values in the finite set $\mathbb{I} = \{I_1, \dots, I_n\}$ and $n$ is fixed. Hence, for some infinite set $Y \subset X$ and for some $I_j$, $1 \leq j \leq n$, we have $f(y) \in I_j$ for all $y \in Y$, but this contradicts the fact that $f \in c_0(X)$ since $I_j$ is bounded away from zero.

We set for $m, n \in \mathbb{N}$

$$\mathcal{R}_{m,n} := \{R_m(\Lambda, \varphi); (\Lambda, \varphi) \in \mathcal{M}_n\}$$

and we have that $\cup_{m,n=1}^{\infty} \mathcal{R}_{m,n}$ is an open basis of the $\| \cdot \|_\infty$-topology in $c_0(X)$ and each family $\mathcal{R}_{m,n}$ is expandable to the family $\mathcal{R}_n$ which is formed by $\tau_p-$open sets and it is a point finite family in $c_0(X)$. Indeed, if $f \in c_0(X)$ and $m$ is a positive integer, if $\|f\|_\infty < \frac{1}{m}$, let $\Lambda = \emptyset$, otherwise let

$$\Lambda = \{x_1, x_2, \dots, x_k\} = \{x \in X : |f(x)| \geq \frac{1}{m}\}$$

and choose $I_{n_i} \in \mathbb{I}$ for $i = 1, 2, \dots, k$, such that

$$f(x_i) \in I_{n_i} \subset (f(x_i) - \frac{1}{m}, f(x_i) + \frac{1}{m}).$$

Let $n = \max\{k, n_1, n_2, \dots, n_k\}$ and define $\varphi : \Lambda \to \{I_1, I_2, \dots, I_n\}$ so that $\varphi(x_i) = I_{n_i}$ for $i = 1, 2, \dots, k$. Then $(\Lambda, \varphi) \in \mathcal{M}_n$ and

$$f \in R_m(\Lambda, \varphi) \subset B_{\|\cdot\|_\infty}(f, \frac{1}{m}).$$

So we have a $\sigma$-point-finitely expandable network in $(c_1(X) \equiv c_0(X), \tau_p)$ when $X$ is a compact space.

Let us show the case when $(X, \tau)$ is $K$-countably determined. So we will have a separable metric space $M$ such that $X = \cup\{X_K; K \in \mathcal{K}(M)\}$ where $X_K$ are compact subsets of $X$ and $X_{K_1} \subset X_{K_2}$ whenever $K_1 \subset K_2$ in $\mathcal{K}(M)$, [9]. If we make the construction we have done in the compact case for every $K \in \mathcal{K}(M)$; i. e. on every $c_0(X_K)$, then we shall arrive to a

$\Sigma$-point-finitely expandable network in $(c_1(X), \tau_p)$. Let us be more precise and for every fixed integer $n$ and every fixed $K \in \mathcal{K}(M)$ we shall consider:

$$\mathcal{M}(n, K) := \{(\Lambda, \varphi); \Lambda \subset X_K; \#\Lambda \leq n \text{ and } \varphi : \Lambda \longrightarrow \mathbb{I}_n\}$$

and we write, as before,

$$R(\Lambda, \varphi) := c_1(X) \cap \prod_{x \in X} R_x \text{ where } R_x = \begin{cases} \varphi(x) & \text{if } x \in \Lambda, \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

and

$$R_m(\Lambda, \varphi, K) := c_1(X) \cap \prod_{x \in X} R_x \text{ where } R_x = \begin{cases} \varphi(x) & \text{if } x \in \Lambda, \\ (-\frac{1}{m}, \frac{1}{m}) & \text{if } x \in X_K \setminus \Lambda, \\ \mathbb{R} & \text{if } x \notin X_K. \end{cases}$$

and we have

$$R_m(\Lambda, \varphi, K) \subset R(\Lambda, \varphi)$$

and the family

$$\mathcal{R}(n, K) := \{R(\Lambda, \varphi); (\Lambda, \varphi) \in \mathcal{M}(n, K)\}$$

is made of $\tau_p-$open subsets of $c_1(X)$ and it is a point finite family in $c_1(X)$ for every fixed $n \in \mathbb{N}$ and $K$ fixed. Indeed, every $h \in c_1(X)$ verifies $h|_{X_K} \in c_0(X_K)$ and therefore

$$\#\{(\Lambda, \varphi) \in \mathcal{M}(n, K); h \in \mathcal{R}(\Lambda, \varphi)\} < +\infty$$

as we have already seen in the compact case.

To describe the network we are looking for we take

$$\mathcal{N} := \{R_m(\Lambda, \varphi, K) : (\Lambda, \varphi) \in \mathcal{M}(n, K), m, n \in \mathbb{N} \text{ and } K \in \mathcal{K}(M)\}$$

$\mathcal{N}$ is a network for the pointwise topology in $c_1(X)$ since $\{R_m(\Lambda, \varphi, K) : n, m \in \mathbb{N}\}$ provides a basis for the topology of uniform convergence on the set $X_K$, as in the compact case. Thus $\mathcal{N}$ is a basis for the topology of uniform convergence on the sets $\{X_K : K \in \mathcal{K}(M)\}$, a topology finer that $\tau_p$ since every finite set $F$ of $X$ is contained in some $X_K$ with $K \in \mathcal{K}(M)$. It remains to show that $\mathcal{N}$ is $\Sigma$-point-finitely expandable. Our set of indexes to describe $\mathcal{N}$ is:

$$I := \{(m, n, K, \Lambda, \varphi) : (\Lambda, \varphi) \in \mathcal{M}(n, K), m, n \in \mathbb{N}\,, K \in \mathcal{K}(M)\}$$

and we set for $i = (m, n, K, \Lambda, \varphi) \in I$ the $\tau_p$-open set

$$G_i := R(\Lambda, \varphi) \supset R_m(\Lambda, \varphi, K) =: N_i$$

Let us consider the metric space $\mathbb{N} \times M$ where $\mathbb{N}$ is endowed with the discrete topology. Let us denote by $\pi_1 : \mathbb{N} \times M \to \mathbb{N}$ and $\pi_2 : \mathbb{N} \times M \to M$ the canonical projections. For a compact subset $S$ of $\mathbb{N} \times M$ we set

$$I_S := \{(m, n, K, \Lambda, \varphi) : (\Lambda, \varphi) \in \mathcal{M}(n, K), m, n \in \{1, 2, \ldots, q\}\}$$

where $q = \max \pi_1(S)$ and $K = \pi_2(S)$.

We can write

(i) $I = \cup\{I_S : S \in \mathcal{K}(\mathbb{N} \times M)\}$. Of course we have $\{G_i : i \in I_{S_1}\}$ is an indexed subfamily of $\{G_i : i \in I_{S_2}\}$ whenever $S_1 \subset S_2$ because $\mathcal{M}(n, \pi_2(S_1)) \subset \mathcal{M}(n, \pi_2(S_2))$ for every $n = 1, 2, \ldots$.

(iii) If $q = \max \pi_1(S)$ and $K = \pi_2(S)$ for the compact subset $S$ of $\mathbb{N} \times M$, we have

$$\#\{(m, n, K, \Lambda, \varphi) = i \in I_S : f \in G_i = R(\Lambda, \varphi)\}$$
$$\leq \sum_{n=1}^{q} q \cdot (\#\{(\Lambda, \varphi) \in \mathcal{M}(n, K) : f \in R(\Lambda, \varphi)\}) < \omega$$

because $\mathcal{R}(n, K)$ was a point finite family in $c_1(X)$ and the proof is over. □

For the corresponding covering property we have:

**Proposition 5.** *Let $(X, \tau)$ be a topological space with a $\Sigma$-point finitely expandable network. Then $X$ is hereditarily weakly $\sigma$-metacompact.*

*Proof.* The hereditarily weakly $\sigma$-metacompactness will follow if we can find, for every arbitrary family $\mathcal{V}$ of open subsets of $X$, a weakly $\sigma$-point finite open refinement of $\mathcal{V}$. So, let us fix $\mathcal{V}$ and $\Omega := \cup\mathcal{V}$. Let $\mathcal{N} = \{N_i : i \in I\}$ be the $\Sigma$-point-finitely expandable network for $(X, \tau)$; i.e. for a suitable separable metric space $M$ we have $I_K \subset I$ for every $K \in \mathcal{K}(M)$ and open sets $G_i \supset N_i$ for every $i \in I$ such that $\{G_i : i \in I\}$ satisfies conditions (i) to (iii) in Definition 6. Given $x \in \Omega$ we can find $i \in I$ with

$$x \in N_i \subset V \in \mathcal{V}$$

by definition of network.

Set $J := \{i \in I : N_i \subset V \text{ for some } V \in \mathcal{V}\}$ and choose, for every $j \in J$, an open set $V(j) \in \mathcal{V}$ with $N_j \subset V(j)$. Now we can define the open refinement of $\mathcal{V}$ by

$$\mathcal{W} := \{G_j \cap V(j) : j \in J\}$$

with $\cup\mathcal{W} = \Omega$. Moreover, since $\{G_i : i \in I\}$ is an index-$\Sigma$-point finite family we know that $I = \cup I_n$ and for every $x \in X$ we also have

$$I = \cup\{I_s : \#\{i \in I_s : x \in G_i\} < \omega\},$$

(see Remark 3). Of course, if we denote by $J_n := J \cap I_n$, we have $J = \cup\{J_n : n = 1, 2, \ldots\}$ and for every $x \in X$

$$J = \cup\{J_s : \#\{j \in J_s : x \in G_j \cap V(j)\} < \omega\}$$

since $\#\{j \in J_s : x \in G_j \cap V(j)\} < \omega$ whenever $\#\{i \in I_s : x \in G_i\} < \omega$. So $\mathcal{W}$ is a weakly $\sigma$-point finite open refinement of $\mathcal{V}$. □

**Corollary 6.** *For every $K$-countably determined topological space $X$, then the space $(c_1(X), \tau_p)$ is hereditarily weakly $\sigma$-metacompact and, in particular, hereditarily submetacompact.*

As a consequence we obtain now Theorem 2 in [23]:

**Corollary 7.** *Every Gul'ko compact space has a $\Sigma$-point-finitely expandable network and it is hereditarily weakly $\sigma$-metacompact too.*

*Proof.* It is a consequence of Mercourakis' theorem ([29, Theorem 3.1]) saying that every Gul'ko compact space is homeomorphically embedded in $(c_1(Y), \tau_p)$ for some $K$-countably determined space $Y$ together with theorem 4 and proposition 5. □

**Remark 4.** *For $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and $\Gamma$ any set, it is defined* [15, 11]

$$c_1(\Sigma \times \Gamma) := \{f \in \ell^\infty(\Sigma \times \Gamma) : f_{|K\times\Gamma} \in c_0(K \times \Gamma) \textit{ for every } K \subset \mathcal{K}(\Sigma)\}$$

*It follows adding one point $\infty$ that $\Sigma \times \Gamma \cup \{\infty\}$ will be $K$-countably determined, see* [29, Definition 1.3]*, then $c_1(\Sigma \times \Gamma)$ can be seen as the subspace of $c_1(\Sigma\times\Gamma\cup\{\infty\})$ formed by the functions vanishing at $\infty$. Thus, for $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and $\Gamma$ any set the space $(c_1(\Sigma \times \Gamma), \tau_p)$ has a $\Sigma$-point-finitely expandable network and it is hereditarily weakly $\sigma$-metacompact too.*

## 4. Covering properties on $X^2 \setminus \Delta$

Following Gruenhage and Michael [24] we say that an open cover $\mathcal{G}$ of a topological space $(X, \tau)$ can be shrunk if there exists an indexed closed cover

$$\{A_G; G \in \mathcal{G}\}$$

such that $A_G \subset G$ for every $G \in \mathcal{G}$.

We shall need the following result in the course of the proof of our theorem 9. The cases of metalindelöf or $\sigma$-metacompact has been considered in [24], now we need the proof for the weakly $\sigma$-metacompact case. Fortunately the same arguments as in [24] also work this time:

**Proposition 8.** *Let $(X, \tau)$ be a weakly $\sigma$-metacompact, locally compact space, and let $\mathcal{B}$ a basis for $(X, \tau)$. Then $X$ has a subcover $\mathcal{B}' \subset \mathcal{B}$ such that the indexed family $\{\overline{B}; B \in \mathcal{B}'\}$ is an index-$\Sigma$-point finite family in $X$.*

*Proof.* Let $\mathcal{G}$ be an open cover of $X$ by open sets with compact closures and let $\mathcal{V}$ be an $M$-point finite open refinement of $\mathcal{G}$ (corollary 2), for a suitable separable metric space $M$. By [24, Theorem 1.1] the cover $\mathcal{V}$ can be shrunk to a closed cover $\{A_V; V \in \mathcal{V}\}$. If $V \in \mathcal{V}$, then $A_V$ is compact, so there is a finite family $\mathcal{B}_V \subset \mathcal{B}$ such that $\mathcal{B}_V$ covers $A_V$ and such that $\overline{B} \subset V$ for every $B \in \mathcal{B}_V$. The collection $\mathcal{B}' = \bigcup\{\mathcal{B}_V; V \in \mathcal{V}\}$ is such that $\{\overline{B}; B \in \mathcal{B}'\}$ is $\Sigma$-point finite. Indeed, since $\mathcal{V}$ es $M$-point finite, we know that for every

$K \in \mathcal{K}(M)$, $\mathcal{V}_K$ is a point finite subfamily of $\mathcal{V}$, $\mathcal{V} = \bigcup\{\mathcal{V}_K; K \in \mathcal{K}(M)\}$ and $\mathcal{V}_{K_1} \subset \mathcal{V}_{K_2}$ whenever $K_1 \subset K_2$. Let us define for $K \in \mathcal{K}(M)$

$$\mathcal{B}'_K := \bigcup\{\mathcal{B}_V; V \in \mathcal{V}_K\}$$

then we have

$$\mathcal{B}' = \cup\{\mathcal{B}'_K; K \in \mathcal{K}(M)\}$$

and $\mathcal{B}'_{K_1} \subset \mathcal{B}'_{K_2}$ whenever $K_1 \subset K_2$ in $\mathcal{K}(M)$. Moreover, for every $K \in \mathcal{K}(M)$ we have $\{\overline{B}; B \in \mathcal{B}'_K\}$ is point finite because $\mathcal{V}_K$ is a point finite family and for every $V \in \mathcal{V}_K$ only a finite number of elements of $\{\overline{B}; B \in \mathcal{B}'_K\}$ has been considered, exactly the ones in the family $\mathcal{B}_V$. □

Finally we are ready for the proof of our main result:

**Theorem 9.** *The following are equivalent for a compact space $X$.*

(i) *$X$ is Gul'ko compact;*
(ii) *$X^2 \setminus \Delta$ is weakly $\sigma$-metacompact;*
(iii) *$X^2$ is hereditarily weakly $\sigma-$metacompact;*
(iv) *$X$ admits a $\Sigma$-point-finitely expandable network.*

*Proof.* (i)⇒(iv) It follows from Corollary 7.

(iv)⇒(iii) Because the property of having a $\Sigma$-point-finitely expandable network is stable by finite products together with our Proposition 5.

(iii)⇒(ii) It is trivial.

(ii)⇒(i) We shall follow the proof of [20, Theorem 2.2] adding the details for our case here. Indeed if $X^2 \setminus \Delta$ is weakly $\sigma$-metacompact, then by the proof of Proposition 8 there is a cover

$$\mathcal{P} = \{U_\gamma \times V_\gamma; \gamma \in A\}$$

of $X^2 \setminus \Delta$ such that:

(a) $U_\gamma$ and $V_\gamma$ are open $\mathcal{F}_\sigma$ in $X$, (take the original cover in Proposition 8 with sets $U \times V$ with $U$ and $V$ being $\mathcal{F}_\sigma$-sets).
(b) $\overline{U_\gamma} \cap \overline{V_\gamma} = \emptyset$, $\forall \gamma \in A$.
(c) $\{\overline{U_\gamma} \times \overline{V_\gamma}; \gamma \in A\}$ is an index-$\Sigma$-point finite family in $X^2 \setminus \Delta$.
(d) $U \times V \in \mathcal{P}$ implies $V \times U \in \mathcal{P}$.

Now if $\operatorname{dens} X = \mu$ and $X = \overline{\{p_\alpha; \alpha < \mu\}}$, we set for each $\alpha < \mu$

$$X_\alpha := \overline{\{p_\beta; \beta < \alpha\}}$$

and

$$\mathcal{U}_\alpha := \{\cap_{\gamma \in F} U_\gamma; F \subset A \text{ and } \{\overline{V_\gamma}; \gamma \in F\} \text{ is a finite minimal cover of } X_\alpha\}.$$

Note that $\mathcal{U}_\alpha$ covers $X \setminus X_\alpha$. Then the family $\bigcup\{\mathcal{U}_\beta; \beta < \mu\}$ is $T_0$-separating as in [21, Theorem 2.2, Claim 2]. And moreover $\bigcup\{\mathcal{U}_\beta; \beta < \mu\}$ is a $\Sigma$-point finite family in $X$. Indeed, by (c) we know that there is a separable metric space $M$ such that $A = \cup\{A_K; K \in \mathcal{K}(M)\}$, with $\{\overline{U}_\alpha \times \overline{V}_\alpha; \alpha \in A_K\}$ point finite for every $K \in \mathcal{K}(M)$ and $A_{K_1} \subset A_{K_2}$ whenever $K_1 \subset K_2$ in $\mathcal{K}(M)$ (that is the case in the proof of Proposition 8).

For $K \in \mathcal{K}(M)$ and $n \in \mathbb{N}$ fixed, let $\mathcal{U}^K_{\alpha,n}$ be all members of $\mathcal{U}_\alpha$ whose corresponding index set $F$ has cardinality $\leq n$, and it is contained in $A_K$. Then $\bigcup\{\mathcal{U}^K_{\alpha,n} : \alpha < \mu\}$ is a point finite family in $X$. Indeed, if there is $x \in X$ that belongs to infinitely many members of $\bigcup_{\alpha<\mu}\mathcal{U}^K_{\alpha,n}$, then for $q \in \mathbb{N}$ we find ordinals $\beta_q < \mu$ and sets $F_q \subset A_K$ such that $\#F_q \leq n$, $x \in \cap\{U_\gamma : \gamma \in F_q\}$, $X_{\beta_q} \subset \cup\{\overline{V}_\alpha : \gamma \in F_q\}$ and $F_q \neq F_r$ if $q \neq r$. By avoiding some $q's$ and relabelling, we may and do assume that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_q \leq \ldots$.

Since $\#F_q \leq n$, $q = 1, 2, \ldots$ and all of them are different, it is possible to assume that $\{F_q; q = 1, 2, \ldots\}$ forms a $\Delta-$system with root $R$ maybe empty. In any case $R \neq F_1$ and there is $y \in X_{\beta_1} \setminus \cup\{\overline{V}_\gamma; \gamma \in R\}$. Then for each $q$ there exists $\delta(q) \in F_q \setminus R$ with $y \in \overline{V}_{\delta(q)}$. But then we have

$$(x, y) \in \cap_{q=1}^{\infty} U_{\delta(q)} \times \overline{V_{\delta(q)}}$$

and $\{\delta(q); q = 1, 2, \ldots\} \subset A_K$ which contradicts the fact that $\{\overline{U_\gamma} \times \overline{V_\gamma}; \gamma \in A_K\}$ is point finite since all $\{\delta(q); q = 1, 2, \ldots\}$ are different elements in $A_K$. Thus we see that $\bigcup\{\mathcal{U}_\alpha; \alpha < \mu\}$ can be written as

$$\bigcup\{\{\mathcal{U}^K_{\alpha,n}; \alpha < \mu\}; K \in \mathcal{K}(M), n \in \mathbb{N}\}$$

and we see that it is a $\Sigma$-point finite family of open $\mathcal{F}_\sigma$ sets in $X$ which is also $T_0$-separating. To finish the proof it is enough to apply Theorem 3 (see Remarks 2 and 4) to conclude that $X$ is a Gul'ko compact indeed. □

**Remark 5.** *As we mentioned in the introduction, Gruenhage* [23, Remark 2] *asks if for a Corson compact $K$, the condition of $K^2$ being hereditarily weakly submetacompact characterizes Gul'ko compacta. The answer in no. An example constructed in* [3, Theorem 3.3] *gives us a Corson compact space $\Omega$ which is not Gul'ko compact but it is a Gruenhage space. Moreover, we have proved in* [36] *that this compact space $\Omega$ admits a $\sigma$-relatively discrete network, i.e. a network $\mathcal{N}$ which can be written $\mathcal{N} = \cup\{\mathcal{N}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$ the family $\mathcal{N}_n$ is discrete in $\cup\mathcal{N}_n$. Since a space that admits such a network must be hereditarily weakly submetacompact* [27]*, the example $\Omega$ provides the answer to Gruenhage's question. The space $\Omega$ is also studied in* [15, Theorem 7.3.2].

## 5. Talagrand compact spaces

There is an analogue of Theorem 9 for Talagrand compact spaces. Of course our previous statements can be adapted to give the proof for that case. The essential change is that the separable metric space $M$ will be now complete too; so a continuous image of the Baire space $\mathbb{N}^{\mathbb{N}}$ where we have the fundamental system of compact subsets given by:

$$\{A_\alpha := \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta(n) \leq \alpha(n), n = 1, 2, \ldots\} \text{ for } \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

So we shall work with the Baire space $\mathbb{N}^{\mathbb{N}}$ and with the order relation

$$\alpha \leq \beta \text{ if, and only if } \alpha(n) \leq \beta(n), n = 1, 2, \ldots \text{ for } \alpha, \beta \in \mathbb{N}^{\mathbb{N}}$$

instead of the lattice of compact subsets $\mathcal{K}(\mathbb{N}^{\mathbb{N}})$.

We shall begin with $\mathbb{N}^{\mathbb{N}}$-point finite families (see Definition 3), then we have the following result.

**Proposition 10.** *A collection $\mathcal{W}$ of subsets of a given set $X$ is $\mathbb{N}^{\mathbb{N}}$-point finite if, and only if, we have subfamilies $\mathcal{W}_\alpha$ of $\mathcal{W}$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that*

(i) $\mathcal{W} = \cup\{\mathcal{W}_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$*;*
(ii) $\mathcal{W}_\alpha \subset \mathcal{W}_\beta$ *whenever* $\alpha \leq \beta$ *in* $\mathbb{N}^{\mathbb{N}}$*;*
(iii) $\mathcal{W}_\alpha$ *is a point finite family in* $X$ *for every* $\alpha \in \mathbb{N}^{\mathbb{N}}$.

For an indexed family $\mathcal{A} = \{A_i : i \in I\}$, $\mathcal{A}$ is index-$\mathbb{N}^{\mathbb{N}}$-point finite if, and only if, $I = \bigcup\{I_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with $\mathcal{A}_\alpha := \{A_i : i \in I_\alpha\}$ an indexed subfamily of $\mathcal{A}_\beta := \{A_i : i \in I_\beta\}$ whenever $\alpha \leq \beta$ in $\mathbb{N}^{\mathbb{N}}$, and for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$ we have $\#\{i \in I_\alpha : x \in A_i\} < \omega$.

Let us remark that every $\sigma$-point finite family is $\mathbb{N}^{\mathbb{N}}$-point finite too because the union of a finite collection of point finite families is point finite too. The following is the analogue to Theorem 1 and describes the combinatorial structure here:

**Theorem 11.** *For a family $\mathcal{W}$ of subsets of a given set $X$ the following conditions are equivalent:*

(i) $\mathcal{W}$ *is* $\mathbb{N}^{\mathbb{N}}$*-point finite;*
(ii) $\mathcal{W}$ *is* $M$*-point finite for some Polish space* $M$*;*
(iii) $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n$ *and for* $n_1, n_2, \ldots, n_k, k \in \mathbb{N}$,

$$\mathcal{W}_{n_1,\ldots,n_k} = \bigcup_{m=1}^{\infty} \mathcal{W}_{n_1,n_2,\ldots,n_k,m}$$

*such that for every* $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$ *and for every* $x \in X$ *there is an integer* $n_0 := n(\alpha, x)$ *such that* $\operatorname{ord}(x, \mathcal{W}_{\alpha|n_0}) < \omega$.

*Proof.* (ii) $\Rightarrow$ (iii) There is a continuous onto map $\varphi : \mathbb{N}^{\mathbb{N}} \to (\mathcal{K}(M), d_H)$, because $(\mathcal{K}(M), d_H)$ is complete too. If we set, for $n_1, n_2, \ldots, n_k, k \in \mathbb{N}$

$$\mathcal{W}_{n_1,\ldots,n_k} := \{W \in \mathcal{W} : W \in \mathcal{W}_{\varphi(\alpha)} \text{ with } \alpha \in \mathbb{N}^{\mathbb{N}}, \alpha|k = (n_1, \ldots, n_k)\}$$

then we have a web of subfamilies

$$\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n \quad \text{and} \quad \mathcal{W}_{n_1,\ldots,n_k} = \bigcup_{m=1}^{\infty} \mathcal{W}_{n_1,\ldots,n_k,m}$$

which verifies (iii) after our claim in Theorem 1 for the proof of (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (i) Given $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$ we set

$$\mathcal{D}_\alpha := \{W \in \mathcal{W} : W \in \mathcal{W}_{a_1,a_2,\ldots,a_n}, n = 1, 2, \ldots\},$$

and we have, by the web conditions in (iii) that $\mathcal{W} = \bigcup\{\mathcal{D}_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

Let us take $\mathcal{W}_\alpha := \bigcup\{\mathcal{D}_\beta : \beta \leq \alpha\}$ and we obviously have (i) and (ii) in Proposition 10. Moreover, for every $x \in X$ we have $\operatorname{ord}(x, \mathcal{W}_\alpha) < \omega$. If this is not true, we will have a sequence of elements $\{W_n\}$ in $\mathcal{W}_\alpha$ with $W_n \neq W_m$ for $n \neq m$ and $x \in \cap_{n=1}^{\infty} W_n$. For every integer $n$ there is $\beta_n \leq \alpha$ such that $W_n \in \mathcal{D}_{\beta_n}$ and we may and do assume that $(\beta_n)$ converges to some $\beta \leq \alpha$ in $\mathbb{N}^{\mathbb{N}}$. Then, for every $p \in \mathbb{N}$, we have $\beta_n|p = \beta|p$ for $n$ large enough, and so $W_n \in \mathcal{W}_{\beta|p}$ for $n$ large enough, and $\operatorname{ord}(x, \mathcal{W}_{\beta|p}) = \omega$ too. This is a contradiction with (iii) which finishes the proof. □

**Remark 6** (Index-$\mathbb{N}^{\mathbb{N}}$-point finite families)**.** *Of course we also have the version of Theorem 11 for index-$\mathbb{N}^{\mathbb{N}}$-point finite families $\mathcal{A} = \{A_i : i \in I\}$. In this case, (iii) reads as follows:*

*There is a web $\{I_{n_1,\ldots,n_k} : (n_1, \ldots, n_k) \in \mathbb{N}^k, k = 1, 2, \ldots\}$ of subsets of $I$; i.e. $I = \cup_{n=1}^{\infty} I_n$ and for $n_1, n_2, \ldots, n_k, k \in \mathbb{N}$ we have*

$$I_{n_1,\ldots,n_k} = \cup_{m=1}^{\infty} I_{n_1,n_2,\ldots,n_k,m}$$

*such that for every $\alpha = (a_n) \in \mathbb{N}^{\mathbb{N}}$ and for every $x \in X$ there is an integer $n_0 := n(\alpha, x)$ such that*

$$\#\{i \in I_{a_1,a_2,\ldots,a_{n_0}} : x \in A_i\} < \omega.$$

*For the proof we use the same arguments as above, using Lemma 1 instead of the claim in (ii) $\Rightarrow$ (i), Theorem 1.*

The covering property in that case is the following:

**Definition 7.** *A topological space $(X, \tau)$ is $\mathbb{N}^{\mathbb{N}}$-metacompact if for every open cover $\mathcal{U}$ in $X$ we have an open refinement $\mathcal{V}$ which is $\mathbb{N}^{\mathbb{N}}$-point finite in $X$.*

Of course we have:

$\sigma$-metacompact $\Rightarrow$ $\mathbb{N}^{\mathbb{N}}$-metacompact $\Rightarrow$ weakly $\sigma$-metacompact

$\Rightarrow$ metalindelöf (*).

and the arrows can not be reversed at all. Indeed, after the characterizations in [20] and our theorems 9 and 16, the examples of compact subsets distinguishing in the relations

$$\text{Eberlein compact} \Rightarrow \text{Talagrand compact} \Rightarrow \text{Gul'ko compact} \\ \Rightarrow \text{Corson compact}$$

provide us with examples to distinguish between the covering properties in (*).

For expandability we now need the following:

**Definition 8.** *Let $\mathcal{A}$ be a family of subsets of a topological space $(X, \tau)$. We shall say that $\mathcal{A}$ is $\mathbb{N}^{\mathbb{N}}$-point-finitely expandable when $\mathcal{A}$ can be indexed as $\mathcal{A} = \{A_i : i \in I\}$ and for every $i \in I$ there exists an open set $G_i \supset A_i$ in $X$ such that the indexed family $\{G_i : i \in I\}$ in index-$\mathbb{N}^{\mathbb{N}}$-point finite.*

Now we have

**Theorem 12.** *Let $(X, \tau)$ be a $K$-analytic topological space. Then the space $(c_1(X), \tau_p)$ has a $\mathbb{N}^{\mathbb{N}}$-point-finitely expandable network.*

*Proof.* As in the proof of Theorem 4, but now we have $X = \cup\{X_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ where $X_\alpha$ is a compact subset of $X$ and $X_\alpha \subset X_\beta$ whenever $\alpha \leq \beta \in \mathbb{N}^{\mathbb{N}}$, . □

Of course we also have the result corresponding to proposition 5;

**Proposition 13.** *Let $(X, \tau)$ be a topological space with a $\mathbb{N}^{\mathbb{N}}$-point-finitely expandable network. Then $X$ is hereditarily $\mathbb{N}^{\mathbb{N}}$-metacompact.*

*Proof.* It follows the arguments of Proposition 5. Now we use the web characterization (iii) in Theorem 11 and the Remark 6, instead of the weakly $\sigma$-point finite characterization for the open expansion $\{G_i : i \in I\}$ of the network $\mathcal{N}$. □

Consequently we have:

**Corollary 14.** *For every $K$-analytic topological space $X$, $(c_1(X), \tau_p)$ is hereditarily $\mathbb{N}^{\mathbb{N}}$-metacompact and, in particular, hereditarily submetacompact.*

**Corollary 15.** *Every Talagrand compact space has a $\mathbb{N}^{\mathbb{N}}$-point-finitely expandable network and it is hereditarily $\mathbb{N}^{\mathbb{N}}$-metacompact too.*

The proof of Proposition 8 can be also adapted to $\mathbb{N}^{\mathbb{N}}$-metacompact spaces. Then we have all the ingredients for the proof of:

**Theorem 16.** *The following are equivalent for a compact space $X$.*

(i) *$X$ is Talagrand compact;*

(ii) *$X^2 \setminus \Delta$ is $\mathbb{N}^{\mathbb{N}}$-metacompact;*
(iii) *$X^2$ is hereditarily $\mathbb{N}^{\mathbb{N}}$-metacompact;*
(iv) *$X$ admits a $\mathbb{N}^{\mathbb{N}}$-point-finitely expandable network.*

*Proof.* It follows the scheme of the proof of Theorem 9 and it is used here the following "Rosenthal-type" theorem for Talagrand compact spaces, that follows from Farmaki [18]: □

**Theorem 17.** *A compact space $X$ is Talagrand compact if, and only if, there exists a $\mathbb{N}^{\mathbb{N}}$-point finite family $\mathcal{A}$ of open $\mathcal{F}_\sigma$-subsets of $X$, which $T_0$-separates the points of $X$.*

*Proof.* It follows from Farmaki's Theorem because the extra assumption in [18] of $\mathcal{A}$ being point countable is not necessary, since every $\mathbb{N}^{\mathbb{N}}$-point finite family is point countable by Theorem 1. □

**Problem 1.** *Find a network characterization for the class of Corson compact spaces. See* [13].

**Acknowledgments.** The authors wish to thank the referee for his suggestions and comments which helped a lot to improve the final presentation of the paper.

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