LOCALLY UNIFORMLY ROTUND RENORMING AND FRAGMENTABILITY

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1. *Introduction*

In this paper we characterize those Banach spaces that admit a locally uniformly rotund renorming by means of a linear topological condition. Let us recall the following definition.

DEFINITION 1. A Banach space E (or the norm in E) is said to be *locally uniformly rotund* (LUR for short) if

$$\lim_k \|x_k - x\| = 0 \quad \text{whenever} \quad \lim_k \|\tfrac{1}{2}(x_k + x)\| = \lim_k \|x_k\| = \|x\| .$$

The spaces with this property are at the core of renorming theory in Banach spaces and consequently have been extensively studied (see, for example, [**3**]). It is well known that the spaces with a LUR norm have the Kadec property.

DEFINITION 2. A Banach space E (or the norm in E) is said to have *Kadec property* (K for short) if the relative norm and the weak topologies coincide on the unit sphere of E.

The third named author showed that every rotund Banach space with the Kadec property admits a LUR equivalent norm (see, for example, [**3**, Chapter IV]). The proof of this result was based on some martingale arguments whose origin can be found in a paper by G. Pisier [**21**].

On the other hand, J. E. Jayne, I. Namioka and C. A. Rogers introduced and studied the class of σ-fragmentable topological spaces and its applications in Banach spaces [**10**, **11**, **12**, **13**], arriving at the concept of spaces having a countable cover by sets of small local diameter. We consider this property in a particular case and call it the JNR property.

DEFINITION 3. Let $(E, \mathcal{T})$ be a Hausdorff topological space and ρ a metric on E. The topological space E is said to have a *countable cover by sets of small local*

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ρ-diameter (ρ-SLD for short) if for each $\varepsilon > 0$, it is possible to write $E = \bigcup_n E_{n,\varepsilon}$, in such a way that for every $n \in \mathbb{N}$ and $x \in E_{n,\varepsilon}$ there exists a neighbourhood V of x such that ρ-diam$(V \cap E_{n,\varepsilon}) < \varepsilon$. Let E be a normed space, w its weak topology, and $\rho(x, y) := \|x - y\|$, for $x, y \in E$, the norm metric. If (E, w) is ρ-SLD then we shall say that E has the property JNR.

In this paper we introduce a concept which is a particular case of JNR, and allows us to characterize the spaces that have a LUR renorming.

If E is a normed space, B_E will stand for the unit ball of E. For $L \subset E$, $f \in E^*$ and $\lambda \in \mathbb{R}$, we denote by $S(L, f, \lambda) = \{u \in L\colon f(u) > \lambda\}$ the open slice of L. If L is the unit ball of E, we write $S(f, \lambda)$ for $S(L, f, \lambda)$.

DEFINITION 4. Let L be a subset of a normed space E and $\varepsilon > 0$. We will say that L has a *countable cover by sets which are a union of slices of diameter less than* ε (ε-sJNR for short) if $L = \bigcup_n L_n$ in such a way that for every $x \in L_n$, $n \in \mathbb{N}$, there exists a slice $S(L_n, f, \lambda)$ containing x and $\operatorname{diam} S(L_n, f, \lambda) < \varepsilon$. If L has ε-sJNR for every $\varepsilon > 0$, then we say that L has sJNR.

Our main result is the following.

MAIN THEOREM. *Let E be a Banach space. The following conditions are equivalent:*

(a) *the unit sphere S_E of E has* sJNR*;*

(b) *E has* sJNR*; and*

(c) *E has an equivalent* LUR *norm.*

As we mentioned, K and rotundity imply the existence of a LUR renorming. Since every point in the unit sphere S_E of such spaces has a base of neighbourhoods in the norm topology made up of slices [**19**], this result follows immediately from the Main Theorem.

In [**17**] Lancien proves that a Banach space E with countable dentability index is LUR renormable. That means there is a countable ordinal τ such that for every $\varepsilon > 0$ there is a decreasing family $(C_{\alpha,\varepsilon})_{\alpha<\tau}$ of convex subsets of B_E, with $C_{0,\varepsilon} = B_E$, $\bigcap\{C_{\alpha,\varepsilon}\colon \alpha < \tau\} = \emptyset$, and such that for every $\alpha < \tau$ the set $C_{\alpha,\varepsilon} \setminus C_{\alpha+1,\varepsilon}$ is a union of slices of diameter less than ε. So we have $B_E = \bigcup_{\alpha<\tau}(C_{\alpha,\varepsilon} \setminus C_{\alpha+1,\varepsilon})$ and his result follows from our main theorem. In the case when τ is a finite ordinal, he obtains a characterization of uniformly rotund renormability and a new proof of the well-known James–Enflo–Pisier renorming theorem [**18**].

If T is a one-to-one bounded linear operator from a Banach space E into a Banach space F and F has a rotund norm, it is well known that E also has a rotund norm. For LUR renorming a similar result does not hold, and it seems natural to ask what conditions it is necessary to impose on the operator T to get a LUR renorming on E. According to the Main Theorem we know that it is enough to pull back the sJNR property. From this point of view it is quite natural to require that $(E, \|\cdot\|_T)$ is $\|\cdot\|$-SLD where $\|x\|_T := \|Tx\|$ (that is, if we consider the metrics $d(x, y) := \|Tx - Ty\|$, $\rho(x, y) := \|x - y\|$, with $x, y \in E$, and $\mathcal{T}$ the topology associated to d, then $(E, \mathcal{T})$ is ρ-SLD). In this way we arrive at the following result which improves the transfer technique of G. Godefroy [**5**; **3**, Chapter

VII.2], at least in obtaining LUR renorming without additional properties for dual norms.

THEOREM 5 (Transfer Technique). *Let T be a one-to-one bounded linear operator from a Banach space $(E, \|\cdot\|)$ into a* LUR *normed space F and define $\|x\|_T = \|Tx\|$ for every $x \in E$. If $(E, \|\cdot\|_T)$ is $\|\cdot\|$*-SLD, *then E has an equivalent* LUR *norm.*

As an application of the transfer technique (Theorem 5) we will deduce, for instance, that every Banach space with separable projectional resolution of the identity has a LUR renorming [**26**]. Now let us state the existence of such a renorming in some new cases.

COROLLARY 6. *Let T be a one-to-one bounded linear operator from the Banach space E into the* LUR *normed space F such that for every bounded sequence (x_n) in E with $\|x_n - x\|_T \to 0$ we have*

$$x \in \overline{\operatorname{span}(x_n)}^{\|\cdot\|_E}$$

(in particular, whenever $x \in \overline{\operatorname{conv}(x_n)}^{\|\cdot\|_E}$, or $\text{w-lim}\, x_n = x$*). Then E is* LUR *renormable.*

The conditions imposed in this corollary are very natural when we are dealing with $C(K)$ spaces, with K compact, since Grothendieck's theorem [**4**, p. 156] asserts that a bounded set L of $C(K)$ is weakly compact if and only if L is compact in the topology of pointwise convergence on K. So from the previous corollary we obtain the following.

COROLLARY 7. *Let T be a one-to-one bounded linear operator from $C(K)$ into the* LUR *normed space F such that for every bounded sequence (x_n) in $C(K)$ with $\|x_n - x\|_T \to 0$ the sequence (x_n) pointwise converges to x. Then $C(K)$ is* LUR *renormable.*

COROLLARY 8. *Let (K_n) be a sequence of closed subsets of a compact space K such that $K = \bigcup K_n$ and $C(K_n)$ has an equivalent* LUR *norm for every $n \in \mathbb{N}$. Then $C(K)$ has an equivalent* LUR *norm.*

This corollary partially answers a question posed by R. Haydon in [**8**]; see also [**7**, Proposition 2.5].

As an application of Corollary 6 we will deduce the following result which is an extension of the classical transfer technique [**3**, Theorem VII.2.8].

COROLLARY 9. *Let T be a one-to-one bounded linear operator from the Banach space E into the* LUR *normed space F such that T^*F^* is norm dense in E^*. Then E is* LUR *renormable.*

A natural application will be the following (see [**3**, Theorem VII.4.10]).

COROLLARY 10. *Let E be a Banach space. Assume there exists a weak*-compact subset $K \subset E^*$ such that the norm-closed linear hull of K is equal to E^*. If $C(K)$ is* LUR *renormable then so is E.*

Moreover, we develop a Decomposition Method which will enable us to deduce from the Main Theorem the three-space property for LUR renormings [**3**] and that $C(K)$ has sJNR and so such a renorming, provided $K^{(\omega_1)} = \varnothing$ [**9**].

From now on the letters $i, j, k, l, m, n,$ will be positive integers, $\varepsilon, \delta, \xi, \eta, \lambda, \mu, \nu, \theta,$ reals, and E, F Banach (or normed) spaces.

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2. *A transfer technique*

We start by proving Theorem 5.

Proof of Theorem 5. Fix $\varepsilon > 0$. We can find subsets $E_{k,n}$, with $k, n \in \mathbb{N}$, such that $E = \bigcup_{n,k} E_{k,n}$ and $\|x - y\| < \varepsilon$ whenever $x, y \in E_{k,n}$ and $\|x - y\|_T < k^{-1}$. Since $\|\cdot\|_T$ is a LUR norm on E and $\|\cdot\|_T \leqslant \|T\|\,\|\cdot\|$, applying the Main Theorem we can find E_k^j, with $j, k \in \mathbb{N}$, such that $E = \bigcup_j E_k^j$ and for every $x \in E_k^j$ there exists a slice $S(E_k^j, f, \lambda)$ containing x with $\|\cdot\|_T$-diam $S(E_k^j, f, \lambda) < k^{-1}$, where $f \in (E, \|\cdot\|_T)^*$. Set $E_{k,n}^j = E_{k,n} \cap E_k^j$. Evidently $E = \bigcup_{j,k,n} E_{k,n}^j$ and every $x \in E_{k,n}^j$ belongs to a slice with $\|\cdot\|$-diameter less than ε. Now the statement follows from the Main Theorem.

Nevertheless the SLD property required in the transfer technique may be not so easy to check, so it will be useful to have easier conditions implying it. There is a recent study of such a property due to L. Oncina [**20**] in the spirit of Spahn [**23**].

Let T be a one-to-one linear map from a normed space $(E, \|\cdot\|_E)$ into a normed space $(F, \|\cdot\|_F)$. Let us mention that when F is separable, then $(E, \|\cdot\|_T)$ is $\|\cdot\|_E$-SLD if and only if E is also separable. Therefore, in the non-separable case, if $(E, \|\cdot\|_T)$ is $\|\cdot\|_E$-SLD and W is a separable subspace of $(TE, \|\cdot\|_F)$, then $T^{-1}W$ must be separable in $\|\cdot\|_E$. On the other hand, if this property of pulling back separable subspaces occurs in a 'continuous way' we can also prove the converse implication.

LEMMA 11. *Let E and F be normed spaces and T: $E \to F$ a one-to-one linear map* (*not necessarily bounded*). *If, for every $x \in E$, there exists a separable subspace Z_x of E with $x \in \overline{\operatorname{span}\{Z_{x_n}\colon n \in \mathbb{N}\}}^{\|\cdot\|_E}$ whenever (x_n) is a bounded sequence in E with (Tx_n) converging to Tx in $\|\cdot\|_F$, then $(E, \|\cdot\|_T)$ is $\|\cdot\|_E$-*SLD.

This lemma has as hypothesis the necessary modification of a sequential property of maps from metric spaces to normed spaces with the weak topologies of [**24**, p. 615, § 1.4], which allows us to use the idea of the proof of Srivatsa's Theorem 2.1 [**24**] to show our conclusion, namely $(E, \|\cdot\|_T)$ is $\|\cdot\|_E$-SLD, which is weaker than his conclusion of being of Baire class 1. Indeed, if T^{-1}: $(TE, \|\cdot\|_F) \to (E, \|\cdot\|_E)$ is only a cluster point of a sequence of continuous functions, it clearly follows that $(E, \|\cdot\|_T)$ is $\|\cdot\|_E$-SLD. Although the converse implication in Lemma 11 also holds, it is not included since it is not within the scope of this paper.

As an application of Lemma 11 and the transfer technique (Theorem 5) it follows easily that every Banach space with separable projectional resolution of the identity has a LUR renorming [**26**].

The simplest application of Lemma 11 is when we deal with the family of one-dimensional subspaces $\{\operatorname{span}(x)\colon x \in E\}$; this, together with the transfer technique, gives us the proof of Corollary 6 stated in the Introduction.

Proof of Corollary 8. For every positive integer let $\|\cdot\|_n$ be an equivalent LUR norm on $C(K_n)$ such that $\|\cdot\|_n \leqslant \|\cdot\|_\infty$. Let F be the ℓ^2-sum of the family of Banach spaces $\{(C(K_n), \|\cdot\|_n)\colon n \in \mathbb{N}\}$. Now the operator $T\colon C(K) \to F$ defined by $Tx = (n^{-1}x|K_n)_{n=1}^{\infty}$ has the property that, given any bounded sequence (x_n) in $C(K)$ for which Tx_n converges to Tx, we have $x_n|K_n \to x|K_n$ in $C(K_n)$, and hence $x_n \to x$ pointwise in K. Applying Corollary 7 we obtain the result.

Proof of Corollary 9. Since T^*F^* is norm dense in E^*, it is well known that on bounded sets of E the weak topology coincides with the topology of pointwise convergence on the elements of T^*F^*. Thus $T^{-1}\colon TB_E \to E$ is continuous from $\|\cdot\|_F$ to the weak topology and by Corollary 6 we have the required result.

It is interesting to discuss a concrete example now.

Example. Let $H = \{u\colon [0,1] \to \{0,1\}\colon u(s) \leqslant u(t) \text{ if } s \leqslant t\}$ endowed with the pointwise convergence topology. Then H is a compact, separable, non-metrizable space such that every point is a G_δ-set. Therefore H is not Corson compact, since separable Corson compact spaces are metrizable. Nor is H Valdivia compact, since a Valdivia compact space in which all the points are G_δ-sets must be Corson compact. Nevertheless, we will deduce from Corollary 7 that $C(H)$ admits a LUR renorming. Let $D[0,1]$ be the space of real functions on $[0,1]$ which are left continuous and have a right limit at each point, endowed with the supremum norm. If $x \in C(H)$, let us define $\hat{x}(s) = x(\mathbb{1}_{[s,1]})$, for all $s \in [0,1]$. Observe that

$$\lim_{s\to t,\, s<t} \hat{x}(s) = \lim_{s\to t,\, s<t} x(\mathbb{1}_{[s,1]}) = x(\mathbb{1}_{[t,1]}) = \hat{x}(t),$$

and

$$\lim_{s\to t,\, s>t} \hat{x}(s) = \lim_{s\to t,\, s>t} x(\mathbb{1}_{[s,1]}) = x(\mathbb{1}_{(t,1]}),$$

so we have $\hat{x} \in D[0,1]$. The operator $T\colon C(H) \to D[0,1]$, defined by $Tx = \hat{x}$, is SLD. Indeed for every bounded sequence (x_n) in $C(H)$ such that $\hat{x}_n$ converges to $\hat{x}$ uniformly on $[0,1]$, we have $\lim_n x_n(\mathbb{1}_{[t,1]}) = \lim_n \hat{x}_n(t) = \hat{x}(t) = x(\mathbb{1}_{[t,1]})$ and, because of the uniform convergence,

$$\begin{aligned}\lim_n x_n(\mathbb{1}_{(t,1]}) &= \lim_n \lim_{s\to t,\, s>t} x_n(\mathbb{1}_{[s,1]}) = \lim_n \lim_{s\to t,\, s>t} \hat{x}_n(s)\\ &= \lim_{s\to t,\, s>t} \lim_n \hat{x}_n(s) = \lim_{s\to t,\, s>t} \hat{x}(s) = x(\mathbb{1}_{(t,1]}),\end{aligned}$$

so (x_n) converges pointwise to x on H. Finally, the space $D[0,1]$ is LUR renormable [**3**, VII.3; **10**, Example 4.1], and by our Corollary 7 so is $C(H)$.

In order to prove Lemma 11 we shall begin with a topological result for metric spaces, and then we will deduce it, taking advantage of the linear structure. A consequence of Stone's proof of the paracompactness of a metric space gives, in any metric space, a σ-discrete base for its topology [**15**, Theorems 4.18 and 4.21]. Essentially we need even less, namely that in any metric space there exists a σ-disjoint base for its topology. This is our main tool for the following lemma.

LEMMA 12. *Let Y be a non-void set with two metrics d and ρ defined on it. If for every $x \in Y$ there is a ρ-separable subset Z_x of Y such that $x \in \overline{\bigcup_{n=1}^{\infty} Z_{x_n}}^{\rho}$ whenever (x_n) is a d-convergent sequence to x, then (Y, d) is ρ-SLD.*

Proof. Let $\mathscr{C} = \bigcup_{n=1}^{\infty} \mathscr{C}_n$ be a base of the topology of (Y, d), where $\mathscr{C}_n$ is a family of disjoint d-open sets for every $n \in \mathbb{N}$. We shall write

$$\mathscr{C}_n = \{V_\gamma^n\colon\ \gamma \in \Gamma_n\}$$

and choose an element $v_\gamma^n \in V_\gamma^n$ in every non-void set of $\mathscr{C}_n$. Let $\{s_{n,\gamma}^m\colon\ m \in \mathbb{N}\}$ be a ρ-dense subset of the ρ-separable subset $Z_{v_\gamma^n}$ for every $\gamma \in \Gamma_n$ and every $n \in \mathbb{N}$; set $D_n = \bigcup \{V_\gamma^n\colon\ \gamma \in \Gamma_n\}$, for $n \in \mathbb{N}$. Since $\mathscr{C}_n$ is a family of disjoint sets, we can define, for every positive integer m, a map $f_{m,n}\colon\ D_n \to Y$ by $f_{m,n}(t) = s_{n,\gamma}^m$ if $t \in V_\gamma^n$. We have

$$Z_{v_\gamma^n} = \overline{\{f_{m,n}(t)\colon\ m \in \mathbb{N}\}}^{\rho}, \quad \text{for all } t \in V_\gamma^n.$$

Now we are ready to prove that (Y, d) is ρ-SLD. Given $\varepsilon > 0$, we define the sets

$$Y_{m,n,\varepsilon} = \{t \in D_n\colon\ \rho(f_{m,n}(t), t) < \tfrac{1}{4}\varepsilon\}$$

for every pair m, n of positive integers. Now we show that

$$Y = \bigcup \{Y_{m,n,\varepsilon}\colon\ m, n \in \mathbb{N}\}$$

and for every set $Y_{m,n,\varepsilon}$ and every point $t \in Y_{m,n,\varepsilon}$ there is a d-neighbourhood V of t such that $\rho\text{-diam}(V \cap Y_{m,n,\varepsilon}) \leqslant \varepsilon$.

Indeed, given $t \in Y$ there are $n_k \in \mathbb{N}$ and $\gamma_k \in \Gamma_{n_k}$ such that $\{V_{\gamma_k}^{n_k}\}_{k=1}^{\infty}$ forms a d-base of neighbourhoods of t. The sequence of points $\{v_{\gamma_k}^{n_k}\}$, previously chosen, converges to t in (Y, d). Then it follows that

$$t \in \overline{\bigcup \{Z_{v_{\gamma_k}^{n_k}}\colon\ k \in \mathbb{N}\}}^{\rho} = \overline{\bigcup \{f_{m,n_k}(t)\colon\ k, m \in \mathbb{N}\}}^{\rho},$$

so

$$t \in \bigcup \{Y_{m,n_k,\varepsilon}\colon\ k, m \in \mathbb{N}\}.$$

To finish the proof let us fix $t \in Y_{m,n,\varepsilon}$. Let $\gamma_t \in \Gamma_n$ be the index for which $t \in V_{\gamma_t}^n$. If $u \in V_{\gamma_t}^n \cap Y_{m,n,\varepsilon}$, we have $f_{m,n}(t) = f_{m,n}(u) = s_{n,\gamma_t}^m$ by the definition of $f_{m,n}$. Therefore we have

$$\rho(t, u) \leqslant \rho(t, f_{m,n}(t)) + \rho(f_{m,n}(t), f_{m,n}(u)) + \rho(f_{m,n}(u), u) < \tfrac{1}{2}\varepsilon$$

and $\rho\text{-diam}(V_{\gamma_t}^n \cap Y_{m,n,\varepsilon}) \leqslant \varepsilon$.

Proof of Lemma 11. It is enough to show that the unit ball B_E of E with $\|\cdot\|_T$

is $\|\cdot\|_E$-SLD, since the same proof works in any ball of E. We shall use the same notation and constructions as in Lemma 12. Indeed, if $\mathscr{C} = \bigcup_{n=1}^{\infty} \mathscr{C}_n$ is a σ-disjoint basis of the metric space $(B_E, \|\cdot\|_T)$, we can construct the functions $f_{m,n}$: $D_n \to E$, with

$$Z_{v_\gamma^n} = \overline{\{f_{m,n}(t):\ m \in \mathbb{N}\}}^{\|\cdot\|_E} \quad \text{for every } t \in V_\gamma^n,$$

as above.

Given a positive integer k, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathbb{N}^k$, $\tau = (\tau_1, \tau_2, \ldots, \tau_k) \in \mathbb{N}^k$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{Q}^k$, we define the function $g_{\sigma,\tau}^{\alpha}$ from $D(\tau) = \bigcap_{j=1}^{k} D_{\tau_j}$ into E by

$$g_{\sigma,\tau}^{\alpha}(t) = \sum_{j=1}^{k} \alpha_j f_{\sigma_j,\tau_j}(t).$$

Let us fix $\varepsilon > 0$ and consider the sets

$$Y_{\sigma,\tau,\varepsilon}^{\alpha} = \{t \in D(\tau):\ \|g_{\sigma,\tau}^{\alpha}(t) - t\|_E < \tfrac{1}{4}\varepsilon\} \quad \text{for } k \in \mathbb{N},\ \sigma, \tau \in \mathbb{N}^k \text{ and } \alpha \in \mathbb{Q}^k.$$

As in the proof of Lemma 12, it follows that

$$B_E = \bigcup \{Y_{\sigma,\tau,\varepsilon}^{\alpha}:\ \sigma, \tau \in \mathbb{N}^k,\ \alpha \in \mathbb{Q}^k,\ k \in \mathbb{N}\}$$

and in every set $Y_{\sigma,\tau,\varepsilon}^{\alpha}$ every point t has a $\|\cdot\|_T$-neighbourhood V such that

$$\|\cdot\|_E\text{-diam}(V \cap Y_{\sigma,\tau,\varepsilon}^{\alpha}) \leqslant \varepsilon.$$

3. *A decomposition method*

Another tool which will enable us to deduce the existence of LUR renormings from the Main Theorem is the following proposition.

PROPOSITION 13 (Decomposition Method). *Let H and L be subsets of a Banach space E and F be a* LUR *normed space with the following properties:*

(a) *H has ε-*sJNR *for some $\varepsilon > 0$;*

(b) *for every $x \in L$ there exist a bounded linear operator T_x: $E \to F$, a continuous (not necessarily linear) map B_x: $T_x E \to E$ and $\delta_x > 0$ such that $x - B_x T_x x \in H$ and $T_x = T_y$, $B_x = B_y$ whenever $y \in L$ and $\|T_x(x-y)\| < \delta_x$;*

(c) *for every $\theta > 0$ we can write $L = \bigcup_k L_{k,\theta}$ in such a way that*

$$\|T_x x\| + \theta > \sup\{\|T_x y\|:\ y \in L_{k,\theta}\}$$

for every $x \in L_{k,\theta}$, and $k \in \mathbb{N}$.

*Then L has ε-*sJNR.

From the decomposition method we can deduce the three-space property for LUR renorming [**6**]. Indeed, let H be a subspace of E such that both H and

$F = E/H$ have a LUR renorming. According to the Bartle–Graves result (cf. for example, [**3**, p. 299]), there exists a continuous (not necessarily linear) map B: $F \to E$ such that $TBz = z$ for every $z \in F$, where T is the canonical quotient map from E onto F. In the decomposition method set $L = E$ and $T_x = T$, $B_x = B$ for every $x \in E$. It is easy to check that the conditions (a), (b), (c) are fulfilled, and we obtain that E has ε-sJNR for every $\varepsilon > 0$.

Let us start by proving the following lemma.

LEMMA 14. *Let E be* LUR. *Then for every $\varepsilon > 0$, we can write $E = \bigcup_n E_{n,\varepsilon}$ in such a way that $E_{n,\varepsilon} \subset E_{(n+1),\varepsilon}$, and for every $z \in E_{n,\varepsilon}$, there exists a functional g_z supporting z and such that the inequality*

$$\|w\| - n^{-1} < \|z\| < g_z(w) + n^{-1} \tag{1}$$

implies $\|w - z\| < \varepsilon$.

Proof. Given $\xi > 0$, since every point x of S_E is strongly exposed, for every functional g_x supporting x we can find $\lambda = \lambda(x, \xi) < 1$ such that $\operatorname{diam} S(g_x, \lambda) < \xi$.

Now let $\varepsilon > 0$ and $n > \varepsilon^{-1}$, and denote by $E_{n,\varepsilon}$ the set of $z \in E$ for which

$$\lambda(z/\|z\|, \varepsilon/\|z\|) < (\|z\| - n^{-1})/(\|z\| + n^{-1}).$$

Let $z \in E_{n,\varepsilon}$ and $w \in E$ satisfy (1). Set $x = z/\|z\|$ and $v = w/\|w\|$. From (1) we obtain

$$g_z(v) > (\|z\| - n^{-1})/\|w\| > \lambda(x, \varepsilon/\|z\|).$$

Then $\|v - x\| < \varepsilon/\|z\|$. Hence

$$\|w - z\| \leq \|z\| \, \|v - x\| + | \, \|w\| - \|z\| \, | < \varepsilon + n^{-1} < 2\varepsilon.$$

Proof of Proposition 13. Since H has ε-sJNR, we can write $H = \bigcup_{j,n} H_n^j$ in such a way that for every $v \in H_n^j$, with $j, n \in \mathbb{N}$, there exist $h_v \in S_{E^*}$ and μ_v such that

$$\min\{h_v(v) - \mu_v, \varepsilon\text{-diam}\, S(H_n^j, h_v, \mu_v)\} > 2/j. \tag{2}$$

From Lemma 14, it follows that for every $m \in \mathbb{N}$ we can write $F = \bigcup_i F_m^i$ in such a way that $F_m^i \subset F_m^{i+1}$ and for every $z \in F_m^i$ there exists a functional g_z supported by z such that

$$\|w - z\| < m^{-1}, \tag{3}$$

whenever $\|w\| - i^{-1} < \|z\| < g_z(w) + i^{-1}$.

For every $x \in L$ and $j \in \mathbb{N}$ we can find $\eta_{x,j} \in (0, \delta_x)$ such that

$$\|B_x T_x x - B_x T_x y\| < j^{-1}, \tag{4}$$

whenever $y \in E$ and $\|T_x x - T_x y\| < \eta_{x,j}$.

Set $\Phi x = B_x T_x x$ and $\Psi x = x - \Phi x$. We note that $\Psi L \subset H$. Let $i, j, k, m, n \in \mathbb{N}$. By $L_{k,m,n}^{i,j}$ we denote the set made up of all $x \in L$ for which $\Psi x \in H_n^j$, $\eta_{x,j} > m^{-1}$, $T_x x \in F_m^i$, $\|x\| < \frac{1}{3}i$, and $x \in L_{k,\theta}$ where $\theta = i^{-2}j^{-1}$. Evidently $L = \bigcup_{i,j,k,m,n} L_{k,m,n}^{i,j}$.

Let $x \in L^{i,j}_{k,m,n}$. Set $u = \Phi x$, $v = \Psi x$, $z = T_x x$, and $f = T_x^* g_z + i^{-2} h_v$. We can find ξ such that

$$\mu_v + 2j^{-1} < \xi < h_v(v) < \xi + \tfrac{1}{3}. \tag{5}$$

From (5) we obtain

$$\begin{aligned} f(x) &= g_z(T_x x) + i^{-2} h_v(x) = \|z\| + i^{-2}(h_v(u) + h_v(v)) \\ &> \|z\| + i^{-2}(h_v(u) + \xi) = \lambda. \end{aligned}$$

So $x \in S(L^{i,j}_{k,m,n}, f, \lambda)$. Let $y \in S(L^{i,j}_{k,m,n}, f, \lambda)$. From (4) we have $\xi > h_v(v) - \frac{1}{3}$, and since $\|x\|$, $\|y\| < \frac{1}{3} i$, we obtain

$$\begin{aligned} g_z(T_x y) &= f(y) - i^{-2} h_v(y) \\ &> \lambda - i^{-2} h_v(y) \\ &= \|z\| + i^{-2}(h_v(u) + \xi - h_v(y)) \\ &> \|z\| + i^{-2}(h_v(x-y) - \tfrac{1}{3}) > \|z\| - i^{-1}. \end{aligned} \tag{6}$$

Since $x \in L_{k,\theta}$, we have

$$\|T_x y\| < \|T_x x\| + i^{-2} j^{-1} = \|z\| + i^{-2} j^{-1} \leqslant \|z\| + i^{-1}. \tag{7}$$

Taking into account the fact that $z \in F^i_m$ and applying (3) for $w = T_x y$ from (6) and (7), we obtain $\|T_x x - T_x y\| = \|z - T_x y\| < m^{-1}$. Since $m^{-1} < \eta_{x,j} < \delta_x$, we get $T_x = T_y$, $B_x = B_y$. This and (3) imply that

$$\|u - \Phi y\| = \|\Phi x - \Phi y\| = \|B_x T_x x - B_x T_x y\| < j^{-1}. \tag{8}$$

Since $f(y) > \lambda$ from (6) and (7), we obtain

$$\begin{aligned} h_v(\Psi y) &= i^2(f(y) - g_z(T_x y)) - h_v(\Phi y) \\ &> i^2(\lambda - \|T_x y\|) - h_v(\Phi y) \\ &= i^2(\|z\| - \|T_x y\|) + \xi + h_v(u - \Phi y) \\ &> -j^{-1} + \xi - \|u - \Phi y\| > \xi - 2j^{-1} > \mu_v. \end{aligned}$$

So $\Psi y \in S(H^j_n, h_v, \mu_v)$. Then from (2) we get $\|v - \Psi y\| < \varepsilon - 2j^{-1}$. Hence $\|x - y\| < \|u - \Phi y\| + \|v - \Psi y\| < \varepsilon$.

EXAMPLE [9]. If K is a compact space such that $K^{(\omega_1)} = \varnothing$ then $C(K)$ has sJNR.

For an ordinal α we set $K_\alpha = K^{(\alpha)} \setminus K^{(\alpha+1)}$. For $t \in K_\alpha$ we can find a clopen set U_t such that $U_t \cap K_\alpha = \{t\}$. Since U_t is clopen, we obtain $u_t = \mathbb{1}_{U_t} \in C(K)$. Let $\{e_t\}$ be the unit vector basis in $l_2(K_\alpha)$. For $x \in C(K)$, $\Lambda \subset K_\alpha$, $|\Lambda| < \infty$ we set

$$P_\Lambda x = \sum_{t \in \Lambda} x(t) u_t, \quad R_\Lambda x = x - P_\Lambda x, \quad Q_\Lambda x = \sum_{t \in \Lambda} x(t) e_t.$$

Fix $\varepsilon > 0$. For $x \in C(K)$, an ordinal α and $m, n \in \mathbb{N}$ we set

$$\Delta^\alpha_x = \{t \in K_\alpha\colon\ |x(t)| \geqslant \varepsilon\}, \quad \eta^\alpha_x = \varepsilon - \max\{|x(t)|\colon\ t \in K_\alpha \setminus \Delta^\alpha_x\},$$

$$E^\alpha_{m,n} = \{v \in C(K), |\Delta^\alpha_v| = n, \eta^\alpha_x > m^{-1}\}, \quad E^\alpha = \bigcup_{m,n} E^\alpha_{m,n}, \quad E_\alpha = \bigcup_{\beta < \alpha} E^\beta.$$

Let us observe that $x \in E_\alpha$ whenever $\Delta_x^\alpha = \emptyset$. In particular, for every $x \in E^\alpha$ we have

$$R_{\Delta_x^\alpha} x \in E_\alpha. \tag{9}$$

We shall prove by transfinite induction that, for every $\alpha < \omega_1$, the set E_α has ε-sJNR. Assume that the inductive assertion has already been proved for all $\beta < \alpha$, that is, E^β has ε-sJNR for $\beta < \alpha$. Since $\alpha < \omega_1$, we find that E_α has ε-sJNR. Fix $m, n \in \mathbb{N}$ and set $H = E_\alpha$, $L = E^\alpha_{m,n}$, $E = C(K)$, $F = l_2(K_\alpha)$ in Proposition 13 (the decomposition method). For $x \in L$ and $y \in E$ we set $T_x y = Q_{\Delta_x^\alpha} y$, $B_x T_x y = P_{\Delta_x^\alpha} y$. Applying (9) we get $x - B_x T_x x = R_{\Delta_x^\alpha} x \in H$. So (b) from Proposition 13 is fulfilled. Let $\theta \in (0, 1]$, and $q = (q_i)_1^n \in \mathbb{Q}^n$ such that $|q_1| \geqslant |q_2| \geqslant \ldots \geqslant |q_n| \geqslant \varepsilon$. By $L_{q,\theta}$ we denote all $x \in E^\alpha_{m,n}$ for which $(t_i)_1^n = \Delta_x^\alpha$, $|x(t_1)| \geqslant |x(t_2)| \geqslant \ldots \geqslant |x(t_n)| \geqslant \varepsilon$ and

$$|x(t_i) - q_i| < \theta/\sqrt{n}. \tag{10}$$

Let $y \in L_{q,\theta}$. Taking into account the fact that $|y(t_i)| < \varepsilon \leqslant |q_i|$ for all $t_i \notin \Delta_y^\alpha$ and using (10) we obtain

$$\| T_x x \| \geqslant \left(\sum q_i^2 \right)^{\frac{1}{2}} - \left(\sum (x(t_i) - q_i)^2 \right)^{\frac{1}{2}} \geqslant \left(\sum q_i^2 \right)^{\frac{1}{2}} - \theta \geqslant \| T_x y \| - 2\theta.$$

This implies that (c) in Proposition 13 is fulfilled. In the same way, it is possible to show that $E^0_{m,n}$ has ε-sJNR. So we have proved that E^α, $\alpha < \omega_1$, has ε-sJNR for every $\varepsilon > 0$.

4. *The main theorem*

We begin by recalling some definitions.

Definition 15. A Banach space E (or the norm in E) is said to:

(a) be *rotund* (R for short) if the unit sphere of E contains no open segment;

(b) be *weakly midpoint locally uniformly rotund* or *midpoint locally uniformly rotund* (wMLUR or MLUR for short, respectively) if given sequences (y_k), (z_k) and x in E we have $\text{w-lim}_k(y_k - z_k) = 0$ or $\lim_k \| y_k - z_k \| = 0$, respectively, whenever $\| y_k \|, \| z_k \| \leqslant \| x \|$ and $\lim_k \| y_k + z_k - 2x \| = 0$.

In order to prove the Main Theorem we add a new equivalence.

Main Theorem (complete version). *Let E be a Banach space. The following conditions are equivalent:*

(a) *the unit sphere S_E of E has* sJNR*;*

(b) *E has* sJNR*;*

(c) *E has an equivalent* LUR *norm; and*

(d) *E has* JNR *and an equivalent* wMLUR *norm.*

In [**10**] it is shown that K$\Rightarrow$JNR. In [**19**] it is proved that K&R imply MLUR. Hence we have K&R$\Rightarrow$JNR&wMLUR. So the metric property K has been replaced by JNR, which is a condition stated in topological terms. The notion of wMLUR is stronger than R. Essentially from [**16**], it follows that wMLUR is equivalent to all points of S_E being extreme points of the bidual ball $B_{E^{**}}$. In fact wMLUR is a stronger condition than R even from the point of view of isomorphism [**1, 2**]. However, in the case of $C(T)$, R. Haydon characterized certain trees T for which $C(T)$ has R renorming and showed that they are the same as those which admit a MLUR renorming. He also characterized the trees T for which $C(T)$ has a K renorming. In this way he characterized the trees T for which $C(T)$ has a LUR renorming. Moreover, he obtained a tree T such that $C(T)$ admits a K renorming but no R equivalent norm. For these and more results about renormings of these spaces see [**7**] and the comments of § 6, Chapter VII in [**3**].

Proof of the complete version of the Main Theorem. (d)$\Rightarrow$(a) Since E has JNR, we must have $S_E = \bigcup_k Q_{k,\varepsilon}$ and for every $z \in Q_{k,\varepsilon}$ there exists a weak open set V_z containing z such that $\operatorname{diam}(V_z \cap Q_{k,\varepsilon}) < \varepsilon$. Since E is wMLUR, the proof of Remark 3 in [**16**] shows that all points of S_E are extreme points for $B_{E^{**}}$. Then it follows that for every $x \in S_E$ the open slices $S(f, \mu)$ of B_E form a base of neighbourhoods for x in the weak topology of B_E (see [**22**, Corollary 1.7]). So we can find a slice $S(f, \mu)$ of B_E, such that $z \in S(f, \mu) \subset V_z \cap B_E$. Since $\operatorname{diam}(V_z \cap Q_{n,\varepsilon}) < \varepsilon$, we have $\operatorname{diam} S(Q_{n,\varepsilon}, f, \mu) < \varepsilon$.

(c)$\Rightarrow$(b) Let $E_{n,\varepsilon}$ satisfy the condition of Lemma 14. For $q \in \mathbb{Q}$ we set $E^q_{n,\varepsilon} = \{z \in E_{n,\varepsilon}\colon \ | \|z\| - q| < 1/2n\}$. We have

$$z \in S(E^q_{n,e}, g_z, q - 1/2n) \quad \text{and} \quad \operatorname{diam} S(E^q_{n,\varepsilon}, g_z, q - 1/2n) < 2\varepsilon.$$

The implication (b)$\Rightarrow$(a) is evident.

(c)$\Rightarrow$(d) Let E be LUR. Then evidently E is wMLUR. From the implication (c)$\Rightarrow$(b) we infer that E has sJNR. Hence E has JNR.

The proof of the existence of a LUR equivalent norm for spaces with sJNR, that is, the implication (a)$\Rightarrow$(c), is based on probabilistic techniques so we require some more notation.

NOTATION. Here and subsequently we will denote by Ω and Ω_n, for $n \in \mathbb{N}$, the sets $\{\omega\colon\ \omega = (\omega_i)_1^\infty,\ \omega_i = \pm 1\}$ and $\{\omega\colon\ \omega = (\omega_i)_1^n,\ \omega_i = \pm 1\}$ respectively. For $\alpha = (\alpha_i)_1^n \in \Omega_n$ we set $T_\alpha = \{\omega \in \Omega\colon\ \omega = (\omega_i)_1^\infty,\ \ \alpha_i = \omega_i,\ \ i = 1, 2, \ldots, n\}$, and $\mathcal{T}_n = \{T_\alpha\}_{\alpha \in \Omega_n}$.

The symbol $\mathcal{A}_0$ will stand for $\{\emptyset, \Omega\}$, and $\mathcal{A}_n$ will be the finite algebra made up by the empty set and the sets $\bigcup_{\alpha \in \Lambda} T_\alpha$ for $\Lambda \subset \Omega_n$. We will denote by $\mathcal{A}$ the σ-algebra generated by the sets of $\mathcal{A}_n$, for $n = 1, 2, \ldots,$ and p will be the probabilistic measure on $\mathcal{A}$.

Here and subsequently, given a subset A of Ω, the symbol A^- will stand for the complement of the set A, that is, $\Omega \setminus A$ (and not the closure of this set).

Let us recall that an E-valued Walsh–Paley martingale (M_n) is a sequence of functions from Ω to a Banach space E such that M_n is $\mathcal{A}_n$-measurable for $n \geqslant 0$ and $\mathbb{E}(M_n | \mathcal{A}_{n-1}) = M_{n-1}$ for $n \geqslant 1$. We denote the 'increments' of the martingale (M_n) by $dM_0 = M_0$ and $dM_n = M_n - M_{n-1}$ for $n \geqslant 1$.

Let H be a cone in E. We set $\gamma_k(H) = \inf\{\sup_n (\mathbb{E}\ \|M_n\|^2)^{\frac{1}{2}}\}$ where the infimum is taken over all E-valued Walsh–Paley martingales (M_n) such that the set

$$\left\{n\colon \int_{M_n^{-1}(H)} \|dM_n\|^2 \geqslant 1\right\}$$

has at least k elements. We put $\gamma(H) = \sup\{\gamma_k(H)\colon\ k \geqslant 1\}$.

PROPOSITION 16. *A Banach space E admits a* LUR *norm if and only if for every $\varepsilon > 0$ there is a sequence $\{E_{n,\varepsilon}\}_{n\geqslant 1}$ of cones in E such that $E = \bigcup_n E_{n,\varepsilon}$ and* $\inf_n \gamma(E_{n,\varepsilon}) \geqslant \varepsilon^{-1}$.

Following the proof in [**3**, pp. 144–148] we obtain a LUR norm $|\cdot|$ that is not necessarily symmetric. Then we set $\|x\| = (|x|^2 + |-x|^2)^{\frac{1}{2}}$ to get a symmetric norm. Since $|\cdot|$ is LUR, it is easy to see that $\|\cdot\|$ is also LUR.

LEMMA 17. *Let $0 < \eta < 1$, $\|x\| = 1$, $x = \mathbb{E}X$, and $\mathbb{E}\,\|X\|^2 \leqslant 1 + \eta^4$. If we set $C = \{\|X\|^2 < 1 - \eta^{\frac{1}{2}}\}$, $D = \{\|X\|^2 > 1 + \eta\}$, then*

$$p(C) \leqslant 7\eta^{\frac{1}{2}}, \quad p(D) \leqslant 3\eta, \quad \left(\int_D \|X\|^2\right)^{\frac{1}{2}} \leqslant (6\eta)^{\frac{1}{2}}, \quad \left(\int_D \|X - x\|^2\right)^{\frac{1}{2}} \leqslant 9\eta^{\frac{1}{2}}.$$

Proof. The last three inequalities are proved in [**3**, p. 136]. Moreover we have

$$\begin{aligned} 1 \leqslant \mathbb{E}\,\|X\|^2 &= \int_C \|X\|^2 + \int_D \|X\|^2 + \int_{D^-\setminus C} \|X\|^2 \\ &< (1 - \eta^{\frac{1}{2}})p(C) + 6\eta + (1 + \eta)p(D^- \setminus C) \\ &\leqslant 1 - \eta^{\frac{1}{2}}p(C) + 7\eta. \end{aligned}$$

DEFINITION 18. A cone $H \subset E$ is said to be (ε, δ)-*admissible* if for every E-valued random variable X we have

$$E\,\|X\|^2 \geqslant (1 + \delta^4)\,\|x\|^2$$

whenever

$$\mathbb{E}X = x \in H \quad \text{and} \quad p(\{\|X - x\| \geqslant \varepsilon\,\|x\|\} \cap X^{-1}(H)) \geqslant \varepsilon^2.$$

We say that H is ε-*admissible* whenever there exists a positive δ such that H is (ε, δ)-admissible.

Now we introduce some notation which will be valid throughout statements 19–23. For fixed $e \in S_E$, $f \in S_{E^*}$ such that $f(e) = 1$ we set

$$d(x) = \inf\{\|x - \lambda e\|\colon\ \lambda \in \mathbb{R}\}, \quad g(x) = (d^2(x) + f^2(x))^{\frac{1}{2}},$$

$$|x| = (\tfrac{1}{3}(\|x\|^2 + g^2(x)))^{\frac{1}{2}}, \quad \text{and} \quad \mathscr{H}(e, \varepsilon) = \{x \in E\colon\ \|x/\|x\| - e\| \leqslant \varepsilon\}.$$

We use $E(\varepsilon, \delta)$ to stand for the set of the elements x in E such that [**3**, p. 139]

$$\inf\{\mathbb{E}\,\|X\|^2\colon\ \mathbb{E}X = x,\ \mathbb{E}\,\|X - x\|^2 \geqslant (\varepsilon\,\|x\|)^2\} \geqslant (1 + \delta)\,\|x\|^2.$$

LEMMA 19. *For every $x \in E$ we have $|x|^2 \leqslant 2g^2(x)$, $(\frac{2}{5})^{\frac{1}{2}}\,\|x\| \leqslant |x| \leqslant \|x\|$.*

Proof. Let $x \in E$ and μ be such that $d(x) = \|x - \mu e\|$. We have

$$\|x\| \leqslant 2\max\{(\|x\| - \mu), (\mu - f(x))\} + f(x) \leqslant 2d(x) + f(x) \leqslant 5^{\frac{1}{2}} g(x).$$

Then $6g^2(x) = 5g^2(x) + g^2(x) \geqslant \|x\|^2 + g^2(x) = 3\,|x|^2$, which proves the first assertion of the statement. To show the second, it is enough to note that $6\|x\|^2 \leqslant 5g^2(x) + 5\,\|x\|^2 = 15\,|x|^2$.

LEMMA 20. *Let $x \in \mathscr{K}(e, \varepsilon)$, $|x| = 1$, $\mathbb{E}X = x$, and $\mathbb{E}\,|X - x|^2 \geqslant (12\varepsilon)^2$. Then $\mathbb{E}\,|X|^2 \geqslant 1 + 4\varepsilon^2$.*

Proof. Since $x \in \mathscr{K}(e, \varepsilon)$, we have $d(x) \leqslant \varepsilon\,\|x\| \leqslant (\frac{5}{2})^{\frac{1}{2}}\varepsilon$. Then from Lemma 19 we have for any $y \in E$, $d(y - x) \leqslant (\frac{5}{2})^{\frac{1}{2}}\,|y - x|$ and

$$\begin{aligned}3\,|y|^2 &\geqslant \|y\|^2 + (d(y - x) - d(x))^2 + f^2(y - x + x)\\ &= \|y\|^2 - \|x\|^2 + g^2(y - x) - 2d(x)d(x - y) + 2f(x)f(y - x) + 3\,|x|^2\\ &\geqslant \|y\|^2 - \|x\|^2 + \tfrac{1}{2}\,|y - x|^2 - 5\varepsilon\,|y - x| + 2f(x)f(x)f(y - x) + 3\,|x|^2.\end{aligned}$$

Taking into account the facts that $\mathbb{E}f(X - x) = 0$ and $\mathbb{E}\,\|X\|^2 \geqslant \|x\|^2$, if in the above inequality we replace y by the random variable X and integrate, we obtain

$$\begin{aligned}3\mathbb{E}\,|X|^2 &\geqslant \tfrac{1}{2}\mathbb{E}\,|X - x|^2 - 5\varepsilon\mathbb{E}\,|X - x| + 3\,|x|^2\\ &\geqslant 3 + (\mathbb{E}\,|X - x|^2)^{\frac{1}{2}}(\tfrac{1}{2}(\mathbb{E}\,|\,X - x|^2)^{\frac{1}{2}} - 5\varepsilon) \geqslant 3 + 12\varepsilon^2.\end{aligned}$$

PROPOSITION 21. *Let $0 < \varepsilon, \delta < 1$, and let (M_n) be an E-valued Walsh–Paley martingale. If $\int_{A_j} \|dM_{n_j}\|^2 \geqslant 36\varepsilon^2$, $M_{n_j}(A_j) \subset E(\varepsilon, \delta)$ for $j = 1, 2, \dots, r$, and $r \geqslant a\varepsilon^{-5}\delta^{-1}$ where a is an absolute positive constant, then $\mathbb{E}\,\|M_{n_r}\|^2 \geqslant 1$.*

This proposition essentially follows from [**3**, Lemmas 3.2, 3.3, pp. 139–144].

COROLLARY 22. *Let $0 < \varepsilon < \frac{1}{24}$, $e \in E$, and (M_n) be an E-valued Walsh–Paley martingale. If $\int_{A_j} \|dM_{n_j}\|^2 \geqslant b\varepsilon^2$ where $b = 5\,.\,12^4$, $M_{n_j}(A_j) \subset \mathscr{K}(e, 2\varepsilon)$ for $j = 1, 2, \dots, r$, and $r \geqslant s(\varepsilon) = [3a(2/\varepsilon)^7] + 1$, then $\mathbb{E}\,\|M_{n_r}\|^2 \geqslant 2$.*

Proof. According to Lemma 20 and Lemma 19 we have $\mathscr{K}(e, 2\varepsilon) \subset E_{|\cdot|}(24\varepsilon, 16\varepsilon^2)$, and

$$\int_{A_j} |dM_{n_j}/\sqrt{2}|^2 \geqslant \frac{1}{5}\int_{A_j} \|dM_{n_j}\|^2 \geqslant 36(24\varepsilon)^2.$$

Now the statement follows from Proposition 21.

LEMMA 23. *Let* $0 < \varepsilon, \delta < \frac{1}{24}$ *and* H *be a* (ε, δ)*-admissible cone in* E, (M_n) *an* E*-valued Walsh–Paley martingale so that* $M_0 \in H$, $\|M_0\| = 1$, $M_{n_j}(A_j) \subset H$, $\int_{A_j} \|dM_{n_j}\|^2 \geq c\varepsilon^2$ *for* $j = 1, 2, \ldots, r$, *and* $r \geq s(\varepsilon)$ *where* $c = b + 36$ *and* b, $s(\varepsilon)$ *are from Corollary* 22. *Then we have* $\mathbb{E}\,\|M_{n_r}\|^2 \geq 1 + \eta^4(\varepsilon, \delta)$ *where* $\eta(\varepsilon, \delta) = \min\{\delta, (\varepsilon/10)^2\}$.

Proof. Suppose the contrary is true. Set $M_0 = e$ and $B_j = A_j \setminus M_{n_j}^{-1}(\mathscr{K}(e, 2\varepsilon))$. We begin by distinguishing two possibilities.

First,

$$\int_{A_j \setminus B_j} \|dM_{n_j}\|^2 \geq b\varepsilon^2 \quad \text{for } j = 1, 2, \ldots, r.$$

In this case, according to Corollary 22 we get $\mathbb{E}\,\|M_{n_r}\|^2 \geq 2$, which is a contradiction since $\eta(\varepsilon, \delta) < 1$.

Second, we have for some i that $\int_{A_i \setminus B_i} \|dM_{n_i}\|^2 < b\varepsilon^2$. Later we will deduce from the above inequality that

$$p(B_i) \geq \varepsilon^2. \tag{11}$$

Since $M_{n_i}(B_i) \cap \mathscr{K}(e, 2\varepsilon) = \varnothing$ for $x \in M_{n_i}(B_i)$, we have

$$\|x - e\| \geq \max\big\{\big|\|x\| - 1\big|,\ \big\|x/\|x\| - e\big\| - \big|\|x\| - 1\big|\big\} \geq \varepsilon. \tag{12}$$

Let us note that once (11) has been proved, the anticipated contradiction follows easily from the inequalities

$$\mathbb{E}\,\|M_{n_i-1}\|^2 \leq \mathbb{E}\,\|M_{n_i}\|^2 \leq \mathbb{E}\,\|M_{n_r}\|^2 < 1 + \eta^4. \tag{13}$$

On the other hand, since $\mathbb{E}M_{n_i} = e$ from the admissibility of H, (11) and (12), we conclude that $\mathbb{E}\,\|M_{n_i}\|^2 \geq 1 + \delta^4$. Hence by (13) we get $\delta < \eta$, which is contrary to the choice of η. We now begin the proof of (11) by observing that

$$\int_{B_i} \|dM_{n_i}\|^2 = \int_{A_i} \|dM_{n_i}\|^2 - \int_{A_i \setminus B_i} \|dM_{n_i}\|^2 > c\varepsilon^2 - b\varepsilon^2 = (6\varepsilon)^2. \tag{14}$$

On the other hand, from (14) and Lemma 17 we deduce that $p(C_i) \leq 3\eta$, $p(D_i) \leq 3\eta$,

$$\left(\int_{C_i} \|M_{n_i-1}\|^2\right)^{\frac{1}{2}} \leq (6\eta)^{\frac{1}{2}}, \quad \left(\int_{D_i} \|M_{n_i}\|^2\right)^{\frac{1}{2}} \leq (6\eta)^{\frac{1}{2}}, \tag{15}$$

where $C_i = \{\|M_{n_i-1}\|^2 \geq 1 + \eta\}$ and $D_i = \{\|M_{n_i}\|^2 \geq 1 + \eta\}$.

Moreover, since B_i is the disjoint union of the sets $B_i \cap C_i \cap D_i$, $B_i \cap C_i \cap D_i^-$, $B_i \cap C_i^- \cap D_i$, $B_i \cap C_i^- \cap D_i^-$, it follows from the triangle inequality that

$$\begin{aligned} \left(\int_{B_i} \|dM_{n_i}\|^2\right)^{\frac{1}{2}} &\leq \left(\int_{B_i \cap C_i \cap D_i} \|dM_{n_i}\|^2\right)^{\frac{1}{2}} + \left(\int_{B_i \cap C_i \cap D_i^-} \|dM_{n_i}\|^2\right)^{\frac{1}{2}} \\ &\quad + \left(\int_{B_i \cap C_i^- \cap D_i} \|dM_{n_i}\|^2\right)^{\frac{1}{2}} + \left(\int_{B_i \cap C_i^- \cap D_i^-} \|dM_{n_i}\|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{16}$$

It follows from the triangle inequality that, for all measurable sets G_1, G_2 containing G,

$$\left(\int_G \|dM_{n_i}\|^2\right)^{\frac{1}{2}} \leqslant \left(\int_{G_1} \|M_{n_i}\|^2\right)^{\frac{1}{2}} + \left(\int_{G_2} \|M_{n_i-1}\|^2\right)^{\frac{1}{2}}.$$

From (15), applying this inequality, we obtain

$$\left(\int_{B_i\cap C_i\cap D_i} \|dM_{n_i}\|^2\right)^{\frac{1}{2}} \leqslant \left(\int_{D_i} \|M_{n_i}\|^2\right)^{\frac{1}{2}} + \left(\int_{C_i} \|M_{n_i-1}\|^2\right)^{\frac{1}{2}} \leqslant 2(6\eta)^{\frac{1}{2}},$$

$$\left(\int_{B_i\cap C_i\cap D_i^-} \|dM_{n_i}\|^2\right)^{\frac{1}{2}} \leqslant \left(\int_{B_i\cap D_i^-} \|M_{n_i}\|^2\right)^{\frac{1}{2}} + \left(\int_{C_i} \|M_{n_i-1}\|^2\right)^{\frac{1}{2}}$$
$$\leqslant (1+\eta)^{\frac{1}{2}} p^{\frac{1}{2}}(B_i) + (6\eta)^{\frac{1}{2}},$$

$$\left(\int_{B_i\cap C_i^-\cap D_i} \|dM_{n_i}\|^2\right)^{\frac{1}{2}} \leqslant \left(\int_{D_i} \|M_{n_i}\|^2\right)^{\frac{1}{2}} + \left(\int_{B_i\cap C_i^-} \|M_{n_i-1}\|^2\right)^{\frac{1}{2}}$$
$$\leqslant (6\eta)^{\frac{1}{2}} + (1+\eta)^{\frac{1}{2}} p^{\frac{1}{2}}(B_i),$$

$$\left(\int_{B_i\cap C_i^-\cap D_i^-} \|dM_{n_i}\|^2\right)^{\frac{1}{2}} \leqslant \left(\int_{B_i\cap D_i^-} \|M_{n_i}\|^2\right)^{\frac{1}{2}} + \left(\int_{B_i\cap C_i^-} \|M_{n_i-1}\|^2\right)^{\frac{1}{2}}$$
$$\leqslant 2(1+\eta)^{\frac{1}{2}} p^{\frac{1}{2}}(B_i).$$

These four inequalities and (16) yield

$$\left(\int_{B_i} \|dM_{n_i}\|^2\right)^{\frac{1}{2}} \leqslant 4(6\eta)^{\frac{1}{2}} + 4(1+\eta)^{\frac{1}{2}} p^{\frac{1}{2}}(B_i) \leqslant \varepsilon + 5p^{\frac{1}{2}}(B_i).$$

Combining this with (14) we obtain (11).

The next lemma has some ideas in common with [**27**, Lemma 4.4].

LEMMA 24. *Let H, ε, δ, $s(\varepsilon)$, $\eta(\varepsilon, \delta)$ and c be as in Lemma 23 and $m \geqslant r\varepsilon^{-2}+1$, where $r = s(\varepsilon)$. Let $(M_n)_{n=0}^{n_m}$ be an E-valued Walsh–Paley martingale such that $n_0 < n_1 < \ldots < n_m$, $\mathbb{E}\|M_{n_0}\|^2 \leqslant 1$, and $\int_{M_{n_j}^{-1}(H)} \|dM_{n_j}\|^2 \geqslant 7c\varepsilon^2$ for $j = 1, 2, \ldots, m$. Then*

$$\mathbb{E}\|M_{n_m}\|^2 \geqslant \left(1 + \varepsilon^2\eta^4(\varepsilon,\delta)/r\binom{m}{r}\right)\mathbb{E}\|M_{n_0}\|^2.$$

Proof. Set $\theta = c^{\frac{1}{2}}\varepsilon$ and τ such that $\mathbb{E}\|M_{n_m}\|^2 = (1+\tau)\mathbb{E}\|M_{n_0}\|^2$. Obviously we may assume that $\tau < \frac{1}{4}$.

Put $A_i = M_{n_i}^{-1}(H)$ and $C_1 = A_1$, $B_j = \bigcup_1^{j-1} A_i$, $C_j = A_j \setminus B_j$, for $j = 2, 3, \ldots, m$.

We denote by $C_{j,k}$, for $j < k$, the union of all $T \in \mathcal{T}_{n_j}$ such that $T \subset C_j$ and

$$\theta^2 \|M_{n_j}(T)\|^2 p(T) \leqslant \int_{A_k\cap T} \|dM_{n_k}\|^2. \tag{17}$$

Evidently (17) implies that

$$\int_{A_k \cap D_{j,k}} \|dM_{n_k}\|^2 \leqslant \theta^2 \int_{D_{j,k}} \|M_{n_j}\|^2 \quad \text{where } D_{j,k} = C_j \setminus C_{j,k}. \tag{18}$$

For $q \leqslant k-j$ let $\Pi_{j,k,q} = \{(n_{i_1}, n_{i_2}, \ldots, n_{i_q}) \colon j < i_1 < i_2 < \ldots < i_q = k\}$, and $C_j(\pi) = \bigcap_{h=1}^{q} C_{j,i_h}$, where $\pi = (n_{i_1}, n_{i_2}, \ldots, n_{i_q}) \in \Pi_{j,k,q}$. Set

$$C_{j,k,q} = \begin{cases} \bigcup \{C_j(\pi) \colon \pi \in \Pi_{j,k,q}\} & \text{if } q \leqslant k-j, \\ \varnothing & \text{if } q > k-j. \end{cases}$$

Let $F_{j,k,q} = C_{j,k,q} \setminus C_{j,k,(q+1)}$. Since $C_{j,k,(q+1)} \subseteq C_{j,k,q} \subseteq \ldots \subseteq C_{j,k,1} = C_{j,k}$, we have

$$C_{j,k} = C_{j,k,r} \cup \left(\bigcup_{q=1}^{r-1} F_{j,k,q} \right). \tag{19}$$

We claim that

$$F_{j_1,k_1,q} \cap F_{j_2,k_2,q} = \varnothing \quad \text{if } (j_1, k_1) \neq (j_2, k_2). \tag{20}$$

Indeed, since $F_{j,k,q} \subset C_j$ and the sets $\{C_j\}_{j=1}^m$ are disjoint, (20) holds when $j_1 \neq j_2$. Suppose now that $j = j_1 = j_2$ and $k = k_1 < k_2 = l$ and, contrary to our claim, that there exists an $\omega \in F_{j,k,q} \cap F_{j,l,q}$. Then since $\omega \in F_{j,k,q}$, there must exist a sequence $\pi = \{i_h\}_1^q \in \Pi_{j,k,q}$ such that $\omega \in C_j(\pi)$. Since $\omega \in F_{j,l,q} \subset C_{j,l}$, we find that $\omega \in C_j(\pi) \cap C_{j,l} = C_j(\sigma)$ where $\sigma = (i_1, i_2, \ldots, i_q = k, l) \in \Pi_{j,l,(q+1)}$. Hence $\omega \in C_{j,l,(q+1)}$, a contradiction.

Denote $N_n = \sup_{s \leqslant n} \|M_s\|$. Then by an inequality due to Doob [**25**, p. 271], we have $\mathbb{E} N_n^2 \leqslant 4\mathbb{E}\, \|M_n\|^2$. Then since $\tau < \frac{1}{4}$ for every $n \leqslant n_m$, we have

$$\mathbb{E} N_n^2 \leqslant 4\mathbb{E}\, \|M_n\|^2 \leqslant 4\mathbb{E}\, \|M_{n_m}\|^2 \leqslant 4(1+\tau) \leqslant 5. \tag{21}$$

Since $\{C_j\}_1^m$ and $\{D_{j,k}\}_{j=1}^m$ are families of disjoint sets, (18) and (21) show that

$$\sum_{j=1}^{k-1} \int_{A_k \cap D_{j,k}} \|dM_{n_k}\|^2 \leqslant \theta^2 \sum_{j=1}^{k-1} \int_{D_{j,k}} \|M_{n_j}\|^2 \leqslant \theta^2 \mathbb{E} N_{n_{k-1}}^2 \leqslant 5\theta^2.$$

Since $B_k = \bigcup_{j=1}^{k-1} C_j$ where $C_j = C_{j,k} \cup D_{j,k}$, according to the previous inequality we have

$$\begin{aligned} \int_{A_k \cap B_k} \|dM_{n_k}\|^2 &= \sum_{j=1}^{k-1} \int_{A_k \cap C_j} \|dM_{n_k}\|^2 \\ &\leqslant \sum_{j=1}^{k-1} \left(\int_{C_{j,k}} \|dM_{n_k}\|^2 + \int_{A_k \cap D_{j,k}} \|dM_{n_k}\|^2 \right) \\ &\leqslant \sum_{j=1}^{k-1} \int_{C_{j,k}} \|dM_{n_k}\|^2 + 5\theta^2. \end{aligned} \tag{22}$$

From (19) we deduce that

$$\sum_{j=1}^{k-1} \int_{C_{j,k}} \|dM_{n_k}\|^2 = \sum_{j=1}^{k-1} \int_{C_{j,k,r}} \|dM_{n_k}\|^2 + \sum_{j=1}^{k-1} \sum_{q=1}^{r-1} \int_{F_{j,k,q}} \|dM_{n_k}\|^2 . \tag{23}$$

Since $\|dM_{n_k}\| \leqslant \|M_{n_k}\| + \|M_{n_k - 1}\| \leqslant 2N_{n_k}$, according (20) and (21) we have

$$\sum_{k=2}^{m} \sum_{j=1}^{k-1} \sum_{q=1}^{r-1} \int_{F_{j,k,q}} \|dM_{n_k}\|^2 \leqslant 4(r-1)\mathbb{E} N_{n_m}^2 \leqslant 20(r-1). \tag{24}$$

Combining (22), (23) and (24) we deduce that

$$\sum_{k=2}^{m}\int_{A_k\cap B_k}\|dM_{n_k}\|^2 \leqslant \sum_{k=2}^{m}\sum_{j=1}^{k-1}\int_{C_{j,k}}\|dM_{n_k}\|^2 + 5\theta^2(m-1)$$

$$\leqslant \sum_{k=r+1}^{m}\sum_{j=1}^{k-r}\int_{C_{j,k,r}}\|dM_{n_k}\|^2 + 20(r-1) + 5\theta^2(m-1).$$

Since the sets $\{A_k \setminus B_k\}$ are disjoint, and $20r \leqslant (m-1)\theta^2$ from the above inequality and (21), we obtain

$$\begin{aligned}\sum_{k=2}^{m}\int_{A_k}\|dM_{n_k}\|^2 &\leqslant \sum_{k=2}^{m}\left(\int_{A_k\cap B_k}\|dM_{n_k}\|^2 + \int_{A_k\setminus B_k}\|dM_{n_k}\|^2\right)\\ &\leqslant \sum_{k=2}^{m}\left(\int_{A_k\cap B_k}\|dM_{n_k}\|^2 + 4\int_{A_k\setminus B_k}N_{n_k}^2\right)\\ &\leqslant \sum_{k=2}^{m}\int_{A_k\cap B_k}\|dM_{n_k}\|^2 + 4\mathbb{E}N_{n_m}^2\\ &\leqslant \sum_{k=2}^{m}\int_{A_k\cap B_k}\|dM_{n_k}\|^2 + 20\\ &\leqslant \sum_{k=r+1}^{m}\sum_{j=1}^{k-r}\int_{C_{j,k,r}}\|dM_{n_k}\|^2\\ &\quad + 20(r-1) + 5\theta^2(m-1) + 20\\ &\leqslant \sum_{k=r+1}^{m}\sum_{j=1}^{k-r}\int_{C_{j,k,r}}\|dM_{n_k}\|^2 + 6(m-1)\theta^2.\end{aligned} \tag{25}$$

Now fix $\pi = (n_{i_1}, n_{i_2}, \ldots, n_{i_r} = n_k) \in \Pi_{j,k,r}$. Let $T \in \mathscr{T}_{n_j}$ and $T \subset C_j(\pi)$. Let us take $K_h = M_{n_j+h}$ for $h = 0, 1, 2, \ldots, n_k - n_j$, and $h_l = n_{i_l} - n_j$ for $l = 1, 2, \ldots, r$. Evidently $\{K_j\}_{j=0}^{h_r}$ is a Walsh–Paley martingale over T and $K_0(T) \in H$, $K_{h_l}(A_{i_l} \cap T) \subset H$ for $l = 1, 2, \ldots, r$. From (17) we obtain

$$\frac{1}{p(T)}\int_{A_{i_l}\cap T}\|dK_{h_l}\|^2 \geqslant \theta^2\,\|K_0\|^2 = c\varepsilon^2\,\|K_0\|^2 \quad \text{for } l = 1, 2, \ldots, r.$$

Now, according to Lemma 23, we have $\int_{C_j(\pi)}\|M_{n_k}\|^2 \geqslant (1+\eta^4)\int_{C_j(\pi)}\|M_{n_j}\|^2$. Then since $C_j(\pi)^- \in \mathscr{A}_{n_j}$, we have

$$\mathbb{E}\,\|M_{n_k}\|^2 \geqslant (1+\eta^4)\int_{C_j(\pi)}\|M_{n_j}\|^2 + \int_{C_j(\pi)^-}\|M_{n_j}\|^2 = \eta^4\int_{C_j(\pi)}\|M_{n_j}\|^2 + \mathbb{E}\,\|M_{n_j}\|^2.$$

Bearing in mind that the sequence $(\mathbb{E}\,\|M_n\|^2)$ is non-decreasing, we obtain

$$\int_{C_j(\pi)}\|M_{n_j}\|^2 \leqslant \eta^{-4}\mathbb{E}(\|M_{n_k}\|^2 - \|M_{n_j}\|^2) \leqslant \tau\eta^{-4}. \tag{26}$$

Let $G_{j,k}$ be the union of all sets $T \in \mathscr{T}_{n_j}$ for which

$$2\,\|M_{n_j}(T)\|^2\,p(T) \leqslant \int_T \|M_{n_k}\|^2. \tag{27}$$

This implies that $\int_{G_{j,k}} \|M_{n_j}\|^2 \leqslant \frac{1}{2}\int_{G_{j,k}} \|M_{n_k}\|^2$, and since $G_{j,k}^- \in \mathscr{A}_{n_j}$, we have

$$\tfrac{1}{2}\int_{G_{j,k}} \|M_{n_k}\|^2 \leqslant \int_{G_{j,k}} (\|M_{n_k}\|^2 - \|M_{n_j}\|^2) \leqslant \mathbb{E}(\|M_{n_k}\|^2 - \|M_{n_j}\|^2) \leqslant \tau.$$

Since $C_j(\pi) \in \mathscr{A}_{n_j}$ and $\eta < \frac{1}{2}$, we conclude from (26) and (27) that

$$\begin{aligned}\int_{C_j(\pi)} \|dM_{n_k}\|^2 &\leqslant 4\int_{C_j(\pi)} \|M_{n_k}\|^2 \leqslant 4\Big(\int_{C_j(\pi)\setminus G_{j,k}} \|M_{n_k}\|^2 + \int_{G_{j,k}} \|M_{n_k}\|^2\Big)\\ &\leqslant 8\Big(\int_{C_j(\pi)\setminus G_{j,k}} \|M_{n_j}\|^2 + \tau\Big) \leqslant 8\tau(1+\eta^{-4}) < 9\tau\eta^{-4}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{C_{j,k,r}} \|dM_{n_k}\|^2 &\leqslant \sum_{\pi\in\Pi_{j,k,r}} \int_{C_j(\pi)} \|dM_{n_k}\|^2\\ &\leqslant 9\binom{k-j-1}{r-1}\tau\eta^{-4} \leqslant 9\binom{m-2}{r-1}\tau\eta^{-4}.\end{aligned}$$

This implies that

$$\sum_{k=r+1}^{m}\sum_{j=1}^{k-r}\int_{C_{j,k,r}} \|dM_{n_k}\|^2 \leqslant 9\binom{m-2}{r-1}\tau\eta^{-4}\sum_{k=r+1}^{m}(k-r) \leqslant 5(m-1)r\binom{m}{r}\tau\eta^{-4}.$$

From the above inequality and (25) we obtain

$$7(m-1)\theta^2 \leqslant \sum_{j=2}^{m}\int_{A_j} \|dM_{n_j}\|^2 \leqslant 6(m-1)\theta^2 + 5(m-1)r\binom{m}{r}\tau\eta^{-4},$$

which implies that

$$\tau \geqslant \theta^2\eta^4 \Big/ 5r\binom{m}{r}.$$

COROLLARY 25. *There exists a positive absolute constant d such that $\gamma(H) \geqslant d\varepsilon^{-1}$, whenever $\varepsilon > 0$ and H is an ε-admissible cone.*

The proof follows from Lemma 24 in the same way as Lemma 3.3 follows from Lemma 3.2 in [**3**, pp. 139–144].

LEMMA 26. *Let $0 < \varepsilon, \sigma \leqslant 1$, $k \in \mathbb{N}$, $Q \subset S_E$, and $L \subset kB_{E^*}$ such that $Q = \bigcup\{S(Q,f,1)\colon f \in L\}$ and $\operatorname{diam} S(Q,f,1) < \varepsilon$ for every $f \in L$. Then the cone $H = \{\lambda u\colon u \in Q,\ |u|^2 > 1+\sigma,\ \lambda \geqslant 0\}$ is 2ε-admissible with respect to the norm $|||x||| = (\|x\|^2 + |x|^2 + |-x|^2)^{\frac{1}{2}}$, where $\|\cdot\|$ is the original norm in E, $|x| = \sup\{f^+(x)\colon f \in L\}$, and $f^+(x) = \max(f(x), 0)$.*

Proof. Set

$$\eta = \varepsilon^2\sigma/14k^2, \quad \theta = \eta^4/17k^2, \quad \delta = \theta/2k.$$

We show that H is $(2\varepsilon, \delta)$-admissible with respect to $|||\cdot|||$.

Fix $x \in Q$. Let $\mathbb{E}X = x$ and

$$\mathbb{E}\, |||X|||^2 \leqslant (1+\delta^4)\, |||x|||^2.$$

We have $|x| \leqslant k$, $|-x| \leqslant k$, and $\theta^4 > (1+2k^2)\delta^4 \geqslant (2+k^2)\delta^4$. Moreover, from the choice of $|\cdot|$ we obtain

$$\mathbb{E}\,|X|^2 \geqslant |x|^2 > 1, \quad \mathbb{E}\,|-X|^2 \geqslant |-x|^2, \quad \mathbb{E}\,\|X\|^2 \geqslant |x\|^2 = 1.$$

Now the choice of $|||\cdot|||$ and the inequalities above imply that

$$\begin{aligned} \mathbb{E}\,\|X\|^2 &\leqslant (1+\delta^4)\, |||x|||^2 - \mathbb{E}(|X|^2 + |-X|^2) \\ &\leqslant (1+\delta^4)\,\|x\|^2 + \delta^4(|x|^2 + |-x|^2) \\ &\leqslant 1+\delta^4 + 2\delta^4k^2 < 1+\theta^4. \end{aligned} \tag{28}$$

Similarly, we obtain

$$\begin{aligned} \mathbb{E}\,|X|^2 &\leqslant (1+\delta^4)\, |||x|||^2 - \mathbb{E}(\|X\|^2 + |-X|^2) \\ &\leqslant (1+\delta^4)\,|x|^2 + \delta^4(\|x\|^2 + |-x|^2) \leqslant (1+\delta^4)\,|x|^2 + \delta^4(1+k^2) \\ &\leqslant (1+2\delta^4+\delta^4k^2)\,|x|^2 < (1+\theta^4)\,|x|^2 < (1+\theta)\,|x|^2. \end{aligned} \tag{29}$$

From Lemma 17 and (28) we obtain

$$p(C_1) \leqslant 7\theta^{\frac{1}{2}}, \quad p(D_1) \leqslant 3\theta, \quad \int_{D_1} \|X\|^2 \leqslant 6\theta, \tag{30}$$

where $C_1 = \{\|X\|^2 < 1 - \theta^{\frac{1}{2}}\}$ and $D_1 = \{\|X\|^2 > 1+\theta\}$.

Set $\xi = \min\{\theta, (|x|^2 - 1)/2\,|x|^2\}$. From the definition of $|\cdot|$, it follows that we can find $g \in L$ such that

$$g^2(x) \geqslant (1-\xi)\,|x|^2. \tag{31}$$

So

$$g^2(x) - 1 \geqslant (1-\xi)\,|x|^2 - 1 \geqslant \tfrac{1}{2}(|x|^2 - 1). \tag{32}$$

Set $B = \{g(X) < 0\}$. Then using the Cauchy inequality we obtain

$$g(x) = \mathbb{E}g(X) \leqslant \int_{B^-} g(X) \leqslant \left(\int_{B^-} g^2(X)\right)^{\frac{1}{2}} (p(B^-))^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} \mathbb{E}\,|X|^2 &\geqslant \int_{B^-} g^2(X) \geqslant g^2(x)/p(B^-) \geqslant (1+p(B))g^2(x) \\ &\geqslant (1+p(B))(1-\xi)\,|x|^2 \geqslant (1-2\xi+p(B))\,|x|^2 \geqslant (1-2\theta+p(B))\,|x|^2. \end{aligned}$$

From (29) we have $1+\theta \geqslant 1-2\theta+p(B)$. So

$$p(B) \leqslant 3\theta. \tag{33}$$

Now from (30) we obtain

$$\int_B g^2(X) \leqslant k^2 \int_B \|X\|^2 \leqslant k^2\Big(\int_{D_1} \|X\|^2 + \int_{B\cap D_1^-} \|X\|^2\Big)$$
$$\leqslant k^2(6\theta + (1+\theta)p(B)) \leqslant 12k^2\theta.$$

Then from (29) and (31) we obtain

$$\mathbb{E}g^2(X) \leqslant \mathbb{E}\,|X|^2 + \int_B g^2(X) \leqslant (1+\theta)\,|x|^2 + 12k^2\theta$$
$$\leqslant (1+\theta)(1+2\xi)g^2(x) + 12k^2\theta$$
$$\leqslant (1+\theta+2\xi+2\xi\theta+12k^2\theta)g^2(x)$$
$$\leqslant (1+17k^2\theta)g^2(x) = (1+\eta^4)g^2(x).$$

From Lemma 17 we have

$$\int_{D_2} g^2(X) \leqslant 6\eta g^2(x), \tag{34}$$

where $D_2 = \{g^2(X) > (1+\eta)g^2(x)\}$.

Set $C = \{g^2(X) \leqslant \|X\|^2 \leqslant 1+\theta\}$. Since $g(x) > 1$ and $\theta < \eta$, we have $C \subset D_2^-$. Then from (34) we obtain

$$g^2(x) \leqslant \mathbb{E}g^2(X) = \int_{D_2} g^2(X) + \int_{D_2^-\setminus C} g^2(X) + \int_C g^2(X)$$
$$\leqslant 6\eta g^2(x) + (1+\eta)g^2(x) - (1+\eta)(g^2(x)-1)p(C).$$

This implies that $p(C) \leqslant 7\eta g^2(x)/(g^2(x)-1)$. From (32) we have

$$p(C) \leqslant \frac{14\eta g^2(x)}{|x|^2-1} \leqslant \frac{14k^2\eta}{|x|^2-1} \leqslant \frac{14k^2\eta}{\sigma} = \varepsilon^2. \tag{35}$$

Set $A = \{X \in \|X\|\,Q,\ \|X-x\| \geqslant 2\varepsilon\}$. We claim that $A \subseteq B \cup C \cup C_1 \cup D_1$. Indeed, assume the contrary and let $y \in X(A \setminus (B \cup C \cup C_1 \cup D_1))$. Then we have

$$1-\theta^{\frac{1}{2}} \leqslant \|y\|^2 \leqslant 1+\theta, \quad g^2(y) > \|y\|^2, \quad g(y) > 0.$$

From the last two inequalities it follows that $g(y) > \|y\|$. Then for $z = y/\|y\|$ we obtain $g(z) > 1$ and $z \in Q$. Since $x,\ z \in S(Q, g, 1)$ and $\operatorname{diam} S(Q, g, 1) < \varepsilon$, we conclude that $\|x-z\| < \varepsilon$. On the other hand, $\|x-z\| \geqslant \|x-y\| - \big|1-\|y\|\big| \geqslant 2\varepsilon - \theta^{\frac{1}{2}} > \varepsilon$, a contradiction. Hence applying (30), (33) and (35), we deduce that $p(A) \leqslant p(B) + p(C) + p(C_1) + p(D_1) < 4\varepsilon^2$.

Proof of the implication (a) $\Rightarrow$ (c) *of the Main Theorem.* For each $i \in \mathbb{N}$ we can find $Q_n^i \subset S_E$, $L_n^i \subset E^*$, $n \in \mathbb{N}$ such that $S_E = \bigcup_n Q_n^i$, $Q_n^i \subset \bigcup\{S(Q_n^i, f, 1)\colon f \in L_n^i\}$ and for every $f \in L_n^i$, $\operatorname{diam}(S(Q_n^i, f, 1)) < i^{-1}$. Set $L_{k,n}^i = kB_{E^*} \cap L_n^i$ and

$$Q_{k,n}^i = Q_n^i \cap \left(\bigcup\{S(Q_n^i, f, 1)\colon f \in L_{k,n}^i\}\right),$$

$$|x|_{i,k,n} = \sup\{f^+(x)\colon f \in L_{k,n}^i\},$$

$$|||\, x \,||| = \left(\|x\|^2 + \sum_{i,k,n} 2^{-(i+k+n)}(|x|_{i,k,n}^2 + |-x|_{i,k,n}^2)\right)^{\frac{1}{2}}.$$

If we take $H_{k,n}^{i,j} = \{\lambda u\colon\ u \in Q_{k,n}^i,\ |u|_{i,k,n}^2 > 1 + j^{-1},\ \lambda \geqslant 0\}$, then from Lemma 26 it follows that $H_{k,n}^{i,j}$ is a 2ε-admissible cone with respect to $|||\cdot|||$. Now combining Corollary 25 and Proposition 16 we deduce that E has an equivalent LUR norm.

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