Lebesgue property for convex risk measures on Orlicz spaces

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To Walter Schachermayer with respect and admiration on the occasion of his 60th birthday.

Abstract

We present a robust representation theorem for monetary convex risk measures $\rho : \mathcal{X} \to \mathbb{R}$ such that

$$\lim_n \rho(X_n) = \rho(X) \text{ whenever } (X_n) \text{ almost surely converges to } X,$$

$|X_n| \leq Z \in \mathcal{X}$, for all $n \in \mathbb{N}$ and $\mathcal{X}$ is an arbitrary Orlicz space. The separable $\langle \mathbb{L}^1, \mathbb{L}^\infty \rangle$ case of Jouini, Schachermayer and Touizi, [14], as well as the non-separable version of Delbaen [6], are contained as a particular case here. We answer a natural question posed by Biagini and Fritelli in [2]. Our approach is based on the study for unbounded sets, as the epigraph of a given penalty function associated with ρ, of the celebrated weak compactness Theorem due to R. C. James [13].

1 Introduction

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with $\mathcal{X}$, a linear space of functions in $\mathbb{R}^\Omega$ that contains the constant functions. We assume here that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, though in practice this is not a restriction since the property of being atomless is equivalent to the fact that on $(\Omega, \mathcal{F}, \mathbb{P})$, one can define a random variable that has a continuous distribution function. The space $\mathcal{X}$ is going to describe all possible financial positions $X : \Omega \to \mathbb{R}$ where $X(\omega)$ is the discounted net worth of the position at the end of the trading period if the scenario $\omega \in \Omega$ is realized. The problem of quantifying the risk of a financial position $X \in \mathcal{X}$ is modeled with function $\rho : \mathcal{X} \to \mathbb{R}$ that satisfy:

- Monotoniticity : If $X \leq Y$, then $\rho(X) \geq \rho(Y)$
- Cash invariance: If $m \in \mathbb{R}$ then $\rho(X+m) = \rho(X) - m$

Such a function ρ is called *monetary measure of risk*, see Chapter 4 in [11]. When it is a convex function too, i.e.

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y) \text{ for } 0 \leq \lambda \leq 1,$$

then ρ is called *convex measure of risk*.

If $\mathcal{X}$ is also a topological space, as it always is in the applications, then it is good to have results on the degree of smoothness of the risk measure ρ. If $\mathcal{X}$ is a Frechet lattice, the convexity and monotoniticity of ρ lead to continuity and subdifferentiability at all positions $X \in \mathcal{X}$ by the Extended Namioka-Klee Theorem of [2], as well as to strong representations of the form

$$\rho(X) = \max_{Y \in (\mathcal{X}')^+} \{\langle Y, -X \rangle - \rho^*(Y)\},$$

for all $X \in \mathcal{X}$ whenever $\mathcal{X}$ is order continuous (see Theorem 1 and Corollary 1 in [2]). Here $(\mathcal{X}')^+$ denotes the positive cone of continuous linear functionals, ρ^* is the Fenchel conjugate to ρ:

$$\rho^*(Y) = \sup_{X \in \mathcal{X}} \{\langle Y, X \rangle - \rho(X)\}$$

for all $Y \in \mathcal{X}'$, and $\langle \cdot, \cdot \rangle$ denotes the bilinear form for the duality between $\mathcal{X}$ and its topological dual $\mathcal{X}'$.

When the Frechet lattice $\mathcal{X}$ is not order continuous, for instance the Lebesgue space of bounded and measurable functions $\mathcal{X} = \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with its canonical essential supremum norm $\|\cdot\|_\infty$, things are more subtle and we have the representation formula with a supremum instead of maximum:

$$\rho(X) = \sup_{Y \in (\mathcal{X}_n^\sim)^+} \{\langle Y, -X \rangle - \rho^*(Y)\}, \tag{1}$$

where $(\mathcal{X}_n^\sim)^+$ denotes the positive cone of the order continuous linear functionals $\mathcal{X}_n^\sim \subset \mathcal{X}'$, whenever the risk measure ρ is $\sigma(\mathcal{X}, \mathcal{X}_n^\sim)$-lower semicontinuous, as seen in Proposition 1 in [2]. A natural question posed in [2] is whether the sup in formula (1) is attained. In general the answer is no, as shown by the essential supremum map on L^∞: see Example 3 in [2].

For risk measures defined on L^∞, the representation formula with supremum is equivalent to the so–called Fatou property, given in Theorem 4.31 in [11]. The fact that the order continuity of ρ is equivalent to turning the sup into a max in (1) i.e:

$$\rho(X) = \max_{Y \in (L^\infty)^+} \{-\mathbb{E}[Y \cdot X] - \rho^*(Y)\},$$

for all $X \in L^\infty$, is the statement of the so called Jouini-Schachermayer-Touzi Theorem in [6] (see Theorem 5.2 in [14]).

S. Biagini and M. Fritelli show (Lemma 7 in [2]) that order continuity of the risk measure ρ is sufficient to turn a maximum in (1) for an arbitrary locally convex Frechet lattice $\mathcal{X}$. Our main contribution here shows that sequential order continuity only is a necessary and sufficient condition for it when $\mathcal{X}$ is an arbitrary Orlicz space L^Ψ and the order continuous linear functionals $\mathcal{X}_n^\sim = \mathbb{L}^{\Psi^*}$ coincides with the Orlicz heart $\mathbb{M}^{\Psi^*}$, i.e., we present the Jouini-Schachermayer-Touizi Theorem for risk measures defined on Orlicz spaces. Let us remark here that Orlicz spaces provide a general framework of Banach lattices for applications in matematical finance, described for instance in [5, 2, 3]. The result reads as follows:

Theorem 1 (Lebesgue Risk Measures)**.** *Let Ψ be a Young function with finite conjugate Ψ^* and*

$$\alpha : (\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P}))^* \to \mathbb{R} \cup \{+\infty\}$$

be a $\sigma((\mathbb{L}^\Psi)^*, \mathbb{L}\Psi)$-lower semicontinuous penalty function representing a finite monetary risk measure ρ as

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{-\mathbb{E}[XY] - \alpha(Y)\}.$$

The following are equivalent:

(i) *For all $c \in \mathbb{R}$, $\alpha^{-1}((-\infty, c])$ is a weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

(ii) *For every $X \in \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$, the supremum in the equality*

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{-\mathbb{E}[XY] - \alpha(Y)\}$$

is attained.

(iii) *ρ is order sequentially continuous.*

Let us remark that order sequential continuity for a map ρ in $\mathbb{L}^\Psi$ is equivalent to having

$$\lim_n \rho(X_n) = \rho(X)$$

whenever (X_n) is a sequence in L^Ψ almost surely convergent to X and bounded by some $Z \in L^\Psi$, i.e. $|X_n| \leq Z$ for all $n \in \mathbb{N}$, see Proposition 1. For that reason we say that a map $\rho : L^\Psi \to (-\infty, +\infty]$ verifies the Lebesgue property whenever it is sequentially order continuous.

As the first author observed, and appears in Appendix in [14], for the L^∞-case the proof requires compactness arguments of perturbed James's type. Indeed, in [6] the Theorem is presented as a generalization of the beautiful result of of R.C. James on weakly compact sets. In that case the penalty function α used has a bounded domain. We have obtained general perturbed James results for coercive functions α in [15]. Non coercive growing conditions for penalty functions in the Orlicz case have been studied in [5], Theorem 4.5, where it is proven that a risk measure ρ, defined by a penalty function α, is finite on the Morse subspace $\mathbb{M}^{\Psi} \subset L^{\Psi}$ if, and only if, α satisfies the growing condition

$$\alpha(Y) \geq a + b\|Y\|_{\Psi^*}$$

for all $Y \in \mathbb{L}^{\Psi^*}$ and fixed numbers a, b with $b > 0$. We look here for statements without restriction on penalty functions α that are easy to apply in the context of Orlicz spaces. We shall prove the following general result that includes the previous separable case in [14] and the non separable approach of F. Delbaen in [6] for the duality $\langle \mathbb{L}^1, \mathbb{L}^\infty \rangle$, as well as new applications here for the duality $\langle \mathbb{M}^{\Psi^*}, \mathbb{L}^{\Psi} \rangle$:

Theorem 2 (Perturbed James's Theorem)**.** *Let E be a real Banach space whose dual unit ball is w^*-sequentially compact and let*

$$\alpha : E \longrightarrow \mathbb{R} \cup \{\infty\}$$

be a proper map such that

$$\textit{for all } x^* \in E^*, \quad x^* - \alpha \textit{ attains its supremum on } E.$$

Then, for all $c \in \mathbb{R}$, the corresponding sublevel set $\alpha^{-1}((-\infty, c])$ is a relatively weakly compact subset of E.

Theorem 2 provides a complement to our study in [15]. It provides answers to questions stated in the Erratum [4] for the case of Banach spaces with a w^*-convex block compact dual unit ball, for instance for all Banach spaces that do not contain a copy of l^1, see Section 3.

Let us remark that in finishing the paper we were kindly informed by F. Delbaen of a surprising result: we always are forced to have the compactness-continuity property ((i)-(iii) in Theorem 1) restricted to the duality $\langle \mathbb{L}^1, \mathbb{L}^\infty \rangle$, whenever the convex risk measure ρ is defined on a rearrangement invariant solid space $\mathcal{X}$ such that $\mathcal{X} \setminus L^\infty \neq \emptyset$, as seen n Section 4.16 in [8]. This result is even true for non-continuous risk measures since F. Delbaen only asks for the property

$$\rho(X) \geq 0 = \rho(0) \text{ whenever } X \geq 0$$

instead of monotonicity. Our Theorem 1 complements Delbaen's results when looking for compactness-continuity properties in the duality $\langle \mathbb{M}^{\Psi^*}, \mathbb{L}^{\Psi} \rangle$.

Acknowledgement We gratefully thank Walter Schachermayer for providing the first author with the compactness problem for non-coherent convex risk measures defined on $\mathbb{L}^\infty$. The coherent case was done by F. Delbaen in [7] and the non-coherent case was presented in [6] also as the Jouini-Schachermayer-Touizi Theorem. The solution given in [14], together with the analysis done by S. Biagini amd M. Fritelli in [2] in the framework of Frechet lattices, motivated the present paper.

2 The unbounded sup–limsup Theorem

This section contains the tools we need to prove the results in the Introduction. To this end we modify results in [17] to obtain versions of them in a pointwise bounded context only, the frame where our results must be formulated to include epigraph sets in the applications later. The main novelty of our approach here is that we are going to use James-boundaries on unbounded sets as the epigraph of a fixed function α.

From now on T will denote a non-empty set. Given a pointwise bounded sequence $\{f_n\}_{n\geq 1}$ in $\mathbb{R}^T$, we define

$$\mathrm{co}_{\sigma_p}\{f_n\colon\ n\geq 1\} := \left\{\sum_{n=1}^{\infty}\lambda_n f_n :\ \text{for all } n\geq 1,\ \lambda_n\geq 0 \text{ and } \sum_{n=1}^{\infty}\lambda_n = 1\right\},$$

where the functions $\sum_{n=1}^{\infty}\lambda_n f_n\in\mathbb{R}^T$ above are pointwise defined on T, i.e. for every $t\in T$ the absolutely convergent series

$$\sum_{n=1}^{\infty}\lambda_n f_n(t)$$

defines the function $\sum_{n=1}^{\infty}\lambda_n f_n : T\to\mathbb{R}$.

Lemma 1. *If $\{f_n\}_{n\geq 1}$ in $\mathbb{R}^X$ is a pointwise bounded sequence and for all $m\geq 1$, $g_m\in \mathrm{co}_{\sigma_p}\{f_n\colon\ n\geq 1\}$, then*

$$\mathrm{co}_{\sigma_p}\{g_m\colon\ m\geq 1\}\subset \mathrm{co}_{\sigma_p}\{f_n\colon\ n\geq 1\}.$$

*P*roof.- It follows from [17, Lemma 2]. □

For $f\in\mathbb{R}^T$ we write

$$I_T(f) := \inf\{f(t) : t\in T\}\in [-\infty,\infty[$$

and

$$S_T(f) := \sup\{f(t) : t\in T\}\in (-\infty,\infty].$$

Definition 1. *If $\{f_n\}_{n\geq 1}$ is a pointwise bounded sequence in $\mathbb{R}^T$, then a sequence $\{g_m\}_{m\geq 1}$ in $\mathbb{R}^T$ is a pointwise–pseudo–subsequence of $\{f_n\}_{n\geq 1}$ provided that for all $m\geq 1$, $g_m\in co_{\sigma_p}\{f_n\colon\ n\geq m\}$.*

Lemma 2. *Let $\{f_n\}_{n\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^T$. Then*

$$\sup I_T(\mathrm{co}_{\sigma_p}\{f_n : n\geq 1\}) < \infty.$$

*P*roof.- Let us prove that if the inequality above does not hold, then for all $x\in X$ the sequence $\{f_n(x)\}_{n\geq 1}$ is not bounded. Thus, let us assume that

$$\sup I_T(\mathrm{co}_{\sigma_p}\{f_n\colon\ n\geq 1\}) = \infty$$

and let $t \in T$. Then, given $m \geq 1$, there exists $g_m \in \mathrm{co}_{\sigma_p}\{f_n : n \geq 1\}$ so that

$$I_T(g_m) > m. \tag{2}$$

But g_m is a pointwise function defined on X as

$$g_m = \sum_{n=1}^{\infty} \lambda_n^{(m)} f_n$$

for some $\{\lambda_n^{(m)}\}_{n\geq 1} \subset [0,1]$ with $\sum_{n^1}^{\infty} \lambda_n^{(m)} = 1$. Then, in view of (2) we have that

$$\sum_{n=1}^{\infty} \lambda_n^{(m)} f_n(t) > m,$$

which implies that there exists $n \geq 1$ such that $f_n(t) > m$. Thus we have shown that given $t \in X$ and $m \geq 1$ there exists $n \geq 1$ with $f_n(t) > m$, i.e.,

$$\sup_{n\geq 1} f_n(t) = \infty.$$

□

The next lemma is a crucial step towards the proof of Theorem 2. Here we are assuming that any sum of the form $\sum_{m=1}^{0} \cdots$ is always defined as 0.

Lemma 3. *Suppose that $\{f_n\}_{n\geq 1}$ is a pointwise bounded sequence in $\mathbb{R}^T$ and that $\rho, \eta \in (0,1)$. Then there exists a pointwise–pseudo–subsequence $\{g_m\}_{m\geq 1}$ of $\{f_n\}_{n\geq 1}$ such that for all $k \geq 0$,*

$$I_T\left(\sum_{m=1}^{\infty} \rho^m g_m\right) \leq I_T\left(\sum_{m=1}^{k} \rho^m g_m\right) + \rho^k \left(I_T\left(\sum_{m=1}^{\infty} \rho^m g_m\right) + \eta\right). \tag{3}$$

*P*roof.- The proof follows from that of [17, Lemma 4(a)], yet now we need to replace the uniform boundedness with pointwise boundedness and to make use of Lemma 2.
For all $q \geq 1$, let $C_q := \mathrm{co}_{\sigma_p}\{f_n\colon\ n \geq q\}$ and choose $g_q \in C_q$ inductively so that

$$I_T\left(\sum_{m=1}^{q-1} \rho^m g_m + \rho^q g_q\right) \geq \sup_{g\in C_q} I_T\left(\sum_{m=1}^{q-1} \rho^m g_m + \rho^q g\right) - \eta(\rho/2)^q \tag{4}$$

(Lemmas 1 and 2 guarantee that $\sup_{g\in C_q} I_T(\sum_{m=1}^{q-1} \rho^m g_m + \rho^q g) < \infty$). Let $h := \sum_{m=1}^{\infty} \rho^m g_m$ pointwise on T and, for all $q \geq 1$, $h_q := \sum_{m=1}^{q} \rho^m g_m$. Then (4) implies that

$$\text{for all } q \geq 1, \quad I_T(h_q) \geq \sup_{g\in C_q} I_T(h_{q-1} + \rho^q g) - \eta(\rho/2)^q. \tag{5}$$

We can assume that $I_T(h) > -\infty$ and $k \geq 1$, since otherwise (3) is clearly valid. So let $k \geq 1$ and $1 \leq q \leq k$. Then, from Lemma 1,

$$(1-\rho)(h - h_{q-1})/\rho^q = \sum_{m=0}^{\infty} \frac{(\rho^m - \rho^{m+1})}{\rho^q} g_{m+q} \in C_q$$

and so, it follows from (5) that

$$\begin{aligned}I_T(h_q) &\geq I_T(h_{q-1} + (1-\rho)(h - h_{q-1})) - \eta(\rho/2)^q\\ &= I_T((1-\rho)h + \rho h_{q-1}) - \eta(\rho/2)^q\\ &\geq (1-\rho)I_T(h) + \rho I_T(h_{q-1}) - \eta(\rho/2)^q.\end{aligned}$$

Finally we divide this inequality by ρ^q, arriving at

$$(1/\rho^q - 1/\rho^{q-1})I_T(h) \leq I_T(h_q)/\rho^q - I_T(h_{q-1})/\rho^{q-1} + \eta/2^q,$$

and adding up these inequalities for $q = 1, 2, \ldots, k$, we conclude that

$$(1/\rho^k - 1)I_T(h) \leq I_T(h_k)/\rho^k + \eta,$$

which is nothing more than (3). □

The following result is the announced generalization of the *sup–limsup theorem* of S. Simons [18], that was stated in terms of sup's and limsup's. It follows ideas in [17, Theorem 7] that can be adjusted to our unbounded case here:

Theorem 3 (Inf-liminf Theorem in $\mathbb{R}^T$)**.** *Let $\{\Phi_k\}_{k\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^T$ and suppose that Y is a subset of T satisfying the following boundary condition:*

$$\textit{for all } \Phi \in \mathrm{co}_{\sigma_p}\{\Phi_k\colon\ k \geq 1\} \textit{ there exists } y \in Y \textit{ with } \Phi(y) = I_T(\Phi).$$

Then

$$I_Y\left(\liminf_{k\geq 1} \Phi_k\right) = I_T\left(\liminf_{k\geq 1} \Phi_k\right).$$

*P*roof.- We only have to prove that $I_Y(\liminf_{k\geq 1} \Phi_k) \leq I_T(\liminf_{k\geq 1} \Phi_k)$. We demonstrate this inequality when $I_T(\liminf_{k\geq 1} \Phi_k) > -\infty$. The proof for the case $I_T(\liminf_{k\geq 1} \Phi_k) = -\infty$ is similar. Let $\eta > 0$. Let $t \in T$ such that $\liminf_{k\geq 1} \Phi_k(t) < I_T(\liminf_{k\geq 1} \Phi_k) + \eta$, and then let $\{f_n\}_{n\geq 1}$ be a subsequence of $\{\Phi_k\}_{k\geq 1}$ such that $\sup_{n\geq 1} f_n(t) \leq I_T(\liminf_{k\geq 1} \Phi_k) + \eta$. In particular we have that

$$I_T\left(\sup_{n\geq 1} f_n\right) \leq I_T\left(\liminf_{k\geq 1} \Phi_k\right) + \eta. \tag{6}$$

If we take $\rho = 1/2$ in Lemma 3, we guarantee the existence of a pointwise–pseudo–subsequence $\{g_m\}_{m\geq 1}$ of $\{f_n\}_{n\geq 1}$ such that for all $k \geq 0$ the inequality

$$I_T\left(\sum_{m=1}^{\infty} g_m/2^m\right) \leq I_T\left(\sum_{m=1}^{k} g_m/2^m\right) + \left(I_T\left(\sum_{m=1}^{\infty} g_m/2^m\right) + \eta\right)/2^k \tag{7}$$

is valid. But

$$\sum_{m=1}^{\infty} g_m/2^m \leq \sup_{m\geq 1} g_m \leq \sup_{n\geq 1} f_n \text{ on } T$$

so, from (6) and (7), we have that for all $k \geq 0$

$$I_T\left(\sum_{m=1}^{\infty} g_m/2^m\right) \leq I_T\left(\sum_{m=1}^{k} g_m/2^m\right) + \left(I_T\left(\liminf_{k\geq 1} \Phi_k\right) + 2\eta\right)/2^k. \tag{8}$$

We know, in view of Lemma 1 and our hypothesis, that there exists $y \in Y$ such that

$$\sum_{m=1}^{\infty} g_m(y)/2^m = I_T(\sum_{m=1}^{\infty} g_m/2^m).$$

Therefore, we deduce from this and (8) that for all $k \geq 0$,

$$\sum_{m=1}^{\infty} g_m(y)/2^m \leq \sum_{m=1}^{k} g_m(y)/2^m + \left(I_T\left(\liminf_{k\geq 1} \Phi_k\right) + 2\eta\right)/2^k,$$

so

$$\sum_{m=k+1}^{\infty} g_m(y)/2^m \leq \left(I_T\left(\liminf_{k\geq 1} \Phi_k\right) + 2\eta\right)/2^k,$$

hence

$$\sum_{m=1}^{\infty} g_{m+k}(y)/2^m \leq I_T\left(\liminf_{k\geq 1} \Phi_k\right) + 2\eta$$

and thus

$$\inf_{m>k} g_m(y) \leq I_T\left(\liminf_{k\geq 1} \Phi_k\right) + 2\eta.$$

In order to conclude the proof it suffices to note that, letting $k \to \infty$, we obtain that

$$\liminf_{m\geq 1} g_m(y) \leq I_T\left(\liminf_{k\geq 1} \Phi_k\right) + 2\eta.$$

But

$$\liminf_{k\geq 1} \Phi_k \leq \liminf_{n\geq 1} f_n \leq \liminf_{m\geq 1} g_m \text{ on } T,$$

and so

$$\liminf_{n\geq 1} \Phi_n(y) \leq I_T\left(\liminf_{k\geq 1} \Phi_k\right) + 2\eta.$$

Then, the inf–liminf equality follows from the arbitrariness of $\eta > 0$. □

The version for the supremum functional reads as follows:

Corollary 1 (Sup-limsup Theorem in $\mathbb{R}^T$)**.** *Let $\{\Phi_k\}_{k\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^T$ and suppose that Y is a subset of T satisfying the following boundary condition:*

for all $\Phi \in \mathrm{co}_{\sigma_p}\{\Phi_k\colon\ k \geq 1\}$ there exists $y \in Y$ with $\Phi(y) = S_T(\Phi)$.

Then

$$S_Y\left(\limsup_{k\geq 1} \Phi_k\right) = S_T\left(\limsup_{k\geq 1} \Phi_k\right).$$

As a consequence of Theorem 1, we arrive at the following extension of [18, Proposition 1] for unbounded boundaries, hearafter E will denote a real Banach space.

Corollary 2. *If $Y \subset T$ are nonempty subsets of E^*, and $\{x_n\}_{n\geq 1}$ is a bounded sequence in the Banach space E such that*

$$\textit{for all } x \in \operatorname{co}_\sigma\{x_n : \ n \geq 1\} \textit{ there exists } y^* \in Y \textit{ with } y^*(x) = I_T(x),$$

(resp. $y^*(x) = S_T(x)$) then

$$I_Y\left(\liminf_{n\geq 1} x_n\right) = I_T\left(\liminf_{n\geq 1} x_n\right).$$

(resp. $S_Y\left(\limsup_{n\geq 1} x_n\right) = S_T\left(\limsup_{n\geq 1} x_n\right)$.)

3 Unbounded James's compactness Theorem

A proof for Theorem 2 in the Introduction is presented here. Nevertheless we can prove the result for Banach spaces with a property more general than w^*-sequential compactness required in Theorem 4, a property used years ago in Banach space Theory and that resembles Komlos's Theorem when it is applied in spaces of measurable functions.

To introduce it let us recall the following

Definition 2. *Given a sequence $\{v_n\}_{n\geq 1}$ in vector space E, we say that another one $\{u_n\}_{n\geq 1}$ is a convex block sequence of $\{v_n\}_{n\geq 1}$ if there isa sequence of finite subsets of integers $\{F_n\}_{n\geq 1}$ such that*

$$\max F_1 < \min F_2 \leq \max F_2 < \min F_3 \cdots < \max F_n < \min F_{n+1} < \cdots$$

together with sets of positive numbers $\{\lambda_i^n : i \in F_n\} \subset (0,1]$ satisfying

$$\sum_{i\in F_n} \lambda_i^n = 1 \textit{ and } u_n = \sum_{i\in F_n} \lambda_i^n v_i.$$

When E is a Banach space and each sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} has a convex block w^*–convergent sequence we say that B_{E^*} is w^*-convex block compact.

It is not difficult to check that if E is separable then the dual unit ball B_{E^*} is w^*–*convex block compact*. Moreover, J. Bourgain proved in [1] that if the Banach space E does not contain a copy of $l^1(\mathbb{N})$, then its dual unit ball is w^*–convex block compact. This result was extended for spaces not containing a copy of $l^1(\mathbb{R})$ under Martin's axiom and the negation of the Continuum Hypothesis in [12].

Lemma 4. *Suppose that the dual unit ball of E is w^*–convex block compact and that A is a nonempty, bounded subset of E. Then A is relatively weakly compact if (and only if) each $\sigma(E^*, E)$–null sequence in E^* is also $\sigma(E^*, \overline{A}^{\sigma(E^{**},E^*)})$–null.*

Proof.- We shall proceed by contradiction, so let us assume that A is not relatively weakly compact. The Eberlein–Smulian theorem provides us a sequence $\{x_n\}_{n\geq 1}$ in A together with an element $x_0^{**} \in \overline{A}^{\sigma(E^{**},E^*)} \backslash E$ which is a cluster point of this sequence in $(E^{**}, \sigma(E^{**}, E^*))$. We can now apply the Hahn–Banach theorem to find $x^{***} \in B_{E^{***}}$ such that

$$x^{***}(x_0^{**}) \neq 0 \quad \text{but} \quad x^{***}(x) = 0, \quad \text{for all } x \in E. \tag{9}$$

Goldstine's theorem and the separability of the subspace E_0 of E^{**} generated by x_0^{**} and $\{x_n : n \geq 1\}$ guarantee the existence of a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*} satisfying

$$\sigma(E^{***}, E_0) - \lim_{n\geq 1} x_n^* = x^{***}. \tag{10}$$

Since B_{E^*} is w^*–convex block compact, there exists a block sequence $\{y_n^*\}_{n\geq 1}$ of $\{x_n^*\}_{n\geq 1}$ and an $x_0^* \in B_{E^*}$ such that

$$\sigma(E^*, E) - \lim_{n\geq 1} y_n^* = x_0^*.$$

Then, by assumption, $\{y_n^*\}_{n\geq 1}$ also converges to x_0^* pointwise on $\overline{A}^{\sigma(E^{**},E^*)}$, and so

$$\sigma(E^*, E_0) - \lim_{n\geq 1} y_n^* = x_0^*. \tag{11}$$

Finally, it follows from (9), (10) and (11) that

$$0 \neq x^{***}(x_0^{**}) = x_0^{**}(x_0^*), \quad \text{but for all } n \geq 1,\ 0 = x^{***}(x_n) = x_0^*(x_n),$$

which is a contradiction, since x_0^{**} is a $\sigma(E^{**}, E^*)$−cluster point of the sequence $\{x_n\}_{n\geq 1}$. □

We can now prove the following perturbed James's compactness theorem without any restriction on the proper perturbation term. It provides an answer to the problem stated in [4] for a wide class of spaces, see [15] for applications of this result to nonlinear variational problems.

Theorem 4 (Perturbed James's Theorem)**.** *Let E be a real Banach space whose dual unit ball is w^*–convex block compact and let $\alpha : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper map such that*

$$\textit{for all } x^* \in E^*, \quad \alpha + x^* \textit{ attains its infimum on } E.$$

Then for all $c \in \mathbb{R}$, the corresponding sublevel set $\alpha^{-1}((-\infty, c])$ is relatively weakly compact.

Proof.- Let us consider the epigraph of α, i.e.

$$\text{Epi}(\alpha) = \{(x, t) \in E \times \mathbb{R} : \alpha(x) \leq t\}$$

and analyze what our hypothesis tell us about it. We claim that for every $(x^*, \lambda) \in E^* \times \mathbb{R}$ with $\lambda > 0$, there exists $x_0 \in \mathrm{Dom}(\alpha) := \{x \in E : \alpha(x) \neq \infty\}$ such that

$$\inf\{(x^*, \lambda)(x, t) : (x, t) \in \mathrm{Epi}(\alpha)\} = x^*(x_0) + \lambda\alpha(x_0). \tag{12}$$

In fact, the optimization problem

$$\inf_{x \in E}\{\langle x, x^*\rangle + \alpha(x)\} \tag{13}$$

may be rewritten as

$$\inf_{(x,t)\in \mathrm{Epi}(\alpha)}\{(x^*, 1), (x, t)\} \tag{14}$$

and the inf in (13) is attained if and only if the inf in (14) is attained.

Let us first observe that every sublevel set $A := \alpha^{-1}((-\infty, c])$ is bounded. Indeed, for every $x^* \in E^*$ there is $x_0 \in E$ such that

$$-x^*(x_0) + \alpha(x_0) \leq -x^*(x) + \alpha(x)$$

for all $x \in E$. Thus we see that

$$x^*(x) \leq x^*(x_0) - \alpha(x_0) + c$$

for all $x \in A$ and the Banach Steinhauss Theorem affirms the boundedness of A. Let us now assume that A is not relatively weakly compact. We derive from Lemma 4 the existence of a sequence $\{x_n^*\}_{n\geq 1}$ in B_{E^*}, a point $x_0^{**} \in \overline{A}^{\sigma(E^{**},E^*)} \backslash E$ and $\beta > 0$ such that

$$\sigma(E^*, E) - \lim_{n\geq 1} x_n^* = 0 \tag{15}$$

and

$$\text{for all } n \geq 1, \quad x_0^{**}(x_n^*) < -\beta. \tag{16}$$

Since we have (12) we can apply the Inf–liminf Theorem to the nonempty sets

$$T := \overline{\mathrm{Epi}(\alpha)}^{\sigma(E^{**},E^*)}, \qquad Y := \mathrm{Epi}(\alpha)$$

and the sequence of pointwise functions on T $\{\Phi_k\}_{k\geq 1}$ given by

$$(x^{**}, t) \in T \mapsto \Phi_k(x^{**}, t) := x^{**}(x_k^*) + \frac{t}{k},$$

obtaining, in view of properties (15) and (16) for the sequence $\{x_n^*\}_{n\geq 1}$

$$0 = I_Y\left(\liminf_{k\geq 1} \Phi_k\right) = I_T\left(\liminf_{k\geq 1} \Phi_k\right) < -\beta,$$

a contradiction which finishes the proof. □

The following consequence will be used later:

Corollary 3. *Let E be a Banach space with B_{E^*} a w^*-convex block compact. Let*

$$\alpha : E \to \mathbb{R} \cup \{+\infty\}$$

be a proper map. If for every $x^* \in E^*$ the minimization problem

$$\inf\{x^*(y) + \alpha(y) : y \in E\}$$

is attained in E, then the epigraph of α is $\sigma(E^{**} \times \mathbb{R}, E^* \times \mathbb{R})$-closed in $E^{**} \times \mathbb{R}$.

*P*roof.- If not, take $(z_0^{**}, \mu_0) \in \overline{\mathrm{Epi}(\alpha)}^{\sigma(E^{**}\times\mathbb{R},E^*\times\mathbb{R})} \setminus E \times \mathbb{R}$ and a net $\{(x_\alpha, t_\alpha) : \alpha \in (A, \preceq)\}$ in $\mathrm{Epi}(\alpha)$ with $\lim_{\alpha \in A, \preceq}(x_\alpha, t_\alpha) = (z_0^{**}, \mu_0)$. Take $\alpha_0 \in A$ so that $\mu_\alpha \leq \mu_0 + 1$ for all $\alpha \geq \alpha_0$. Then (z_0^{**}, μ_0) belongs to the $\sigma(E^{**} \times \mathbb{R}, E^* \times \mathbb{R})$-closure of the truncated epigraph

$$\mathrm{Epi}(\alpha, \mu_0 + 1) = \{(x, t) \in E \times \mathbb{R} : \alpha(x) \leq \mu_0 + 1\},$$

which coincides with it since $\alpha^{-1}(-\infty, m_0 + 1]$ is a relatively weakly compact subset of E by Theorem 4, a contradiction. □

4 Convex risk measures on Orlicz spaces

A Young function Ψ is an even, convex function $\Psi : E \to [0, +\infty]$ with the properties:

1. $\Psi(0) = 0$
2. $\lim_{x\to\infty} \Psi(x) = +\infty$
3. $\Psi < +\infty$ in a neighborhood of 0.

The Orlicz space L^Ψ is defined as:

$$L^\Psi(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : \exists \alpha > 0 \text{ with } \mathbb{E}_\mathbb{P}[\Psi(\alpha X)] < +\infty\}$$

and we consider the Luxemburg norm on it:

$$N_\Psi(X) := \inf\{c > 0 : \mathbb{E}_\mathbb{P}[\Psi(\frac{1}{c}X)] \leq 1\}$$

for all $X \in L^\Psi(\Omega, \mathcal{F}, \mathbb{P})$. With the usual pointwise lattice operations, $L^\Psi(\Omega, \mathcal{F}, \mathbb{P})$ is a Banach lattice and we have inclusions:

$$L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset L^\Psi(\Omega, \mathcal{F}, \mathbb{P}) \subset L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Moreover, $(L^\Psi)^* = L^{\Psi^*} \oplus G$ where G is the singular band and L^{Ψ^*} is the order continuous band identified with the Orlicz space L^{Ψ^*}, where

$$\Psi^*(y) := \sup_{x\in\mathbb{R}}\{yx - \Psi(x)\}$$

is the Young function conjugate to Ψ.

We are interested in risk measures defined on $L^\Psi(\Omega,\mathcal{F},\mathbb{P})$ and their robust representation. Given a convex risk measure

$$\rho : \mathbb{L}^\Psi(\Omega,\mathcal{F},\mathbb{P}) \to (-\infty,+\infty]$$

we look for penalty functions

$$\alpha : (\mathbb{L}^\Psi(\Omega,\mathcal{F},\mathbb{P})^* \to (-\infty,+\infty]$$

representing ρ as

$$\rho(X) = \sup_{Y\in\mathbb{L}^{\Psi^*}} \{-\mathbb{E}[XY] - \alpha(Y)\}$$

for all $X \in L^\Psi$. We shall study when it is possible to reduce it to a robust representation with a maximum instead of a supremum, so that for every $X \in \mathbb{L}^\Psi(\Omega,\mathcal{F},\mathbb{P})$, the supremum in the equality

$$\rho(X) = \sup_{Y\in\mathbb{L}^{\Psi^*}} \{-\mathbb{E}[XY] - \alpha(Y)\}$$

would be finite and attained. Genuine examples of Young function that induce spaces different from the L^p are associated to utility maximization problems as described in [2]. For example, if we fix the utility function

$$u_\gamma(t) := -\exp(-\gamma t) + 1,$$

we may consider the associated Young function

$$\Psi_\gamma(t) := -u(-|t|) = \exp(\gamma|t|) - 1$$

and the dual function

$$\Psi^*_\gamma(t) = (|\frac{t}{\gamma}| \ln |\frac{t}{\gamma}| - |\frac{t}{\gamma}|)\chi_{\{|\frac{t}{\gamma}|\geq 1\}}.$$

In this example we have $\mathbb{L}^{\Psi^*_\gamma} = \mathbb{M}^{\Psi^*_\gamma}$, since Ψ^*_γ verifies the so-called Δ_2 condition (see [2]). Let us remember here that a Young function Φ verifies the Δ_2 condition if there are $t_0 > 0$ and $K > 0$ such that:

$$\Phi(2t) \leq K\Phi(t) \text{ for all } t > t_0.$$

The study of risk measures in the Orlicz heart M^Ψ, the Morse subspace of all $X \in L^\Psi$ such that

$$\mathbb{E}_\mathbb{P}[\Psi(\beta X)] < +\infty$$

for all $\beta > 0$, is different because for finite functions Ψ we have $(M^\Psi)^* = L^{\Psi^*}$ and we will have robust representation formulas (1) with a maximum instead of a supremum by using w^*-compactness of dual unit balls (see [5]). As byproduct of Theorem 1 we will see in Corollary 7 how to reduce it to $\mathbb{M}^{\Psi^*}$.

Let us state here the Jouini–Schachermayer–Touizi Theorem which motivates our study, [14]. We remind the reader that a monetary utility function U is nothing else but a function

$$U : \mathbb{L}^\infty(\Omega,\mathcal{F},\mathbb{P}) \to \mathbb{R}$$

such that $-U$ is a monetary risk measure:

Theorem 5 (Theorem 5.2 in [14]). *Let* $U : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *be a monetary utility function with the Fatou property and let*

$$V : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$$

be its Fenchel–Legendre transform. The following are equivalent:

(i) *For all* $c \in \mathbb{R}$, $V^{-1}((-\infty, c])$ *is a weakly compact subset of* $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) *For every* $X \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, *the infimum in the equality*

$$U(X) = \inf_{Y \in \mathbb{L}^1} \{V(Y) + \mathbb{E}[XY]\}$$

is attained.

(iii) *For every uniformly bounded sequence* (X_n) *tending a.s. to* X, *we have*

$$\lim_{n \geq 1} U(X_n) = U(X).$$

The proof presented in [14] is based on the particular case of Theorem 4 which only concerns to separable Banach spaces. Later on, F. Delbaen gave a different proof, as seen in Theorem 2 in [6], which is valid for arbitrary $L^1(\Omega, \mathcal{F}, \mathbb{P})$ spaces. Delbaen's approach is based on a homogenisation trick to reduce the matter to a direct application of the classical James's Theorem, as well as the Dunford–Pettis Theorem characterizing weakly compact sets in $L^1(\Omega, \mathcal{F}, \mathbb{P})$, see Section 4.12 in [8] for a detailed description of his reduction technique.

We shall present in this section a complete proof for Theorem 1 in the Introduction together with two corollaries to analyze the case where either Ψ or Ψ^* verifies the Δ_2 condition. They represent the extension of Theorem 5 for risk measures on Orlicz spaces. They moreover provide an answer, for Orlicz spaces, to a question posed by S. Biagini and M. Fritelli in [2], an seen in Example 3 and Lemma 7.

To begin with, let us prove the following result. It motivates to use the name of Lebesgue property for the order sequential continuity in the lattice L^Ψ. Let us remember that a sequence (X_n) in L^Ψ is order convergent to $X \in \mathbb{L}^\Psi$ if there is a decreasing sequence $(Z_n) \searrow 0$, i.e., almost surely pointwise convergent to zero, such that $|X_n - X| \leq Z_n$ for all $n \in \mathbb{N}$.

Proposition 1. *A sequence* (X_n) *is order convergent to* X *in the Banach lattice* $\mathbb{L}^\Psi$ *if, and only if, there is* $Z \in \mathbb{L}^\Psi$ *such that* $|X_n| \leq Z$ *for all* $n \in \mathbb{N}$ *and* (X_n) *almost surely converges to* X.

*P*roof.- If (X_n) converges to X in order, then there is a sequence (Z_n) pointwise decreasing to zero in the lattice $\mathbb{L}^\Psi$ with $|X_n - X| \leq Z_n$ for all $n \in \mathbb{N}$. It follows that $|X_n| \leq Z_1 + |X|$ for all $n \in \mathbb{N}$ and (X_n) is almost surely convergent to X. On the other hand, if there is $Z \in \mathbb{L}^\Psi$ such that $|X_n| \leq Z$ and (X_n) is almost surely convergent to X, Egoroff's Theorem give us, for every $\epsilon > 0$, a subset $A_\epsilon \in \Sigma$ with $\mathbb{P}(A_\epsilon) < \epsilon$ and (X_n) uniformly convergent to X on $\Omega \setminus A_\epsilon$. In particular we choose $n_1 < n_2 < \cdots < n_k < \cdots$ so that $|X - X_n|\chi_{\Omega\setminus A_{2^{-k}}} \leq \frac{1}{2^k}$ for all $n \geq n_k$. Without loosing of generality we may assume $A_{2^{k+1}} \subset A_{2^k}$ for all $k \in \mathbb{N}$. Let us define

$$W_n = \frac{1}{2^k}\chi_{\Omega\setminus A_{2^{-k}}} + 2Z\chi_{A_{2^{-k}}}$$

for $n_k \leq n < n_{k+1}$. We have $|X_n - X| \leq W_n$ for all $n \in \mathbb{N}$. Since $(W_n)_{n=n_1}^{\infty}$ is almost surely decreasing to zero, we conclude that (X_n) is order convergent to X. □

The proof of Theorem 1 needs a little more on weak compactness in Banach spaces:

Proposition 2. *Let C be a weakly compact convex subset of a Banach space E and (x_n^*) a bounded sequence in the dual space E^*. Then we have*

$$\lim_{n\to\infty}[\sup_{x^*\in \mathrm{co}\{x_m^*:m\geq n\}} \inf_{y\in C}\langle x^*, y\rangle] = \inf_{y\in C} \limsup_{n\to\infty}\langle x_n^*, y\rangle$$

Proof.- The weak compactness of set C tells us, by the minimax Theorem [19, Theorem 3.2], that

$$\sup_{x^*\in \mathrm{co}\{x_m^*:m\geq n\}} \inf_{y\in C}\langle x^*, y\rangle = \inf_{y\in C} \sup_{x^*\in \mathrm{co}\{x_m^*:m\geq n\}}\langle x^*, y\rangle$$

for every $n \in \mathbb{N}$. Since

$$\limsup_{n\to\infty}\langle x_n^*, y\rangle \leq \sup_{m\geq n}\langle x_m^*, y\rangle$$

for every $n \in \mathbb{N}$, we have

$$\inf_{y\in C}\limsup_{n\to\infty}\langle x_n^*, y\rangle \leq \inf_{y\in C}\sup_{m\geq n}\langle x_m^*, y\rangle = \sup_{x^*\in \mathrm{co}\{x_m^*:m\geq n\}} \inf_{y\in C}\langle x^*, y\rangle.$$

Thus we see that

$$\inf_{y\in C}\limsup_{n\to\infty}\langle x_n^*, y\rangle \leq \lim_{n\to\infty}[\sup_{x^*\in \mathrm{co}\{x_m^*:m\geq n\}} \inf_{y\in C}\langle x^*, y\rangle].$$

On the other hand,

$$\sup_{x^*\in \mathrm{co}\{x_m^*:m\geq n\}} \inf_{y\in C}\langle x^*, y\rangle \leq \sup_{x^*\in \mathrm{co}\{x_m^*:m\geq n\}}\langle x^*, y_0\rangle = \sup_{m\geq n}\langle x_m^*, y_0\rangle$$

for every $y_0 \in C$ and $n \in \mathbb{N}$ fixed. So we have

$$\lim_{n\to\infty}[\sup_{x^*\in \mathrm{co}\{x_m^*:m\geq n\}} \inf_{y\in C}\langle x^*, y\rangle] \leq \lim_{n\to\infty}[\sup\{\langle x_m^*, y_0\rangle : m \geq n\}] = \limsup_{n\to\infty}\langle x_n^*, y_0\rangle$$

for any $y_0 \in C$, and the reverse inequality now follows:

$$\lim_{n\to\infty}[\sup_{x^*\in \mathrm{co}\{x_m^*:m\geq n\}} \inf_{y\in C}\langle x^*, y\rangle] \leq \inf_{y\in C}\limsup_{n\to\infty}\langle x_n^*, y\rangle$$

and the proof is over. □

Corollary 4. *Let D be a relatively weakly compact subset of the Banach space E and (x_n^*) a sequence in the dual E^* that converges to $x^* \in E^*$ in the w^*-topology and uniformly on D. Then we have*

$$\lim_{n\to\infty}\inf_{y\in D}\langle x_n^*, y\rangle = \inf_{y\in D}\lim_{n\to\infty}\langle x_n^*, y\rangle.$$

*P*roof.- Let C the closed convex hull of D. Krein–Smulian's Theorem says that C remains weakly compact. When the involved sequence w^*-converges the former Proposition says:

$$\inf_{y\in C}\lim_{n\to\infty}\langle x_n^*, y\rangle = \lim_{n\to\infty}[\sup_{x^*\in\mathrm{co}\{x_m^*:m\geq n\}}\inf_{y\in C}\langle x^*, y\rangle] \geq$$

$$\geq \limsup_n \inf_{y\in C}\langle x_n^*, y\rangle \geq \liminf_n \inf_{y\in C}\langle x_n^*, y\rangle \geq \liminf_n[\inf_{y\in C}\langle x_n^* - x^*, y\rangle + \inf_{y\in C}\langle x^*, y\rangle] =$$

$$= \liminf_n \inf_{y\in C}\langle x_n^* - x^*, y\rangle + \inf_{y\in C}\langle x^*, y\rangle = \inf_{y\in C}\lim_{n\to\infty}\langle x_n^*, y\rangle,$$

since (x_n^*) converges to x^* uniformly on C and

$$\liminf_n \inf_{y\in C}\langle x_n^* - x^*, y\rangle = 0,$$

from which the result follows since infima involved are the same on C or D. □

Now we arrive to a proof for Lemma 5.1 in [14] in the frame of Orlicz spaces:

Proposition 3. *Let C be a relatively weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$ where Ψ^* is finite. Let $(X_n)_{n=1}^{\infty}$ be a bounded sequence in the dual space $\mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P})$ that order converges to $X \in \mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P})$. Then we have:*

$$\lim_{n\to\infty}\inf_{Y\in C}\mathbb{E}[X_nY] = \inf_{Y\in C}\mathbb{E}[XY].$$

*P*roof.- The dual space of $\mathbb{M}^{\Psi^*}$ coincides with $\mathbb{L}^{\Psi}$ and the duality is given by

$$\langle Y, X\rangle = \mathbb{E}[YX]$$

for all $Y \in \mathbb{M}^{\Psi^*}$ and $X \in \mathbb{L}^{\Psi}$, as seen in Theorem 4.1.6, p. 105 in [16]. The relatively weakly compact subsets B of $\mathbb{M}^{\Psi^*}$ are characterized as bounded sets B such that for each $X \in L^{\Psi}(\Omega, \mathcal{F}, \mathbb{P})$, and each $A_n \in \mathcal{F}$, with $A_n \downarrow A$ and with $\mathbb{P}(A) = 0$ one has

$$\lim_{n\to\infty}\int_{A_n}|X \cdot Y|d\mathbb{P} = 0$$

uniformly in $Y \in B$ (see Corollary 4.5.2, Chapter IV in [16]). Let us fix Y in $\mathbb{M}^{\Psi^*}$, and thus we have $\lim_{n\to\infty}\mathbb{E}[X_nY] = \mathbb{E}[XY]$. Indeed, given the ordered convergent sequence (X_n) to X in $\mathbb{L}^{\Psi}$, we have

$$|X_n - X| \leq Z_n \downarrow 0$$

where $Z_n \in \mathbb{L}^{\Psi}, n = 1, 2, \cdots$. Once $Y \in \mathbb{M}^{\Psi^*}$ is fixed we will have

$$|Y \cdot X_n - Y \cdot X| \leq |Y| \cdot |Z_n| \leq |Y| \cdot |Z_1| \in \mathbb{L}^1$$

and the Monotone Convergence Theorem tells us:

$$\lim_{n\to\infty}\mathbb{E}[YX_n] = \mathbb{E}[YX]. \tag{17}$$

Moreover, the former convergence is going to be uniform on $Y \in C$ by the compactness assumption we have on C. Indeed, Egoroff's Theorem tells us that for every $\delta > 0$ there is a set $A_\delta \in \mathcal{F}$, with $\mathbb{P}(A_\delta) < \delta$ and such that (X_n) converges to X uniformly on $\Omega \setminus A_\delta$. We then have

$$|\mathbb{E}[(X_n - X)Y]| \leq \int_{\Omega\setminus A_\delta} |X_n - X||Y| d\mathbb{P} + \int_{A_\delta} |X_n - X||Y| d\mathbb{P}$$

$$\leq \int_{\Omega\setminus A_\delta} |X_n - X||Y| d\mathbb{P} + \int_{A_\delta} |Z_1||Y|.$$

Now we apply the compactness for $\epsilon > 0$ fixed, and we find $\delta > 0$ so that

$$\int_A |Z_1||Y| \leq \epsilon/2$$

whenever $\mathbb{P}(A) < \delta, Y \in C$. Since (X_n) converges to X uniformly on the set $\Omega \setminus A_\delta$, there exists $n_0 \in \mathbb{N}$ so that $\sup_{\omega \notin A_\delta} \|X_n(\omega) - X(\omega)\| \leq \epsilon/2l$ where $l = \sup_{Y \in C} \|Y\|_1$. Altogether this tells us that:

$$|\mathbb{E}[(X_n - X)Y]| \leq \epsilon$$

whenever $n \geq n_0$ and for all $Y \in C$. Therefore, we may apply Corollary 4 to get:

$$\inf_{Y\in C} \mathbb{E}[XY] = \inf_{Y\in C} \lim_{n\to\infty} \mathbb{E}[X_n Y] = \lim_{n\to\infty} \inf_{Y\in C} \mathbb{E}[X_n Y]$$

as we wanted to prove. □

Another tool we shall need is the following variant of the uniform boundness principle:

Proposition 4. *Let $\alpha : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper map such that for all $x^* \in E^*$*

$$x^* + \alpha$$

is bounded from below on E. Then there exist $\xi > 0$ and $K > 0$ such that we have the uniform boundedness from below:

$$-K \leq x^*(x) + \alpha(x), \qquad \text{for all } x \in E \text{ and all } x^* \in \xi B_{E^*}.$$

*P*roof.- The bounded from below hypothesis tell us that:

$$E^* = \bigcup_{n=1}^{\infty} \{x^* \in E^* : \epsilon x^*(x) + \alpha(x) \geq -n \text{ for all } x \in E, \epsilon \in \{-1, +1\}\}.$$

For every $n \in \mathbb{N}$ the set

$$\{x^* \in E^* : \epsilon x^*(x) + \alpha(x) \geq -n \text{ for all } x \in E, \epsilon \in \{-1, +1\}\}$$

is norm closed in E^*, so the Baire Category Theorem provides us with an integer $n_0 \in \mathbb{N}$, an element $x_0^* \in E^*$ and a number $\xi > 0$ such that

$$x_0^* + \xi B_{E^*} \subset \{x^* \in E^* : \epsilon x^*(x) + \alpha(x) \geq -n_0 \text{ for all } x \in E, \epsilon \in \{-1, +1\}\}.$$

If we take x^* with $\|x^*\| \leq \xi$, we have

$$\epsilon(x_0^* + \epsilon x^*)(x) + \alpha(x) \geq -n_0 \text{ for all } x \in E, \epsilon \in \{-1, +1\},$$

thus $x_0^*(x) + x^*(x) + \alpha(x) \geq -n_0$ and $-x_0^*(x) + x^*(x) + \alpha(x) \geq -n_0$ and we get that:

$$x^*(x) + \alpha(x) \geq -n_0, \quad \text{ for all } x \in E \text{ and all } x^* \in \xi B_{E^*},$$

as announced. □

Corollary 5. *Let E be a real Banach space and let*

$$\alpha : E \longrightarrow \mathbb{R} \cup \{\infty\}$$

be a proper map such that

$$\textit{for all } x^* \in E^*, \quad x^* - \alpha \textit{ is bounded from above on } E.$$

Then there are numbers $a, b \in \mathbb{R}$ with $b > 0$ such that

$$\alpha(x) \geq a + b\|x\|$$

for all $x \in E$.

*P*roof.- By the former result there is $\xi > 0$ (small enough) and $K > 0$ (big enough) such that we have the uniform boundedness from above:

$$\sup\{x^*(x) - \alpha(x) : x \in E, x^* \in \xi B_{X^*}\} \leq K.$$

Let us take $x \in E$ and select $x^* \in B_{E^*}$ so that $\|x\| = x^*(x)$, then we see that $\xi x^*(x) - \alpha(x) \leq K$ from which it follows $\alpha(x) \geq -K + \xi\|x\|$ as we wanted to prove. □

The last Corollary is a Banach space version for Theorem 4.5 of P. Cheridito and T. Li, [5], that we will use later.

Now we are ready for the proof of our main Theorem. We closely follow [14] for arguments non related with Theorem 4.

Proof of Theorem 1 We assume in all the proof that $\rho(0) = 0$ and thus $\alpha \geq 0$ too. Doing so we do not lose generality, since we can always add a constant to ensure it. $(ii) \Rightarrow (i)$ Let us remark we are in conditions to apply our Theorem 4 to the penalty function α restricted to the Banach space $\mathbb{M}^{\Psi^*}$. Indeed, the dual of the Banach space $\mathbb{M}^{\Psi^*}$ coincides with L^Ψ and the duality is given by

$$\langle Y, X \rangle = \mathbb{E}[YX]$$

for all $Y \in \mathbb{M}^{\Psi^*}$ and $X \in \mathbb{L}^\Psi$, as seen in Theorem 4.1.6, p. 105 in [16]. If we consider the inclusion maps

$$i : \mathbb{L}^\Psi \mapsto L^1 \text{ and } j : \mathbb{L}^\infty \mapsto \mathbb{M}^{\Psi^*}$$

we see that for every $X \in \mathbb{L}^\infty = (\mathbb{L}^1)^*$ the linear form defined by it verifies:

$$\mathbb{E}[X \ \cdot \ ] \circ i = j(X) \in M^{\Psi^*}.$$

So the inclusion map i is going to be $\sigma(\mathbb{L}^\Psi, M^{\Psi^*})$ to $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$ continuous. Thus the $\sigma(\mathbb{L}^\Psi, M^{\Psi^*})$ compact subsets of $\mathbb{L}^\Psi$ are homeomorphic to weakly compact subsets of $\mathbb{L}^1$ and the Eberlein–Smulian's Theorem, see Chapter II in [9], tells us they are sequentially compact. It now follows that the dual unit ball B_{L^Ψ} is w^* sequentially compact. Thus we can apply Theorem 4 to derive the implication.

(ii) $\Rightarrow$ *(iii)* The representation formula for ρ describes it as the supremum of affine continuous maps for $\sigma(\mathbb{L}^\Psi, \mathbb{M}^{\Psi^*})$, hence ρ is $\sigma(\mathbb{L}^\Psi, \mathbb{M}^{\Psi^*})$ lower semicontinuous. Since every sequence (X_n) in $\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$ order convergent to X is also convergent in $\sigma(\mathbb{L}^\Psi, \mathbb{M}^{\Psi^*})$ as seen in the proof of Proposition 3 above, we will have

$$\rho(X) \le \liminf_n \rho(X_n).$$

We now see that we also have:

$$\rho(X) \ge \limsup_n \rho(X_n)$$

from which the conclusion follows. Let us fix $\epsilon > 0$ and for every $n \in \mathbb{N}$ we select $Y_n \in \mathbb{M}^{\Psi^*}$ so that

$$\rho(X_n) = -\mathbb{E}[X_n Y_n] - \alpha(Y_n). \tag{18}$$

We claim that $(\alpha(Y_n))_{n=1}^\infty$ must be a bounded sequence of real numbers. Indeed, if we set $Z \le X_n \le Y$ for all $n \in \mathbb{N}$ in the lattice order of $\mathbb{L}^\Psi$, then we have

$$\rho(Y) \le \rho(X_n) \le \rho(Z). \tag{19}$$

In the case that $(\alpha(Y_n))_{n=1}^\infty$ is not a bounded sequence of real numbers, we may take subsequences if necessary, (18) and (19) force that

$$\lim_n \mathbb{E}[X_n Y_n] = -\infty$$

since $\lim_n \alpha(Y_n) = +\infty$ because α is bounded from below. Even more, since the risk measure ρ is finite we can apply Corollary 5 to see that $\alpha(Y) \ge a + b\|Y\|_{\Psi^*}$ for some $a, b \in \mathbb{R}, b > 0$. Then we will have

$$\rho(2X_n) \ge -\mathbb{E}[X_n Y_n] - \mathbb{E}[X_n Y_n] - \alpha[Y_n] = -\mathbb{E}[X_n Y_n] + \rho(X_n)$$

and $\lim \rho(2X_n) = +\infty$, which is a contradiction with $\rho(2Y) \le \rho(2X_n) \le \rho(2Z)$. Let us write:

$$-\rho(X_n) \le \inf_{\{\alpha(Y) \le c_n\}} \mathbb{E}[X_n Y] + c_n$$

where $c_n = \alpha(Y_n) \ge 0$, and fix a point $c \ge 0$ and a sequence

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$$

with $\lim_k c_{n_k} = c$ and $|c - c_{n_k}| < \epsilon$, for all $k \in \mathbb{N}$ by compactness. We will have by Proposition 3, since the level sets of α are going to be weakly compact by the former implication $(ii) \Rightarrow (i)$:

$$-\rho(X) \leq \inf_{\{\alpha(Y) \leq c + \epsilon\}} \mathbb{E}[XY] + c + \epsilon = \lim_k \inf_{\{\alpha(Y) \leq c + \epsilon\}} \mathbb{E}[X_{n_k} Y] + c + \epsilon \leq$$

$$\leq \lim_k \inf_{\{\alpha(Y) \leq c_{n_k}\}} \mathbb{E}[X_{n_k} Y] + c + \epsilon \leq \lim_k \inf_{\{\alpha(Y) \leq c_{n_k}\}} \mathbb{E}[X_{n_k} Y] + c_{n_k} + 2\epsilon \leq$$

$$\leq \lim_k -\rho(X_{n_k}) + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary we have the reverse inequality:

$$\rho(X) \geq \lim_k \rho(X_{n_k})$$

as we wanted to prove. In fact, the former reasoning is valid for any subsequence (X_{m_k}) of (X_n) and thus we arrive at

$$\rho(X) \geq \limsup_n \rho(X_n).$$

$(iii) \Rightarrow (i)$ Let us suppose that for some $c > 0$ the norm bounded set $\{\alpha \leq c\}$ fails to be relatively $\sigma(\mathbb{M}^{\Psi^*}, \mathbb{L}^{\Psi})$-compact in $\mathbb{M}^{\Psi^*}$. Indeed the level sets are bounded by Corollary 5. Following Corollary 4.5.2, p. 144 in [16], we arrive to the existence of a decreasing sequence (A_n) in $\mathcal{F}$ with $\lim_n \mathbb{P}(A_n) = 0$, an element $Z \in \mathbb{L}^{\Psi}$ and a sequence of elements $Y_n \in \{\alpha \leq c\}$ such that

$$\int_{A_n} |Y_n \cdot Z| d\mathbb{P} \geq \mu > 0$$

for every $n \in \mathbb{N}$ and some $\mu > 0$. We set $X_n = \epsilon_n \frac{2c}{\mu} \chi_{A_n} Z$ where $\epsilon_n(\omega) = -1$ if $Z(\omega)Y_n(\omega) \geq 0$ and $\epsilon_n(\omega) = +1$ if $Z(\omega)Y_n(\omega) < 0$ for every $\omega \in \Omega$. We find that (X_n) is order convergent to zero in $\mathbb{L}^{\Psi}$ but

$$\rho(X_n) \geq \mathbb{E}[-X_n \cdot Y_n] - \alpha(Y_n) \geq \mu \frac{2c}{\mu} - c = c > 0$$

while $\rho(\lim_n X_n) = \rho(0) = 0$, a contradiction with the order sequential continuity assumption we have in (iii).

$(i) \Rightarrow (ii)$ Here we use the fact that for all $X \in L^{\Psi} = (\mathbb{M}^{\Psi^*})^*$ the function

$$\langle -X, \cdot \rangle - \alpha(\cdot)$$

is bounded from above on M^{Ψ^*}, indeed the representation formula

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}[-XY] - \alpha(Y)\} < +\infty \tag{20}$$

implies it and we have by Corollary 5 the growing condition $\alpha(Y) \geq a + b\|Y\|_{\Psi^*}$ for all $Y \in \mathbb{M}^{\Psi^*}$. If the level sets of the penalty function α are weakly compact sets in M^{Ψ^*}, the optimization problem

$$\sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}[-XY] - \alpha(Y)\}$$

has a solution for all $X \in \mathbb{L}^{\Psi}$. Indeed, let us fix $X \in \mathbb{L}^{\Psi}$ and take a maximization sequence $Y_n \in \mathbb{M}^{\Psi^*}$ with

$$\rho(X) \geq \mathbb{E}[-XY_n] - \alpha(Y_n) \geq \rho(X) - 1/n. \tag{21}$$

The sequence of real numbers $(\alpha(Y_n))$ must be bounded. Otherwise

$$\lim_n \mathbb{E}[-X \cdot Y_n] = +\infty,$$

at least taking an adequate subsequence. Now it follows that

$$\rho(2X) \geq \mathbb{E}[-2XY_n] - \alpha(Y_n) =$$

$$= \mathbb{E}[-XY_n] + \mathbb{E}[-XY_n] - \alpha(Y_n) \geq \mathbb{E}[-XY_n] + \rho(X) - 1.$$

So $\rho(2X) \geq \lim_n \mathbb{E}[-XY_n] + \rho(X) - 1 = +\infty$, a contradiction with the fact our risk measure ρ is assumed to be finite. Then we have $c \in \mathbb{R}$ such that $\alpha(Y_n) \leq c$ for all $n \in \mathbb{N}$. Since the level set $\{\alpha \leq c\}$ is assumed to be weakly compact in $\mathbb{M}^{\Psi^*}$, a $\sigma(\mathbb{M}^{\Psi^*}, \mathbb{L}^{\Psi})$ convergent subsequence (Y_{n_k}) to some $Y_0 \in \mathbb{M}^{\Psi^*}$ can be selected. Thus

$$\lim_{k \to \infty} \mathbb{E}[-XY_{n_k}] = \mathbb{E}[-XY_0].$$

From (20), (21) and the $\sigma(\mathbb{M}^{\Psi^*}, \mathbb{L}^{\Psi})$ -lower semicontinuity assumption for the penalty function α we arrive at

$$\rho(X) = \mathbb{E}[-X \cdot Y_0] - \alpha(Y_0)$$

as we wanted to prove.

□

Let us finally mention the fact that the lower semicontinuity of the risk measure ρ with respect to $\sigma(L^{\Psi}, L^{\Psi^*})$ is equivalent to order lower semicontinuity by the results of Biagini and Fritelli (see Proposition 1 and p. 18 in [2]). This property is called the Fatou property and for a decreasing sequence (X_n) almost surely convergent to X it implies that $\lim_n \rho(X_n) = \rho(X)$, see Section 3.1 in [2]. Under the Fatou property, the sequential order continuity is equivalent to the monotone convergence fact

$$\lim_n \rho(X_n) = \rho(X) \text{ whenever } X_n \nearrow X \text{ in } \mathbb{L}^{\Psi}.$$

Let us summarize the consequences below.

Corollary 6. *Let Ψ be a Young function with conjugate Ψ^* that verifies the Δ_2 condition. Let $\rho : \mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a finite convex risk measure with the Fatou property and*

$$\rho^* : (\mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P})^* \to \mathbb{R} \cup \{+\infty\}$$

its Fenchel–Legendre conjugate defined on the dual space. The following are equivalent:

(i) *For all* $c \in \mathbb{R}$, $(\rho^*)^{-1}((-\infty, c])$ *is a weakly compact subset of* $\mathbb{L}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) *For every* $X \in \mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P})$, *the supremum in the equality*

$$\rho(X) = \sup_{Y \in \mathbb{L}^{\Psi^*}} \{\mathbb{E}[-XY] - \rho^*(Y)\}$$

is attained.

(iii) ρ *is sequentially order continuous.*

(iv) $\lim_n \rho(X_n) = \rho(X)$ *whenever* $X_n \nearrow X$ *in* $\mathbb{L}^{\Psi}$.

(v) $Dom(\rho^*) \subset \mathbb{L}^{\Psi^*}$.

*P*roof.- The Δ_2-condition for the conjugate function Ψ^* implies that it is finite and moreover, the Orlicz space $\mathbb{L}^{\Psi^*}$ coincides with the Morse subspace $\mathbb{M}^{\Psi^*}$; see Corollary 5, p. 77 in [16]. The equivalence with (iv) follows from the previous discussion, let us take (X_n) a sequence almost surely convergent to X with $|X_n| \le Z \in \mathbb{L}^{\Psi}$ and denote with

$$G_n := \sup\{X_m : m \ge n\} \ge X_n \ge H_n := \inf\{X_m : m \ge n\}.$$

Then we have $G_n \searrow X$, $H_n \nearrow X$ and the Fatou property together with (iv) implies that $\lim_{n\to\infty} \rho(X_n) = \rho(X)$. The implication with (iii) now follows from Proposition 1. Let us show the equivalence with (v). We denote with β the restriction of ρ^* on the subspace $\mathbb{L}^{\Psi^*}$. Then we have $\rho = \beta^*$ and $\rho^* = \beta^{**}$, Proposition 7.31 in [10] tell us that

$$\text{Epi}(\rho^*) = \overline{\text{Epi}(\beta)}^{\sigma((\mathbb{L}^{\Psi})^* \times \mathbb{R}, \mathbb{L}^{\Psi} \times \mathbb{R})}$$

and (v) is equivalent to have $\text{Epi}(\rho^*) = \text{Epi}(\beta)$. Thus (v) implies that β has w^*-closed epigraph in the bidual and therefore the level sets in (i) are going to be w^*-closed in the bidual. Thus weakly compact sets since they are bounded, indeed $\rho = \rho^{**}_{\restriction E}$ finite implies boundness for $(\rho^*)^{-1}(-\infty, c]$ and every $c \in \mathbb{R}$ by Corollary 5, that is (i) is fulfilled. From Corollary 3 it follows that $(ii) \Rightarrow (v)$. The proof is over.

□

When Ψ is a Young function that verifies the Δ_2 condition we have, with the same proof, a result for the risk measures studied by P. Cheredito and T. Li, see [5]:

Corollary 7. *Let* Ψ *a Young function that verifies the* Δ_2 *condition and finite* Ψ^*. *Let* $\rho : \mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *be a finite convex risk measure with the Fatou property and*

$$\rho^* : \mathbb{L}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \cup \{+\infty\}$$

its Fenchel–Legendre conjugate defined on the dual space. The following are equivalent:

(i) *For all* $c \in \mathbb{R}$, $(\rho^*)^{-1}((-\infty, c])$ *is a weakly compact subset of* $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) *For every* $X \in \mathbb{L}^{\Psi}(\Omega, \mathcal{F}, \mathbb{P})$*, the supremum in the equality*

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{\mathbb{E}[-XY] - \rho^*(Y)\}$$

is attained.

(iii) ρ *is sequentially order continuous.*

(iv) $\lim_n \rho(X_n) = \rho(X)$ *whenever* $X_n \nearrow X$ *in* $\mathbb{L}^{\Psi}$.

(v) $Dom(\rho^*) \subset \mathbb{M}^{\Psi^*}$.

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