POINTWISE COMPACTNESS IN SPACES OF CONTINUOUS FUNCTIONS

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SUMMARY

In this paper we describe a class of topological spaces X such that $C_p(X)$, the space of continuous functions on X endowed with the topology of pointwise convergence, is an angelic space. This class contains the topological spaces with a dense and countably determined subspace; in particular the topological spaces which are K-analytic in the sense of G. Choquet. Our results include previous ones of A. Grothendieck, J. L. Kelley and I. Namioka, J. D. Pryce, R. Haydon, M. De Wilde, K. Floret and M. Talagrand. As a consequence we obtain an improvement of the Eberlein–Šmulian theorem in the theory of locally convex spaces. This result allows us to deduce, for instance, that (LF)-spaces and dual metric spaces, in particular (DF)-spaces of Grothendieck, are weakly angelic. In this way the answer to a question posed by K. Floret about the weak angelic character of (LF)-spaces is given.

I. *Introduction and preliminary results*

All the topological spaces we shall use here are assumed to be Hausdorff. Standard references for notation and concepts are **[8, 13]**.

A subset M of a topological space X is said to be countably compact (briefly NK), or relatively countably compact (briefly RNK), if every sequence in M has an adherent point in M, or in X, respectively; M is said to be sequentially compact (briefly SK), or relatively sequentially compact (briefly RSK), if every sequence in M has a subsequence convergent to an element of M, or of X, respectively. We shall use the abbreviations K for compact and RK for relatively compact. Naturally we have the following relationships:

$$\begin{array}{ccccc} \mathrm{K} & \Longrightarrow & \mathrm{NK} & \Longrightarrow & \mathrm{SK} \\ \swarrow\!\!\!\swarrow & & \Downarrow & & \searrow\!\!\!\searrow \\ \mathrm{RK} & \Longrightarrow & \mathrm{RNK} & \Longrightarrow & \mathrm{RSK} \end{array}$$

and these are the only implications that generally hold **[6, 8]**.

V. L. Šmulian showed in 1940 **[16]** that RK $\Rightarrow$ RSK in the weak topology of a Banach space. He also proved that RNK $\Leftrightarrow$ RSK if the weak-$*$ dual is separable. The last result was extended by J. Dieudonné and L. Schwartz **[4]** to locally convex spaces with a coarser metrizable topology. But the converse to Šmulian's theorem was to wait until W. F. Eberlein **[5]** who in 1947 proved for the weak topology of a Banach space that RK $\Leftrightarrow$ RNK. Soon after Eberlein's proof, A. Grothendieck **[10]** in 1952 provided a considerable generalization by showing that weakly-RNK $\Leftrightarrow$ weakly-RK in any locally convex space that is quasicomplete for its Mackey topology. Grothendieck's result is based upon a similar one on the space $C_p(K)$ of continuous functions on a compact space K endowed with the pointwise convergence topology. As J. L. Kelley and I. Namioka have pointed out **[12]**, these results can be obtained using a refinement

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of a theorem of Kaplansky's to see that a cluster point of an RNK subset A of $C_p(K)$ is the limit of a sequence in A. Whitley's proof [20] of the Eberlein–Šmulian theorem uses this idea and J. D. Pryce [14] has extended it to spaces $C_p(X)$ where X is a topological space with a dense and σ-compact subset. Fremlin's notion of angelic space [14] and some of its consequences provide us with the necessary tools for proving those results in a clear-cut way (see K. Floret [8]).

A topological space is called *angelic* (or has a countably determined compactness) if for every RNK subset A of X the following holds:

(a) A is RK;
(b) for each x belonging to $\overline{A}$ there is a sequence in A which converges to x.

In angelic spaces K = NK = SK and RK = RNK = RSK.

The following lemma will constitute the basic idea of the joint investigation of all the compactness notions as well as for the question whether the sequential closure of an RNK set is its closure.

ANGELIC LEMMA (K. Floret [8, p. 27]). *Let X and Y be topological spaces, let X be regular, and let $\phi: X \to Y$ be continuous and injective. If $A \subset X$ is* RNK *and for all $B \subset \phi(A)$ the sequential closure of B is closed, that is, $\overline{B} = \{y \in Y$: there is (y_n) in B with $\lim_n y_n = y\}$, then $\phi(\overline{A})$ is closed in Y and the restriction of ϕ on $\overline{A}$ is a homeomorphism.*

As a consequence of this result, if Y is angelic, then X is angelic too. In particular any topology on Y which is regular and finer than the original one is angelic; and subspaces of regular angelic topological spaces are also angelic.

On the other hand, the study of compact subsets of $C_p(X)$, where X is a K-analytic space, or more generally a countably determined space, called Talagrand and Gul'ko compact spaces, respectively, has become of great interest in the descriptive theory of Banach spaces and they have been studied by M. Talagrand [17], S. P. Gul'ko [10] and L. Vasák [20] among others. Nevertheless, as far as the author knows, given a K-analytic space X and an RNK subset A of $C_p(X)$, it is only known that if A is countable then the closure of A in $\mathbb{R}^{\mathbb{R}}$ is an angelic compact subset of $C_p(X)$, in fact much deeper results about countable sets of K-analytic functions on X have been proved by Bourgain, Fremlin and Talagrand [1]. Furthermore, if A is any compact subset of $C_p(X)$ then every closure point of a subset of A is the limit of a sequence of points in the subset [17]. In fact, $C_p(X)$ is an angelic space as we are going to prove here. In any case, more can be done and we shall introduce a class of topological spaces X, called web-compact spaces, that contains many of the topological spaces described in the descriptive theory of sets, and such that $C_p(X)$ will be angelic. We shall improve the Eberlein–Šmulian theorem in such a way that (LF)-spaces and dual metric spaces, in particular (DF)-spaces of Grothendieck, are checked as weakly angelic spaces. That contains the answer to a problem posed by K. Floret about the weak angelic character of (LF)-spaces [7, question 7.6].

II. *Accessibility by sequences*

We present in this section a result that ensures the accessibility by sequences of closure points of certain subsets in a space of functions. It has its origin in a result of De Wilde [3, Lemma 4.6] that is based upon an argument of Kelley and Namioka

[12, Theorem 8.20] related to the classical result of Kaplansky [8, p. 37]. However, on dealing with a 'web' in the base space instead of a countable cover, we shall obtain a result that will be applicable to spaces that arise in the descriptive theory of sets. This result will be the basis of all the applications that follow. Its proof is strongly based upon Floret's proof of the theorem of De Wilde [8, p. 33].

Grothendieck introduced the following notion, which is an extremely useful tool for dealing with compactness. Let Z be a topological space, let X be a set and $A \subset Z^X$. If $(f_n) \subset A$ and $(x_m) \subset X$, it is said that (f_n) has the interchangeable double limit property (in Z) with (x_m) if

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m)$$

whenever all the limits involved exist.

In the following theorem X will be a non-void set, Σ will be a non-void subset of $\mathbb{N}^{\mathbb{N}}$, the space of sequences of positive integers, and S will be the subset of the set of finite sequences defined by

$$S = \{(a_1, a_2, \ldots, a_n) \colon \exists \alpha = (a_m) \in \Sigma, n \in \mathbb{N}\}.$$

Let us suppose that there is a family $\{A_\alpha \colon \alpha \in \Sigma\}$ of non-void subsets of X that covers X. Given $\alpha = (a_m)$ in Σ and n in $\mathbb{N}$, we put

$$C_{a_1, a_2, \ldots, a_n} = \bigcup \{A_\beta \colon \beta \in \Sigma, \beta = (b_m), b_j = a_j, j = 1, 2, \ldots, n\}.$$

The family of subsets $\{C_{a_1, a_2, \ldots, a_n} \colon (a_1, a_2, \ldots, a_n) \in S\}$ constructed in this way is clearly countable.

THEOREM 1. *Let (Z, d) be a compact metric space and let A be a set of functions from X into Z. We suppose that for every $\alpha = (a_m) \in \Sigma$ and every sequence (x_n) in X, that is, eventually in every set $C_{a_1, a_2, \ldots, a_n}$ for $n = 1, 2, \ldots$, we have for every sequence in A the interchangeable double limit property with (x_n) in Z. Then for every f in the closure of A in the product space Z^X there is a sequence (f_n) in A such that (f_n) converges pointwise to f on X.*

Proof. *Step* I. Given functions $g_1, g_2, \ldots, g_n$ in $Z^X, \varepsilon > 0$, and a subset C of X, there is a finite subset L of C such that

$$\min_{y \in L} \max_{k \leqslant n} \{d(g_k(x), g_k(y))\} \leqslant \varepsilon \quad \text{for every } x \in C.$$

Indeed, it is enough to consider the mapping G from C into Z^n defined by

$$G(x) = (g_1(x), g_2(x), \ldots, g_n(x))$$

and to use the compactness of Z^n.

Step II. The interchangeable double limit property together with step I will now enable us to find a sequence of functions (f_n) in A that converges pointwise to f on X. The idea is to construct a countable subset L in X together with a sequence of functions (f_m) in A satisfying $\lim_m f_m(y) = f(y)$ for every y belonging to L; we shall do it in such a way that for every x in X we shall have enough points of the countable set L 'close to x' such that we shall obtain the convergence of the sequence (f_m) at the point x by iteration of limits.

Since S is countable there is a bijection $\psi \colon \mathbb{N} \to S$. For every positive integer n

let D_n be equal to $C_{\psi(n)}$ and let f_1 be equal to f. By step I there is a finite subset $L_1^1 \subset D_1$ such that

$$\min_{y \in L_1^1} \{d(f_1(x), f_1(y))\} \leqslant 1 \quad \text{for every } x \in D_1.$$

But f is in the closure of A, so there is $f_2 \in A$ such that

$$\max_{y \in L_1^1} \{d(f_2(y), f(y))\} \leqslant \tfrac{1}{2}.$$

Proceeding by recurrence, for every positive integer n, we find finite subsets $L_n^i \subset D_i$ for $i \leqslant n$ and functions f_i, $i = 1, 2, \ldots, n+1$ such that

$$\min_{y \in L_n^i} \max_{k \leqslant n} \{d(f_k(x), f_k(y))\} \leqslant 1/n \quad \text{for every } x \in D_i$$

and

$$\max \{d(f_{n+1}(y), f(y)) : y \in \bigcup \{L_j^i : i \leqslant j \leqslant n\}\} \leqslant 1/(n+1).$$

Step III. The sequence $\{f_n : n = 1, 2, \ldots\}$ selected above clearly satisfies

$$\lim_n f_n(y) = f(y) \quad \text{for every } y \in \bigcup \{L_j^i : i \leqslant j = 1, 2 \ldots\}.$$

We are now going to see that $\lim_n f_n(x) = f(x)$ whatever x in X we take. Indeed, let us take $x \in X$ and $\alpha \in \Sigma$, $\alpha = (a_m)$, such that $x \in A_\alpha$. We set

$$P = \psi^{-1}(\{(a_1, a_2, \ldots, a_n) : n = 1, 2, \ldots\}),$$

which is an infinite subset of positive integers because ψ is a bijection. The point x clearly belongs to every D_p for $p \in P$ and so, given $p \in P$ and $n \geqslant p$, by step II there is $y_{n,p} \in L_n^p$ such that $\max_{k \leqslant n} \{d(f_k(x), f_k(y_{n,p}))\} \leqslant 1/n$.

We put $y_p = y_{p,p}$ and we have

$$\max_{k \leqslant p} \{d(f_k(x), f_k(y_p))\} \leqslant 1/p. \tag{1}$$

Since $y_p = y_{p,p} \in L_p^p \subset D_p$ for every $p \in P$, the sequence $\{y_p : p \in P\}$ is eventually in every $C_{a_1, a_2, \ldots, a_n}$ for $n = 1, 2, \ldots$. Indeed, let m be a positive integer and p_j be equal to $\psi^{-1}((a_1, a_2, \ldots, a_j))$ for $j = 1, 2, \ldots, m$. If $p \in P$ and $p \neq p_j$, $j = 1, 2, \ldots, m$, we have $y_p \in D_p = C_{a_1, a_2, \ldots, a_{m_0}}$ with $m_0 > m$ and the conclusion follows when we bear in mind that $C_{a_1, a_2, \ldots, a_{m_0}} \subset C_{a_1, a_2, \ldots, a_m}$.

We shall now use the permutability of limits to conclude that $\lim_n f_n(x) = f(x)$. It will be enough to prove that $f(x)$ is the only adherent point of the sequence $(f_n(x))$ in Z, since Z is compact. Let y be an adherent point of $(f_n(x))$ and (n_k) be a strictly increasing sequence of positive integers such that $\lim_k f_{n_k}(x) = y$. Since P is infinite, it is a cofinal subset in $\mathbb{N}$ and there is a strictly increasing sequence (p_j) in P. Property (1) ensures that $\lim_j f_k(y_{p_j}) = f_k(x)$ for $k = 1, 2, \ldots$. Thus we have

$$y = \lim_k f_{n_k}(x) = \lim_k \lim_j f_{n_k}(y_{p_j}) = \lim_j \lim_k f_{n_k}(y_{p_j})$$
$$= \lim_j f(y_{p_j}) = \lim_j f_1(y_{p_j}) = f_1(x) = f(x).$$

III. *Web-compact spaces*

Let X be a topological space and let Z be a compact metric space. Let A be an RNK subset of $C_p(X, Z)$, the space of all continuous functions from X into Z, endowed with the topology T_p of pointwise convergence on X. A subset A is RK in $C_p(X, Z)$ if and only if the closure of A in the topological product space Z^X is contained in $C(X, Z)$. If we can find sequences in A converging to the closure points in T_p, then because of the relative countable compactness of A in $C_p(X, Z)$, we shall have the closure points as continuous functions. So in order to prove the angelic character of $C_p(X, Z)$ it seems to be natural to ask for conditions on the topological space X that will force A to verify the hypothesis of Theorem 1. An easy way to do that is to demand that the sequences involved in it have an adherent point in X. This idea motivates us to introduce the web-compact spaces.

DEFINITION. A topological space will be called a *web-compact space* if there is a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha: \alpha \in \Sigma\}$ of subsets of X such that, if we denote by

$$C_{a_1, a_2, \ldots, a_n} = \bigcup \{A_\beta: \beta \in \Sigma, \beta = (b_m), b_j = a_j, j = 1, 2, \ldots, n\}$$

for every $\alpha = (a_m)$ in Σ and n in $\mathbb{N}$, the following two conditions are satisfied:

(i) $\overline{\bigcup \{A_\alpha: \alpha \in \Sigma\}} = X$;

(ii) if $\alpha = (a_n) \in \Sigma$ and $x_n \in C_{a_1, a_2, \ldots, a_n}, n = 1, 2, \ldots$, then the sequence (x_n) has an adherent point in X.

This notion, that at a first glance seems to be very technical, contains some of the topological spaces described through the descriptive theory of sets, at least those that usually appear in functional analysis. First we describe these spaces in terms of their RNK subsets. In what follows $\mathbb{N}^{\mathbb{N}}$ will be considered to be endowed with its usual topology, making it a Polish space.

PROPOSITION 2. *For a topological space X the following statements are equivalent:*

(i) *X is web-compact;*

(ii) *there is a metrizable and separable space P together with a mapping T from P into $\mathcal{P}(X)$, the set of all the subsets of X such that*

(a) $\overline{\bigcup \{Tx: x \in P\}} = X$,

(b) *if (x_n) converges in P, then $\bigcup \{Tx_n: n = 1, 2, \ldots\}$ is* RNK *in X.*

Proof. (i) $\Rightarrow$ (ii) We can take $P = \Sigma$ with the topology induced by the Polish space $\mathbb{N}^{\mathbb{N}}$, and the mapping T defined by $T\alpha = A_\alpha$. Indeed, conditions (a) and (b) are easily followed by conditions (i) and (ii) within the definition of web-compact space.

(ii) $\Rightarrow$ (i) Every Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$ and so there is a continuous mapping φ from a subset Σ of $\mathbb{N}^{\mathbb{N}}$ onto P. Given $\alpha \in \Sigma$ we put $A_\alpha = T\varphi(\alpha)$ and the family $\{A_\alpha: \alpha \in \Sigma\}$ obtained in this way gives us the representation of X as a web-compact space.

EXAMPLES. (A) Let X be a topological space and $\{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ a family of RNK subsets of X such that

(a) $\overline{\bigcup \{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}} = X$;

(b) for $\alpha = (a_n)$ and $\beta = (b_n)$ with $a_n \leqslant b_n$, $n = 1, 2, \ldots$ ($\alpha \leqslant \beta$ for short), we would have $A_\alpha \subset A_\beta$.

Then X is a web-compact space.

Indeed, let (α_n) be a sequence in $\mathbb{N}^{\mathbb{N}}$ that converges to $\alpha \in \mathbb{N}^{\mathbb{N}}$. It easily follows that there is $\beta \in \mathbb{N}$ such that $\beta \geqslant \alpha_n, n = 1, 2, \ldots$, and so $\bigcup \{A_{\alpha_n} : n = 1, 2, \ldots\} \subset A_\beta$ and Proposition 2(ii)(b) holds.

In particular, every topological space with a sequence $\{K_n : n = 1, 2, \ldots\}$ of RNK subsets such that $\bigcup \{K_n : n = 1, 2, \ldots\} = X$ is a web-compact space.

(B) Any countably determined topological space X [**17**] is a web-compact space.

For such a space X there is a mapping T from a subset Σ of $\mathbb{N}^{\mathbb{N}}$ into the family of compact subsets of X such that

(a) $\bigcup \{Tx : x \in \Sigma\} = X$;

(b) given a neighbourhood V of Tx there is a neighbourhood U of x in Σ such that $T(U) \subset V$.

It is easily seen that the continuity condition (b) implies condition (b) of Proposition 2(ii) and so X is a web-compact space.

In particular, every K-analytic space, in the sense of Choquet, is a web-compact space [**17**]; and every K-Suslin space, in the sense of Martineau (see [**18**, p. 59]), is also a web-compact space. As C. A. Rogers has shown [**15**] K-analytic and K-Suslin spaces coincide in the category of completely regular spaces.

(C) Any quasi-Suslin space, in the sense of Valdivia [**18**, p. 52], is a web-compact space.

A quasi-Suslin space is a topological space X for which there is a mapping T from a Polish space P into $\mathscr{P}(X)$ such that

(a) $\bigcup \{Tx : x \in P\} = X$;

(b) if (x_n) converges to x in P and $z_n \in Tx_n$, $n = 1, 2, \ldots$, then the sequence (z_n) has an adherent point in Tx.

It is quite obvious that T gives us the web-compact structure of X after Proposition 2.

A countably compact subset of a topological space is compact if and only if it is Lindelöf. Spaces which are K-Suslin and countably determined are Lindelöf [**18**, **20**]. Therefore a countably compact and non-compact space is a quasi-Suslin and not K-Suslin or countably determined space. In any case, for a regular topological space X it is easy to realize that X is K-Suslin if and only if it is quasi-Suslin and Lindelöf. M. Valdivia has shown in [**18**, p. 67] an example of a Fréchet space E such that $E''[\sigma(E'', E')]$ is a quasi-Suslin and not a K-Suslin space. Of course such an example is neither countably determined nor a Lindelöf space.

On the other hand there are separable spaces and therefore web-compact spaces that are not quasi-Suslin. For instance the topological product $\mathbb{R}^{\mathbb{R}}$ is separable but it is not quasi-Suslin, otherwise it would be a K-Suslin and Baire space, thus metrizable through a result of De Wilde and Sunyach (see [**18**, p. 64]).

Note 1. In the special case where a web-compact space allows a representation with $\Sigma = \mathbb{N}^{\mathbb{N}}$, then it also allows a representation satisfying property (b) of example (A) above. For if $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a representation of web-compact space on X, we set $B_\alpha = \bigcup \{A_\beta : \beta \in \mathbb{N}^{\mathbb{N}}, \beta \leqslant \alpha\}$ for every α in $\mathbb{N}^{\mathbb{N}}$ and Proposition 2 obviously implies that B_α is RNK.

IV. *Angelic spaces of continuous functions*

The following main result contains, as a particular case, the theorems of A. Grothendieck [9] for X a compact space; J. D. Pryce [14] for X having a dense σ-compact subspace; K. Floret [8] for X having a dense σ-RNK subset; and M. Talagrand [17, Theorem 6.4] who shows that for a Gul'ko compact space K, the points of the closure of a subset of K are limit points of sequences in this subset. It must be pointed out here that Talagrand's proof makes extensive use of the Lindelöf character of countably determined spaces.

THEOREM 3. *Let X be a web-compact space. The space $C_p(X)$ is angelic.*

Proof. Let $\overline{\mathbb{R}}$ be the compactification of $\mathbb{R}$ with the two points $+\infty$ and $-\infty$. The inclusion mapping from $C_p(X, \mathbb{R})$ into $C_p(X, \overline{\mathbb{R}})$ is continuous and injective. The angelic lemma ensures that it is sufficient to prove the theorem for the space $C_p(X, Z)$, where Z is a compact metric space. Let $\{A_\alpha : \alpha \in \Sigma\}$ the family of subsets of X giving it a web-compact structure. Let Y be equal to the union of the sets of the family $\{A_\alpha : \alpha \in \Sigma\}$. We consider the mapping

$$\phi : C_p(X, Z) \longrightarrow Z^Y[T_p]$$

defined by restriction on Y, $\phi(f) := f|_Y$, that is continuous and injective because of the density of Y in X. Let A be an RNK subset of $C_p(X, Z)$. Every sequence in A has the interchangeable double limit property with every sequence in X having adherent point in X [8, p. 11]. Therefore $\phi(A)$ is a set of functions that satisfies the conditions of Theorem 1 in Z^Y. Thus for any B contained in $\phi(A)$ we have that

$$\overline{B} = \{f \in Z^Y : \text{ there is a sequence } (f_n) \text{ in } B \text{ with } \lim_n f_n = f \text{ in } T_p\}.$$

A resort to the angelic lemma informs us that $\phi(\overline{A})$ is closed in Z^Y and so compact, and that the restriction of ϕ on $\overline{A}$ is a homeomorphism. After all the closure of A in $C_p(X, Z)$ is compact and every point in this closure is accessible by sequences in A.

COROLLARY 1.3. *If X is a web-compact space and Z is a metric space, the space $C_p(X, Z)$ of continuous functions from X into Z endowed with the pointwise convergence topology is an angelic space.*

Proof. This follows from Theorem 3 and Fremlin's theorem [8, p. 32].

Let X be an arbitrary topological space and let $\{X_i : i \in I\}$ be the family of all the subspaces of X which are web-compact. We shall say that X is a $\mathscr{W}$-space if any function from X into $\mathbb{R}$ is continuous if and only if its restriction on every X_i is continuous. For instance, a regular space with a K-analytic neighbourhood of every point is a $\mathscr{W}$-space.

Following ideas of J. D. Pryce [14] and as a straightforward consequence of Theorem 3 we have the following.

THEOREM 4. *Let X be a $\mathscr{W}$-space. Then every* RNK *subset of $C_p(X)$ is* RK.

Proof. Let A be an RNK subset of $C_p(X)$. Then the closure B of A in $\mathbb{R}^X$ is compact and Theorem 3, together with the fact of being X a $\mathscr{W}$-space, ensures that B is contained in $C(X)$, and so A is RK in $C_p(X)$.

Note 2. Theorem 4 can be looked at as an extension to spaces of continuous functions of the theorem of Grothendieck that gives the equivalence of weakly RNK and weakly RK subsets of a locally convex space quasi-complete for its Mackey topology.

Note 3. R. Haydon extends in [11] the results of Pryce about the angelic character of $C_p(X)$ to spaces X which are completely regular and have a dense σ-bounding subset. De Wilde also obtains results of this kind. In [2, 3] he shows that a subset A of a completely regular space X is bounding if and only if it has the interchangeable double limit property in $\overline{\mathbb{R}}$ with all the RNK subsets of $C_p(X)$. As a consequence, the Haydon theorem can be derived from our Theorem 1 in the same way as we have proved Theorem 3. Furthermore, we can say that *a completely regular space X is web-bounding* when there is a dense family $\{A_\alpha\colon \alpha\in\Sigma\}$ of subsets of X, where $\Sigma\subset\mathbb{N}^{\mathbb{N}}$, and for every $\alpha=(a_m)\in\Sigma$, and every sequence (x_n) with

$$x_n\in C_{a_1,a_2,\ldots,a_n}=\bigcup\{A_\beta\colon \beta=(b_n)\in\Sigma, b_j=a_j, j=1,2,\ldots,n\},\qquad n=1,2,\ldots,$$

we have $\{x_n\colon n=1,2,\ldots\}$ as a bounding subset of X. We have a similar characterization to that of Proposition 2 and of course *for every web-bounding space X the space $C_p(X)$ is angelic too.* Also a result similar to Theorem 4 holds.

V. *Applications to weak compactness in locally convex spaces*

Let $E[\mathscr{L}]$ be a locally convex space on the field $\mathbb{K}$ of real or complex numbers, and let E' be its topological dual. Clearly $E[\sigma(E,E')]$ is a subspace of $C_p(E'[\sigma(E',E)],\mathbb{K})$. Theorem 3 provides us with the following extension of the Eberlein and Šmulian theorem [8, p. 38].

THEOREM 5. *Let E be a locally convex space such that $E'[\sigma(E',E)]$ is web-compact. Then $E[\sigma(E,E')]$ is an angelic space.*

We remark on the following important particular case.

COROLLARY 1.5. *Let E be a locally convex space with a family $\{K_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}$ of $\sigma(E',E)$-*RNK *subsets of E' such that*

(i) $\overline{\bigcup\{K_\alpha\colon \alpha\in\mathbb{N}^{\mathbb{N}}\}}^{\sigma(E',E)}=E'$;

(ii) *for α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha\leqslant\beta$ we have $K_\alpha\subset K_\beta$.*

Then $E[\sigma(E,E')]$ is angelic.

Proof. Example (A) in Section III shows that $E'[\sigma(E', E)]$ is web-compact.

Now we shall obtain important classes of locally convex spaces for which, as far as the author knows, the angelic character of the weak topology was not previously known.

(A) *Let* $E[\mathscr{L}] = \varinjlim E_n[\mathscr{L}_n]$ *be an inductive limit of a sequence of metrizable spaces* $\{E_n[\mathscr{L}_n]: n = 1, 2, \ldots\}$. *Then* $E[\sigma(E, E')]$ *is an angelic space.*

Proof. Let $U_1^n \supset U_2^n \supset \ldots \supset U_j^n \supset \ldots$ be a fundamental system of neighbourhoods of the origin in $E_n[\mathscr{L}_n]$. Given $\alpha = (a_n)$ in $\mathbb{N}^{\mathbb{N}}$ we write

$$K_\alpha = \bigcap \{(U_{a_n}^n)^\circ : n = 1, 2, \ldots\},$$

where we denote by $^\circ$ the polars in the duality $\langle E, E' \rangle$. Every K_α is an equicontinuous subset of E' and therefore a compact subset of $E'[\sigma(E', E)]$. Certainly

$$\bigcup \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = E'$$

and for α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha \leqslant \beta$ we have that $K_\alpha \subset K_\beta$.

Note 4. This example answers a question posed by Floret [**9**, question 7.6] in which it was asked what (*LF*)-spaces are weakly angelic.

(B) *Let* $E[\mathscr{L}]$ *be a dual metric space, that means a locally convex space with a fundamental sequence* $\{B_n : n = 1, 2, \ldots\}$ *of bounded subsets and such that every sequence* (x_n) *in* E' *which is strongly bounded is also equicontinuous. Then* $E[\sigma(E, E')]$ *is an angelic space.*

Proof. Given $\alpha = (a_n)$ in $\mathbb{N}^{\mathbb{N}}$ we write

$$K_\alpha = \bigcap \{a_n B_n^\circ : n = 1, 2, \ldots\}.$$

Obviously $E' = \bigcup \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and every K_α is strongly bounded and consequently $\sigma(E', E)$-RNK. Of course $K_\alpha \subset K_\beta$ for α and β in $\mathbb{N}^{\mathbb{N}}$ with $\alpha \leqslant \beta$.

In particular, every (*DF*)-space of Grothendieck is weakly angelic.

Another interesting consequence of Theorem 5 is the following.

THEOREM 6. *Let E be a locally convex space that is* $\sigma(E, E')$*-web-compact. Then* $E'[\sigma(E', E)]$ *is angelic.*

Finally, let us remark now that a locally convex space with a $\mathscr{C}$-web [**13**, §35], in the sense of De Wilde, has a bidual $E''[\sigma(E'', E')]$ which is web-compact. Indeed every bounded subset of E is $\sigma(E'', E')$-RK and E is $\sigma(E'', E')$-dense in E''. Therefore we have the following.

THEOREM 7. *Let E be a* $\mathscr{C}$*-webbed space. Then* $E'[\sigma(E', E'')]$ *is angelic.*

Note added in proof. After the preparation of this paper we have shown the following (with B. Cascales). If X is a web-compact space where the RNK subsets are RK, then X contains a dense and countably determined subspace. As a consequence, for every web-compact space X, the compact subsets of $C_p(X)$ are Gul'ko compact spaces. These results will appear elsewhere.

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