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LUR renormings through Deville's master lemma

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Abstract. A completely geometrical approach for the construction of locally uniformly rotund norms and the associated networks on a normed space X is presented. New proof with a quantitative estimate for a central theorem by M. Raja, A. Moltó and the authors is given with the only external use of Deville-Godefory-Zizler decomposition method.

Renormamientos LUR a través del lema maestro de Deville

Resumen. Presentamos una aproximación completamente geométrica para la construcción de normas localmente uniformemente convexas y sus network asociadas sobre un espacio normado X. Una nueva demostración, con estimaciones cuantitativas, para un resultado central de M. Raja, A. Moltó y los autores se da con el único uso externo del método de descomposición de Deville-Godefroy-Zizler.

1. Introduction

Let $(X, \|\cdot\|)$ be a normed space. The norm $\|\cdot\|$ in X is said to be locally uniformly rotund (**LUR** for short) if

$$[\lim_n (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0] \Rightarrow \lim_n \|x - x_n\| = 0$$

for any sequence (x_n) and x in X. The construction of this kind of norms in separable Banach spaces lead Kadec to the proof of the existence of homeomorphisms between all separable Banach spaces, [1]. For a non separable Banach space is not always possible to have such an equivalent norm, for instance the space l^∞ does not have it, see for instance p.74 in [2]. When such a norm exists its construction is usually based on a good system of coordinates that we must have on the normed space X from the very beginning, for instance a biorthogonal system

$$\{(x_i, f_i) \in X \times X^* : i \in I\}$$

with some additional properties such as being a strong Markusevich basis, [18]. Sometimes there is not such a system and the norm is constructed modelling enough convex functions on the given space X to add all of them up with the powerful lemma of Deville, see lemma VII 1.1 in [2]. Deville's lemma has been extensively used by R. Haydon in his seminal papers [5], [6], as well as in [7]. It is based on the construction of an equivalent **LUR** norm on a weakly compactly generated Banach space by the second named author in [17], where the convex functions are measuring distances to suitable finite dimensional subspaces as well as evaluations on some coordinate functionals in the dual space X^*; see [18], theorem 7.3. We have been able to show the connection between biorthogonal systems and LUR renormings in [15]. Using Deville's lemma we have proved the following:

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Theorem 1 ([15]) *Let X be a Banach space and $F \subset X^*$ a norming subspace for X. X has an equivalent $\sigma(X,F)$-lower semicontinuous and locally uniformly rotund norm if, and only if, there are countably many families of convex and $\sigma(X,F)$-lower semicontinuous functions $\{\varphi_i^n : X \to \mathbb{R}^+; i \in I_n\}_{n=1}^{\infty}$ such that there are open subsets*

$$G_i^n \subset \{\varphi_i^n > 0\} \cap \{\varphi_j^n = 0 : j \neq i, j \in I_n\}$$

with $\{G_i^n : i \in I_n, n \in \mathbb{N}\}$ a basis for the norm topology of X

Our method is mainly based on Stone's theorem about paracompactness of metric spaces, [14] . The σ-discrete basis for the norm topology of a normed space X can be refined to obtain the basis described in theorem 1. More recent contributions show an interplay between both methods too,[6, 10, 11]. It is our intention here to give a straightforward proof of the main renorming construction in [12, 16]. This result is in the core of the theory, and we shall prove it with a geometrical approach based on Deville's lemma only, without any use of paracompactness at all. Indeed, the theorem we are going to prove reads as follows:

Theorem 2 ([16, 12]) *Let X be a normed space and F a norming subspace in the dual X^*. X admits a $\sigma(X,F)$-lower semicontinuous and equivalent locally uniformly rotund norm if, and only if, there is a sequence (A_n) of subsets of X such that for every $x \in X$ and every $\epsilon > 0$ there is a $\sigma(X,F)$-open half space H and a positive integer p with $x \in A_p \cap H$ and diam$(A_p \cap H) \leq \epsilon$.*

The known proofs of this result show a difficult task when they arrive to a convexification process of the sets A_n needed to construct a countable family of seminorms, and they involve Stone's theorem if additional information on the structure of the sets A_n is required, see [14, 16, 15]. We are going to present here a different approach where either Stone's theorem or the convexification process are not needed any more. We shall do it by developing our main result here with the use of Deville's master lemma only, indeed we are going to prove the following localization result showing that for any family of slices on a bounded set A of a normed space X, we can always construct an equivalent norm such that the **LUR** condition for a sequence (x_n), and a fixed point x in A, implies that the sequence eventually belongs to slices containing the point x too. When the involved slices have small diameter, then the sequence is eventually close to x. If the diameter can be made small enough, then the sequence (x_n) converges to x and the norm will be locally uniformly rotund at the point x. The precise statement reads as follows:

Theorem 3 (Slice localization theorem) *Let X be a normed space with a norming subspace F in X^*. Let A be a bounded subset in X and $\mathcal{H}$ a family of $\sigma(X,F)$-open half spaces such that for every $H \in \mathcal{H}$ the set $A \cap H$ is non empty. Then there is an equivalent $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{H},A}$ such that for every sequence $(x_n)_{n\in\mathbb{N}}$ in X and $x \in A \cap H$ for some $H \in \mathcal{H}$, the condition*

$$\lim_n (2\|x_n\|_{\mathcal{H},A}^2 + 2\|x\|_{\mathcal{H},A}^2 - \|x + x_n\|_{\mathcal{H},A}^2) = 0$$

implies that there is a sequence of open half spaces $\{H_n \in \mathcal{H} : n = 1, 2, ...\}$ such that

1. *There is $n_0 \in \mathbb{N}$ such that $x, x_n \in H_n$ for $n \geq n_0$ if $x_n \in A$.*
2. *For every $\delta > 0$ there is some n_δ such that*

$$x, x_n \in \overline{(co(A \cap H_n) + B(0,\delta))}^{\sigma(X,F)}$$

for all $n \geq n_\delta$

We use a standard Geometry of Banach spaces and topology notation which can be found in [8, 4] and [3, 9]. In particular, B_X (resp. S_X) is the unit ball (resp. the unit sphere) of a normed space X. If F is a subset of X^*, then $\sigma(X,F)$ denotes the topology of pointwise convergence on F. Given $x^* \in X^*$ and $x \in X$, we write $\langle x^*, x\rangle$ and $x^*(x)$ to indistinctively denote the evaluation of x^* at x. If D is a subset of a normed space X we denote by $co(D)$ the convex hull of D. If $x \in X$ and $\delta > 0$ we denote by $B(x,\delta)$ the norm open ball centered at x of radius δ. A subspace $F \subset X^*$ is said to be a norming subspace for X when

$$\|x\|_F := sup\{\langle x, f\rangle : f \in B_{X^*} \cap F\}$$

define an equivalent norm on X. When the original norm coincides with $\|\cdot\|_F$ the subspace F is called 1-norming.

2. The tool

A main result here is theorem 3 above. It is a refinement of the Conection Lemma we develop in [15]. The main difference in the present context is that we do not have any rigidity condition here for the family of slices. In [15] we have slices describing a discrete family of sets, here we have an arbitray family of slices without any additional assumption at all. We need the following definition:

Definition 1 ([15]) *Let X be a normed space and F a norming subspace in the dual space X^*. For a bounded and convex subset C of X^{**} we define*

$$F - dist(x, C) := inf\,\{sup\,\{| < x - c^{**}, f > | : f \in B_{X^*} \cap F\} : c^{**} \in C\}$$

It has been proved in [15] that the $F{-}dist(\cdot, C)$ is a convex, $\sigma(X,F)$-lower semicontinuous and 1-Lipschitz map from X to $\mathbb{R}^+$.. We are going to make extensive use of this kind of functions in our construction of the **LUR** norm.

Proof of theorem 3.-

We shall consider $\sigma(X,F)$-lower semicontinuous and convex functions (φ_H) and (ψ_H) for every $H \in \mathcal{H}$ defined as follows:

$$\varphi_H(x) := F - dist(x, \overline{H^c \cap co(A)}^{\sigma(X^{**},X^*)})$$

for every $x \in X$, where we are denoting by H^c the closed half space equal to the complementary of the open half space H. Let us choose a point $a_H \in H \cap A$ and set $D_H = co(H \cap A)$ for every $H \in \mathcal{H}$, and $D_H^\delta := D_H + B(0,\delta)$, where $B(0,\delta) := \{x \in X : \|x\| < \delta\}$ for every $\delta > 0$ and $H \in \mathcal{H}$. We are going to denote with p_H^δ the Minkowski functional of the convex body $\overline{D_H^\delta}^{\sigma(X,F)} - a_H$. Then we define the $\sigma(X,F)$-lower semicontinuous norm p_H by the formula

$$p_H(x)^2 = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} (p_H^{1/n}(x))^2$$

for every $x \in X$. Finally we define the nonnegative, convex, and $\sigma(X,F)$-lower semicontinuous function ψ_H as $\psi_H(x)^2 := p_H(x - a_H)^2$ for every $x \in X$. We are now in position to apply R. Deville's master lemma; see [2], lemma VII.1.1, p. 279, to get an equivalent norm $\|\cdot\|_{\mathcal{H},A}$ on X such that the condition

$$\lim_n \left(2\,\|x_n\|^2_{\mathcal{H},A} + 2\,\|x\|^2_{\mathcal{H},A} - \|x_n + x\|^2_{\mathcal{H},A}\right) = 0$$

for a sequence $\{x_n : n \in \mathbb{N}\}$ and x in X implies that there exists a sequence of indexes (H_n) in $\mathcal{H}$ such that:

1. $lim_n \varphi_{H_n}(x) = lim_n \varphi_{H_n}(x_n) = lim_n \varphi_{H_n}((x + x_n)/2) = sup\,\{\varphi_H(x) : H \in \mathcal{H}\}$ and
2. $\lim_n [1/2\psi^2_{H_n}(x_n) + 1/2\psi^2_{H_n}(x) - \psi^2_{H_n}((x_n + x)/2)] = 0$

If the given point x belongs to one of the open half spaces $H_0 \in \mathcal{H}_\epsilon$ then we have that $\varphi_{H_0}(x) > 0$ and so we have that:

$$sup\,\{\varphi_H(x) : H \in \mathcal{H}_\epsilon\} \geq \varphi_{H_0}(x) > 0,$$

so the condition 1 provide us with an integer n_0 such that

$$\varphi_{H_n}(x) > 0, \varphi_{H_n}(x_n) > 0, \varphi_{H_n}((x + x_n)/2) > 0$$

whenever $n \geq n_0$, from where our conclusion 1 in the theorem follows. Moreover, condition 2 above and the convex arguments imply now that for every positive integer q we have that

$$\lim_n [1/2(p^{1/q}_{H_n}(x_n - a_{H_n}))^2 + 1/2(p^{1/q}_{H_n}(x - a_{H_n}))^2 - (p^{1/q}_{H_n}((x_n + x)/2 - a_{H_n}))^2] = 0,$$

and consequently

$$\lim_n [p^{1/q}_{H_n}(x_n - a_{H_n}) - p^{1/q}_{H_n}(x - a_{H_n})] = 0, \forall q \in \mathbb{N}$$

.

If we fix a positive number δ, an open half space $H \in \mathcal{H}$ and any $y \in A \cap H$ we have that

$$y - a_H + (y - a_H)\delta\|y - a_H\|^{-1} \in B(0, \delta) + (y - a_H) \subset D^\delta_H - a_H,$$

thus

$$[(1 + \delta)\|y - a_H\|^{-1}](y - a_H) \in (D^\delta_H - a_H)$$

and therefore

$$p^\delta_H(y - a_H) < [(1 + \delta\|y - a_H\|^{-1}]^{-1}$$

because $D^\delta_H - a_H$ is a norm open set.

Let us choose now the integer q such that $1/q < \delta$, and take an integer $n \geq n_0$, we know that $x \in A \cap H_n$ since $\varphi_{H_n}(x) > 0$ and the given point x belongs to A, therefore

$$p^{1/q}_{H_n}(x - a_{H_n}) < [(1 + (1/q)\|x - a_{H_n}\|^{-1}]^{-1},$$

and we can find a number $0 < \xi < 1$ such that

$$p^{1/q}_{H_n}(x - a_{H_n}) < 1 - \xi,$$

for all $n \geq n_0$ by the boundness of A. If we now take the integer n big enough to have

$$p^{1/q}_{H_n}(x_n - a_{H_n}) < 1 - \xi$$

too, we arrive to the fact that $x_n - a_{H_n} \in D^\delta_{H_n} - a_{H_n}$, and indeed $x_n \in \overline{(co(A \cap H_n) + B(0, \delta))}^{\sigma(X,F)}$, so the proof is over. ■

Thus, given any family of slices on a given set A of a normed space, we have seen how it is always possible to construct equivalent norms such that the **LUR** condition on a given sequence (x_n) and a fixed point x implies that the sequence eventually belongs to halfspaces of the given family containing the point x too.

3. LUR renormings

We can prove now a quantitative version for main results in [12] and [16]. It corresponds with the renorming implication of theorem 2 where the hypothesis provide the conclusion for every $\epsilon > 0$ and every $x \in X$. Let us remember that a subspace $F \subset X^*$ is said to be 1-norming for X when $\overline{B_{X^*} \cap F}^{\sigma(X^*,X)} = B_{X^*}$.

Theorem 4 *Let X be a normed space and $F \subset X^*$ be 1-norming subspace for X. Given ϵ we assume that there are subsets A_n such that for every $x \in \cup_{n=1}^{\infty} A_n$ we can find $p \in \mathbb{N}$ and a $\sigma(X,F)$-open half space H such that $x \in A_p \cap H$ and diam-$A_p \cap H \leq \epsilon$. Then X admits an equivalent $\sigma(X,F)$-lower semicontinuous norm $|||\cdot|||$ such that the condition*

$$\lim_n (2|||x_n|||^2 + 2|||x|||^2 - |||x + x_n|||^2) = 0$$

implies that for every $\delta > 0$ there is some integer n_δ such that for all $n \geq n_\delta$ we have $|||x_n - x||| < \epsilon + \delta$ whenever $x \in \cup_{n=1}^{\infty} A_n$

Proof.- Let us consider $\mathcal{H}_n$ the family of all $\sigma(X,F)$-open half spaces such that $A_n \cap H \neq \emptyset$ and diam-$(A_n \cap H) \leq \epsilon$. If there is not such slice for some set A_n we do not consider it at all. If we apply the former theorem for the family $\mathcal{H}_n$ and the set A_n we get the equivalent norm $\|\cdot\|_n$ that verifies conditions 1 and 2 of theorem 3 for any sequence (x_m) and x such that

$$\lim_m (2\|x_m\|_n^2 + 2\|x\|_n^2 - \|x + x_m\|_n^2) = 0.$$

Let us take the c_n such that $\|\cdot\|_n \leq c_n \|\cdot\|$. If we set

$$|||x|||^2 := \sum_{n=1}^{\infty} \frac{1}{c_n 2^n} \|x\|_n^2$$

for every $x \in X$, we obtain the renorming we are looking for. Indeed, if

$$\lim_n (2|||x_n|||^2 + 2|||x|||^2 - |||x + x_n|||^2) = 0$$

by the convex arguments we know that

$$\lim_n (2|||x_n|||_p^2 + 2|||x|||_p^2 - |||x + x_n|||_p^2) = 0$$

for every positive integer p. If $x \in A_q$ and there is a $\sigma(X,F)$-open half space H such that $x \in A_q \cap H$ and diam-$A_q \cap H \leq \epsilon$, we have that $H \in \mathcal{H}_q$ and diam-$co(A_p \cap H) \leq \epsilon$ too. Moreover, the condition 2 of theorem 3 tell us that there is a sequence of half spaces $H_n \in \mathcal{H}_q$ such that for every $\delta > 0$ there is some n_δ with

$$x, x_n \in \overline{(co(A_q \cap H_n) + B(0,\delta))}^{\sigma(X,F)}$$

for all $n \geq n_\delta$. Since F is 1-norming the original norm is $\sigma(X,F)$-lower semicontinuous and we have $\|x - x_n\| \leq \epsilon + \delta$ for every $n \geq n_\delta$ as we wanted to prove. ∎

Corollary 1 *Let X be a Banach space and $F \subset X^*$ a norming subspace for X. Let us assume that $Z \subset X$ is a subspace of X with a sequence of subsets $(A_n) \subset Z$ such that for every $\epsilon > 0$ and $z \in Z$ there is some $p \in \mathbb{N}$ together with a $\sigma(X,F)$-open half space H such that $z \in H \cap A_p$ and diam-$A_p \cap H \leq \epsilon$. Then the whole space X admits an equivalent $\sigma(X,F)$-lower semicontinuous norm $|||\cdot|||$ such all points in the subspace Z are* **LUR** *points for the new norm in the whole of X, i.e. for every point $z \in Z$ and every sequence (x_n) in X such that*

$$\lim_n (2|||x_n|||^2 + 2|||z|||^2 - |||z + x_n|||^2) = 0$$

we will have that $\lim_n x_n = z$ in norm.

Proof.- Without lose of generality we can and do assume that the original norm is $\|\cdot\|_F$; i.e F i a 1-norming subspace. If we perform the construction of theorem 4 for a fixed $\epsilon > 0$ we obtain the equivalent $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_\epsilon$. Let us take the d_n such that $\|\cdot\|_{1/n} \leq d_n \|\cdot\|_{1/n}$. If we set

$$|||x|||^2 := \sum_{n=1}^{\infty} \frac{1}{d_n 2^n} \|x\|_{1/n}^2$$

for every $x \in X$, we obtain the renorming we are looking for. Indeed, as above, if

$$\lim_n (2|||x_n|||^2 + 2|||z|||^2 - |||z + x_n|||^2) = 0,$$

by the convex arguments we know that

$$\lim_n (2|||x_n|||_{1/p}^2 + 2|||z|||_{1/p}^2 - |||z + x_n|||_{1/p}^2) = 0$$

for every positive integer p, and the theorem says that $|||x_n - z||| < 2/p$ for $n \geq n_{1/p}$ whenever $z \in Z$. ■

Remark 1 *The corollary provide us with a geometrical proof of the renorming implication in theorem 2 based on the Deville-Godefroy-Zizler decomposition method only.*

Corollary 2 *Let X and Y normed spaces with 1-norming subspaces $F \subset X^*$, $G \subset Y^*$ and*

$$T : (X, \sigma(X,F)) \longrightarrow (Y, \sigma(Y,G))$$

a continuous linear map. Given $\epsilon > 0$ we assume there are subsets $A_n \subset Y$ such that for every $y \in \cup_{n=1}^{\infty} A_n$ we can find $p \in \mathbb{N}$ and a $\sigma(Y,G)$-open half space L such that $y \in A_p \cap L$ and diam-$(A_p \cap L) \leq \epsilon$. Then X admits an equivalent $\sigma(X,F)$-lower semicontinuous norm $|||\cdot|||_T$ such that the condition

$$\lim_n (2|||x_n|||_T^2 + 2|||x|||_T^2 - |||x + x_n|||_T^2) = 0$$

implies that for every $\delta > 0$ there is some integer n_δ such that for all $n \geq n_\delta$ we have

$$|||T(x_n) - T(x)||| < \epsilon + \delta \|T\|$$

whenever $T(x) \in \cup_{n=1}^{\infty} A_n$. In particular when Y admits an equivalent $\sigma(Y,G)$-lower semicontinuous and **LUR** *norm we will have that the condition*

$$\lim_n (2|||x_n|||_T^2 + 2|||x|||_T^2 - |||x + x_n|||_T^2) = 0$$

implies that $\lim_n T(x_n) = T(x)$ in the norm of Y.

Proof.- Let us fix the integer n and apply theorem 3 to the set $T^{-1}(A_n)$ together the family $\mathcal{H}_n$ of $\sigma(X,F)$-open half spaces given by $T^{-1}(L)$ for every L, $\sigma(Y,G)$-open half space, such that $A_n \cap L \neq \emptyset$ and diam-$(A_n \cap L) \leq \epsilon$. We will get an equivalent norm $\|\cdot\|_n$ on X such that, the condition

$$\lim_m (2|||x_m|||_n^2 + 2|||x|||_n^2 - |||x + x_m|||_n^2) = 0$$

implies that $x_m$and x are in sets

$$\overline{co(T^{-1}(A_n) \cap T^{-1}(L_m)) + B(0,\delta)}^{\sigma(Y,G)}$$

for $m \geq m_\delta$, where diam-$(A_n \cap L_m) \leq \epsilon$, and therefore $\|T(x_m) - T(x)\| \leq \epsilon + \delta\|T\|$ for all $m \geq m_\delta$ whenever $T(x) \in A_n \cap L$ with some $\sigma(Y,G)$-open half space L and diam-$(A_n \cap L) \leq \epsilon$. Adding all this norms we get the equivalent norm $\|\cdot\|_T$ we are looking for. Indeed, let us take the h_n such that $\|\cdot\|_n \leq h_n\|\cdot\|$; if we set

$$|||x|||^2 := \sum_{n=1}^{\infty} \frac{1}{h_n 2^n} \|x\|_n^2$$

for every $x \in X$, we obtain the renorming we are looking for, the proof follows the same arguments as above. When Y admits the **LUR** norm we have the former conditions for all $\epsilon > 0$ by theorem 2, and the conclusion then follows ■

4. The network construction

Our approach for **LUR** renormings is also based on the topological concept of network. A family of subsets $\mathcal{N}$ in a topological space $(T, \mathcal{T})$ is a network for the topology $\mathcal{T}$ if for every open set $W \in \mathcal{T}$, and every $x \in W$, there is some $N \in \mathcal{N}$ such that $x \in N \subset W$.

Let us recall precise definitions and results:

Definition 2 *Let X be a normed space and F be a norming subspace in the dual X^*. A family $\mathcal{B} := \{B_i : i \in I\}$ of subsets in X is called $\sigma(X,F)$-slicely isolated (or $\sigma(X,F)$-slicely relatively discrete) if it is a disjoint family of sets such that for every*

$$x \in \bigcup \{B_i : i \in I\}$$

there exist a $\sigma(X,F)$-open half space H and $i_0 \in I$ such that

$$H \bigcap \bigcup \{B_i : i \in I, i \neq i_0\} = \emptyset \text{ and } x \in B_{i_0} \cap H.$$

A main result, with the approach of [14], is the following one; it is equivalent to theorem 2 if we have in mind Stone's theorem on the paracompactness of a metric space, see [14], chapter III.

Theorem 5 ([14], chapter III, theorem 3.1, pag 49) *Let X be a normed space and F a norming subspace in the dual X^*. The space X admits an equivalent $\sigma(X,F)$-lower semicontinuous and locally uniformly rotund norm if, and only if, the norm topology has a network $\mathcal{N}$ that can be written as $\mathcal{N} = \cup_{n=1}^{\infty}\mathcal{N}_n$ where each one of the families $\mathcal{N}_n$ is $\sigma(X,F)$- slicely isolated.*

In the monograph [14] the network point of view for locally uniformly rotund renormings is the central one. The approach to construct them make extensive use of Stone's theorem on the paracompactness of metric spaces. We shall construct in this section the network that characterize the property of being locally uniformly rotund renormable, but our approach will be completely geometrical as the one presented in [13] for the weak topology, see lemma 3.19 in [14] too. We have presented next result in [15] but using Stone's theorem in the construction.

Theorem 6 *Let X be a normed space with a $\sigma(X,F)$-lower semicontinuous and locally uniformly rotund norm for some subspace $F \subset X^*$. Then the norm topology admits a network $\mathcal{N}$ such that $\mathcal{N} = \cup_{n=1}^{\infty}\mathcal{N}_n$ where the families $\mathcal{N}_n$ are norm discrete, $\sigma(X,F)$-slicely isolated, and formed with sets which are the diference of convex and $\sigma(X,F)$-closed subsets of X for every $n \in \mathbb{N}$.*

Proof.- In a **LUR** norm all points in the unit sphere are denting points, then for $\epsilon > 0$ fixed we will have a family of $\sigma(X,F)$-open half spaces $\mathcal{H}_\epsilon$, covering the unit sphere S_X of our $\sigma(X,F)$-lower semicontinuous and LUR norm, and such that $\|\cdot\| -$ diam $(H \cap B_X) < \epsilon$ for all $H \in \mathcal{H}_\epsilon$. Let us choose a well order relation for the elements in $\mathcal{H}_\epsilon$ and let us write

$$\mathcal{H}_\epsilon = \{H_\gamma : \gamma < \Gamma\}$$

where we denote by $H_\gamma = \{x \in X : f_\gamma(x) > \lambda_\gamma\}, f_\gamma \in B_{X^*} \cap F$.

We set

$$M_\gamma := H_\gamma \cap B_X \setminus \left(\bigcup \{H_\beta \cap B_X : \beta < \gamma\right\}$$

for every $\gamma < \Gamma$, and let us define the sets $M^n_\gamma := \{x \in M_\gamma : f_\gamma(x) \geq \lambda_\gamma + 1/n\}$. It follows that when $x \in M^n_\gamma$ and $y \in M^n_\beta$ for $\gamma \neq \beta$ then we have either

$$f_\gamma(x) - f_\gamma(y) \geq 1/n \quad (\text{ when } \gamma < \beta) \tag{1}$$

or

$$f_\beta(y) - f_\beta(x) \geq 1/n \quad (\text{ when } \beta < \gamma), \tag{2}$$

but in any case

$$\|x - y\| \geq 1/n \tag{3}$$

since the linear functionals f_γ, f_β are assumed to be in $B_{X^*} \cap F$. If we fix $x \in S_X$ the **LUR** condition of the norm gives a slice

$$G = \{y \in B_X : g(y) > \mu\}$$

with $g(x) > \mu$, $g \in B_{X^*} \cap F$ and $\|\cdot\| -$ diam $(G) < 1/n$, thus G meets at most one member of the family of sets $\{M^n_\gamma : \gamma < \Gamma\}$ by (3).

These families of closed and convex subsets of X cover the unit sphere S_X and they suffice to describe the network there. Nevertheless, to go over the whole space X we need to make the difference of closets convex sets. Indeed, take $x \in X \setminus \{0\}$ and $y := x/\|x\|$. If we take $\gamma_0 < \Gamma$ so that $y \in M_{\gamma_0}$ and n big enough to have $f_{\gamma_0}(y) > \lambda_{\gamma_0} + 1/n$, we will have a rational number $0 < \mu_x < 1$, but close enough to one, such that $f_{\gamma_0}(\mu_x y) > \lambda_{\gamma_0} + 1/n$. The **LUR** condition of the norm tell us that there is $\delta_x > 0$ such that $\|(y + z)/2\| > 1 - \delta_x$ implies that $\|y - z\| < 1/n$ whenever the condition $\|z\| \leq 1$ holds .

Let us take a rational number ρ such that

$$\rho > \|x\| > \rho(1 - \delta_x) \text{ and } \rho\mu_x < \|x\|.$$

Then $x \in \rho M^n_{\gamma_0}$ and $\|\cdot\| -$ diam $(\rho M^n_{\gamma_0}) < \rho\epsilon$. Moreover, if we choose $g_x \in B_{X^*} \cap F$ such that $g_x(x) > \rho(1 - \delta_x)$, then for any $z \in \cup\{\rho M^n_\gamma : \gamma < \Gamma\}$, with $g_x(z) > \rho(1 - \delta_x)$, we will have

$$g_x(z/\rho) > 1 - \delta_x \text{ and } g_x(y) > \rho(1 - \delta_x)/\|x\| > 1 - \delta_x,$$

thus $\|\frac{y+z/\rho}{2}\| > 1 - \delta_x$ and we will have$\|y - z/\rho\| < 1/n$, from where it follows that $\gamma = \gamma_0$. Therefore, if we consider the sets $M^{n,p}_\gamma := \{x \in M^n_\gamma \cap S_X : \delta_x > 1/p\}$ and we take the family

$$\{\rho M^{n,p}_\gamma \setminus \rho(1 - 1/p)B_X : \gamma < \Gamma\}$$

for rational numbers ρ and integers p, n fixed, we form an slicely isolated family of sets. All together, with the same construction done for every $\epsilon > 0$ give us a family

$$\bigcup\{\{\rho M^{n,p}_\gamma(\epsilon) \setminus \rho(1 - 1/p)B_X : \gamma < \Gamma\} : \rho \in \mathbb{Q}, n, p \in \mathbb{N}, \epsilon > 0\}$$

which is a network for the norm topology. Taking $\epsilon = 1/r, r = 1, 2, ...$ we get the network for the norm we are looking for. ■

Remark 2 *Let us observe that we have completed a geometrical proof of theorem 2. Indeed, theorem 6 provides us the $\sigma(X, F)$-slicely isolated network $\mathcal{N} = \cup_{n=1}^\infty \mathcal{N}_n$ for the norm topology. Setting $A_q := \cup\{N : N \in \mathcal{N}_q\}$ for $q \in \mathbb{N}$ and given $x \in X$ and $\epsilon > 0$, if we take p and $M \in \mathcal{N}_p$ with $x \in M \subset B(x, \epsilon/2)$, then by the slicely isolatedness property of the family $\mathcal{N}_p$ there is a $\sigma(X, F)$-open half space H with $x \in H \cap A_p \subset M$. Thus $\|\cdot\| - diam(H \cap A_p) \leq \|\cdot\| - diam(M) \leq \epsilon$. The reverse implication follows from corollary 1.*

The network provides us criteria to see when the closure of a **LUR** renormable space could be **LUR** renormabel too. For instance, we can prove the following:

Theorem 7 *Let X be a Banach space with a norming subspace $F \subset X^*$ and a dense subspace E of X. There is a network $\mathcal{N} = \cup \mathcal{N}_n$ of the norm topology on E where every one of the families $\mathcal{N}_n$ is going to be $\sigma(E,F)$-slicely isolated and such that the family of sets:*

$$\mathcal{B} := \{N + \epsilon B_X : N \in \mathcal{N}, \epsilon > 0\}$$

is a basis of the norm topology of X if, and only if, the whole X admits an equivalent $\sigma(X,F)$-lower semicontinuous and LUR equivalent norm.

In the proof we are going to use the following result we have obtained in [15]

Proposition 1 *Let X be a normed space with a norming subspace $F \subset X^*$ and $\|\cdot\|_F$ the equivalent norm associated with it. Given a $\sigma(X,F)$-slicely isolated family $\mathcal{A} := \{A_i : i \in I\}$ there exist decompositions with increasing sequences of subsets $(A_i^n)_n$, $A_i = \cup_{n=1}^{\infty} A_i^n$ for every $i \in I$, such that the families*

$$\{A_i^n + B_{|||\cdot|||}(0, 1/4n) : i \in I\}$$

are $\sigma(X,F)$-slicely isolated and norm discrete for every $n \in \mathbb{N}$.

Proof of theorem 7.- If the normed space X admits an equivalent $\sigma(X,F)$-lower semicontinuous and **LUR** norm, we have proved in [15] that it has a basis of the norm topology $\mathcal{B} = \cup \mathcal{B}_n$ such that every one of the families of open sets $\mathcal{B}_n$ is $\sigma(X,F)$-slicely isolated and norm discrete. It now follows that $\mathcal{N}_n := \mathcal{B}_n \cap F$ are families of non void subsets in E since E is dense in X, and they are $\sigma(E,F)$-slicely isolated and norm discrete with $\mathcal{N} := \cup_{n=1}^{\infty} \mathcal{N}_n$ a basis of the norm topology of E. It is clear that the family of sets:

$$\mathcal{B} := \{N + \epsilon B_X : N \in \mathcal{N}, \epsilon > 0\}$$

is a basis of the norm topology of X. Indeed, since every set $B \in \mathcal{B}$ is open and E is dense we have $B \subset \overline{B \cap E}$, this fact together with the regularity of the norm topology complete the proof for this implication.

Let us prove the converse result now. Let us fix the σ-slicely isolated (for $\sigma(E,F)$) network $\mathcal{N}$ of the norm topology in E such that that the family of sets:

$$\mathcal{B} := \{N + \epsilon B_X : N \in \mathcal{N}, \epsilon > 0\}$$

is a basis of the norm topology of X. Let us write $\mathcal{N} = \cup \mathcal{N}_n$ where every one of the families $\mathcal{N}_n$ is a $\sigma(E,F)$-slicely isolated family of sets in E, thus $\sigma(X,F)$-slicely isolated in X too. We apply the proposition 1 and we can write:

$$\mathcal{N}_n := \{N_j^n : j \in I_n\}$$

$N_i^n = \cup_{m=1}^{\infty} N_i^n(m)$ where $N_i^n(1) \subset N_i^n(2) \subset ... \subset N_i^n(m) \subset ...$ and the families

$$\{N_i^n(m) + (1/4m)B_X) : i \in I_n\}$$

for every fixed integer m are $\sigma(X,F)$-slicely isolated and norm discrete. Moreover, the families

$$\bigcup_{n,m \in \mathbb{N}} \{\overline{N_i^n(m)} : i \in I_n\}$$

form a $\sigma(X,F)$-slicely isolated network of the norm topology on the whole space X as we are going to see now. Let us take $x \in X$ and $\epsilon > 0$, then there is some pair of positive integers p, q such that $x \in \overline{N_i^p}(q) \subset B(x,\epsilon)$ for some $i \in I_p$. Indeed, if not we will have some point $x_{p,q} \in \overline{N_i^p(q)} \cap (X \setminus B(x,\epsilon))$ whenever

$$x \in N_i^p(q) + \delta B_X \subset B(x,\epsilon)$$

for some $p, q \in \mathbb{N}$, some $i \in I_p$ and some $\delta > 0$. Let us begin with the first integers p_1 such that

$$x \in N_i + \delta_1 B_X \subset B(x, \epsilon),$$

for some $i \in I_{p_1}$ and some $\delta_1 > 0$. Thus we can select the first integer q_1 such that

$$x \in N_i^{p_1}(q_1) + \delta_1 B_X \subset B(x, \epsilon)$$

and take $x_1 \in \overline{N_i^{p_1}(q_1)} \cap (X \setminus B(x, \epsilon))$ by our assumption. Taking $0 < \delta_2$ small enough we will have $B(x, \delta_2) \subset B(x, \epsilon)$ too. Let us take again first integers p_2 such that

$$x \in N_i + \delta_3 B_X \subset B(x, \delta_2),$$

for some $i \in I_{p_2}$ and some $\delta_3 > 0$ together with the first integer q_2 such that

$$x \in N_i^{p_2}(q_2) + \delta_3 B_X \subset B(x, \delta_2),$$

then we can take again a point $x_2 \in \overline{N_i^{p_1}(q_1)} \cap (X \setminus B(x, \epsilon))$ together with $0 < \delta_4 < \delta_2/2$. If we continue in that way by induction we obtain a sequence (x_n) in the closed set $X \setminus B(x, \epsilon)$ with a decreasing sequence $(\delta_{2n}) \downarrow 0$ such that $x_n \in B(x, \delta_{2n})$, a contradiction and the proof is over. ■

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