BOUNDARIES OF ASPLUND SPACES

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ABSTRACT. We study the relationship between the classical combinatorial inequalities of Simons and the more recent (I)-property of Fonf and Lindenstrauss. We obtain a characterization of strong boundaries for Asplund spaces using the new concept of finitely self predictable set. Strong properties for w^*-K-analytic boundaries are established as well as a $\sup - \limsup$ theorem for Baire maps.

1. INTRODUCTION

All Banach spaces are real in this paper. Let $K \subset X^*$ be a w^*-compact convex subset of a dual Banach space X^*. A subset $B \subset K$ is called a boundary of K, if for any $x \in X$ there is $f \in B$ with $\max x(K) = f(x)$. If $K = B_{X^*}$ then a boundary B of K is called a boundary of X. ¿From the Krein-Milman theorem follows that the set $\mathrm{ext}K$ of all extreme points of K is a boundary. Easy examples show that a boundary may be a proper subset of $\mathrm{ext}K$. It is also possible (for a non-separable X) that a boundary is disjoint with $\mathrm{ext}K$. Hahn-Banach separation theorem shows that

$$\overline{\mathrm{co}\, B}^{w^*} = K. \tag{W}$$

One of the main problems in studying boundaries is to find conditions under which a boundary B has property

$$\overline{\mathrm{co}\, B}^{\|\cdot\|} = K. \tag{S}$$

A boundary with (S) is also called strong. For instance, if a boundary B is separable then it has (S) (see [30, 14, 13]). In a non-separable case this is not true: think of $\mathrm{ext}B_{C([0,1])^*}$. Although not all boundaries are strong, it has been proved in [13] that any boundary has the following property (I):

For any increasing sequence $\{B_n\}_{n=1}^{\infty}$ of subsets of B such that $B = \cup_n B_n$ we have

$$K = \overline{\bigcup_{n \in \mathbb{N}} \overline{\mathrm{co}\, B_n}^{w^*}}^{\|\cdot\|}. \tag{I}$$

Property (I) is weaker than (S). However in some cases (I) implies (S). For instance (I) implies (S) for separable boundaries and for any boundary when X is separable without copies of ℓ^1, see [13]. Therefore the validity of (I) for any boundary yields results by Rodé [30] and by Godefroy [14].

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A classical and important tool for the investigation of boundaries is the following Simons' inequality:

For any boundary B of the w^*-compact set K in X^* and every bounded sequence (z_n) in X the following inequality holds:

$$\sup_{b^*\in B}\{\limsup_{n\to\infty} z_n(b^*)\} \geq \inf\big\{\sup_K w : w\in \mathrm{co}\{z_n : n\in\mathbb{N}\}\big\}. \tag{SI}$$

Simons's inequality and (I)-property look different and certainly their proofs are different. Nonetheless, Kalenda has implicitly proved, using some additional Simons' result, that the (I)-property is equivalent to the following $\sup-\limsup$ Theorem by Simons, see [31, 32], see lemma 2.1 and remark 2.2 in [22]:

For any boundary B of the w^*-compact set K in X^* and every bounded sequence (z_n) in X the following equality holds:

$$\sup_{b^*\in B}\{\limsup_{n\to\infty} z_n(b^*)\} = \sup_{x^*\in K}\{\limsup_{n\to\infty} z_n(x^*)\}. \tag{SLS}$$

In Section 2 we give a proof of the fact that (I)-property, the $\sup-\limsup$ Theorem (SLS) and Simons' inequality (SI) are indeed equivalent for any subset B of w^*-compact convex $K\subset X^*$, see Theorem 2.2; as a consequence we obtain Corollary 2.3 that, in particular, shows that neither for Simons' inequality nor for (I)-property is important that B is a boundary (what matters is that $\overline{\mathrm{co}\,B}^{\|\cdot\|}$ contains a boundary). This observation could be useful for further applications. We stress that we also prove, at the end of the paper, see Theorem 5.9, a version of the Simons' $\sup-\limsup$ Theorem, when $\ell^1\not\subset X$, for bounded sequences (z_n) in X^{**}, instead of sequences in X, but requiring that each z_n is a Baire map when restricted to (B_{X^*}, w^*).

In the remaining of the paper we are mostly interested in boundaries of Asplund spaces. One of the main results, see Theorem 3.4, is a necessary and sufficient condition for a boundary to have (S). The main tool here is a new notion we introduce, namely, the notion of *finitely-self-predictable* set, FSP in short, see Definition 3.1. The definition of FSP-set is inspired by properties of boundaries of polyhedral spaces [11], and by properties of boundaries described with σ-fragmentable maps [2, 4, 8, 17]. The discussion on σ-fragmentable maps needs some background which is done in Section 4. As to the boundaries of polyhedral spaces we can do it now just to give the reader a feeling what an FSP-set is. Recall that a Banach space X is called *polyhedral* [24] if the unit ball of each of its finite-dimensional subspace is a polytope. The following theorem was proved in [11].

Theorem 1.1 ([11]). *Let X be a polyhedral space of the density character w. Then X has a boundary $B\subset S_{X^*}$ of cardinality w such that for any $h\in B$ we have*

$$\mathrm{int}_{\{x\in X:\, h(x)=1\}}\,\{x\in S_X :\ h(x)=1\}\neq\emptyset \tag{1.1}$$

In particular the boundary B is a minimum among all boundaries, i.e.*, it is contained in any other boundary of X.*

For a finite subset $\sigma\subset X$ denote $E_\sigma=\mathrm{span}\,\sigma$. Since the unit ball B_{E_σ} of a subspace E_σ is a polytope, it follows that there is a finite subset $A_\sigma\subset B$ (A_σ may not be unique) such that $A_\sigma|_{E_\sigma}$ is a boundary of E_σ (which is easily seen to be equivalent to $\mathrm{ext}B_{E^*_\sigma}=A_\sigma|_{E_\sigma}$). Thus we can define a map $\xi:\mathcal{F}_X\rightrightarrows\mathcal{F}_B$ from the family $\mathcal{F}_X$ of all finite subsets of X into the family $\mathcal{F}_B$ of all finite subsets of B such that $\xi(\sigma)=A_\sigma$. Let $\sigma_n, n=1,2,...$, be an increasing sequence of finite subsets of X and $E=[\cup_{n=1}^\infty\sigma_n]$. By using (1.1) in Theorem 1.1 above it is not difficult to prove that the set $D=\cup_{n=1}^\infty\xi(\sigma_n)|_E$ is a boundary of E. We can look at this result in the following way. We form a subspace E by using countably many steps (on the n-th step we add a finite-dimensional subspace E_{σ_n}), and on each step we are allowed to choose a finite subset (A_{σ_n}) of the boundary. Finally, "at the end", we get a boundary of E (just as the union of the sets A_{σ_n}'s): in a sense, the boundary is *finitely-predictable*. Clearly, this property is very strong, and it is held for polyhedral spaces only (just take all σ_n's are equal to the same finite set σ). However, a small modification of this property (which we call FSP, see Definition 3.1) allows us to prove the following:

Theorem 3.9. *Let X be an Asplund space and B be a boundary of X. Then B has (S) if and only if B is FSP.*

To prove the "if" part of Theorem 3.9 we use a separable reduction (similar to ones used in [2, 4, 8, 12]); in the proof of the "only if" part we use the existence of the so-called Jayne-Rogers selector for the duality mapping in Asplund spaces.

We also give a characterization of Asplund spaces involving FSP boundaries, Theorem 3.10, and $\sigma-$fragmented selectors, Section 4.

In Section 5 we strengthen the property (I) of boundaries (see [13]) for Asplund spaces (see Proposition 5.3), by using the γ-topology instead of the w^*-topology. By using a Haydon's result [16] and a γ-topology technique developed in Section 5 we prove:

Theorem 5.6. *Let X be a Banach space. The following statements are equivalent:*

(i) *X does not contain ℓ^1;*
(ii) *for every w^*-compact subset K of X^* any w^*-K-analytic boundary B of K is strong.*

We use standard Geometry of Banach spaces and topology notation which can be found in [21] and [7, 23]. In particular, B_E (resp. S_E) is the unit ball (resp. the unit sphere) of a normed space E. If S is a subset of E^*, then $\sigma(E, S)$ denotes the topology of pointwise convergence on S. Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x\rangle$ and $x^*(x)$ to indistinctively denote the evaluation of x^* at x. If (X, ρ) is a metric space, $x \in X$ and $\delta > 0$ we denote by $B_\rho(x, \delta)$ (or $B(x, \delta)$ if no confusion arises) the open ρ-ball centered at x of radius δ. The notation $B[x, \delta]$ is reserved to denote the corresponding closed balls. If E is a Banach space and $A \subset E$, then $\overline{A}$ (or $\overline{A}^{\|\cdot\|}$) means closure for the norm topology; we will explicitly indicate when closures are taken in some other topology.

2. The (I)-property and Simons' inequality

Let us recall now the combinatorial principle that lies behind in James' compactness theorem as it was found by S. Simons [31], and described in his famous lemma:

Lemma 2.1 (Simons, [31, Lemma 2 and Theorem 3])**.** *Let K be a set and $(z_n)_n$ a uniformly bounded sequence in $\ell^\infty(K)$. If B is a subset of K such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $b^* \in B$ such that*

$$\sup\{\sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in K\} = \sum_{n=1}^{\infty} \lambda_n z_n(b^*),$$

then

$$\sup_{b^* \in B} \{\limsup_{n\to\infty} z_n(b^*)\} \geq \inf \left\{ \sup_K w : w \in \mathrm{co}\{z_n : n \in \mathbb{N}\} \right\} \tag{2.1}$$

and

$$\sup_{b^*\in B}\{\limsup_{n\to\infty} z_n(b^*)\} = \sup_{x^*\in K}\{\limsup_{n\to\infty} z_n(x^*)\}. \tag{2.2}$$

Note that (SI) and (SLS) are particular cases of the thesis in Lemma 2.1 above. As commented in the introduction our main result in this section analyzes the coincidence of the above (I)-property, the $\sup - \limsup$ Theorem and Simons' inequality (2.1) in an arbitrary setting. Our proof is self-contained and uses elementary facts.

Theorem 2.2. *Let X be a Banach space and K a w^*- compact and convex subset of X^*. For a given subset $B \subset K$, the following statements are equivalent:*

(i) *For every covering $B \subset \bigcup_{n=1}^{\infty} D_n$ by an increasing sequence of w^*-closed convex subsets $D_n \subset K$, we have*

$$\overline{\cup_{n=1}^{\infty} D_n}^{\|\cdot\|} = K. \tag{2.3}$$

(ii) *For every bounded sequence (x_k) in X*

$$\sup_{f\in B}(\limsup_k f(x_k)) = \sup_{g\in K}(\limsup_k g(x_k)). \tag{2.4}$$

(iii) *For every bounded sequence (x_k) in X*

$$\sup_{f\in B}(\limsup_k f(x_k)) \geq \inf_{\sum\lambda_i=1,\lambda_i\geq 0}(\sup_{g\in K} g(\sum \lambda_i x_i)). \tag{2.5}$$

(iv)

$$\overline{\operatorname{co} B}^{w^*} = K, \tag{2.6}$$

and for every bounded sequence (x_k) in X

$$\sup_{f\in B}(\limsup_k f(x_k)) \geq \inf_{\sum\lambda_i=1,\lambda_i\geq 0}(\sup_{g\in B} g(\sum \lambda_i x_i)). \tag{2.7}$$

In particular all of them happen when the subset B is a boundary of the compact K after Lemma 2.1 or (I)-property.

Proof. (i)⇒(ii). For a given bounded sequence (x_k) in X let us put

$$l := \sup_{f\in B}(\limsup_k f(x_k))$$

and fix $\varepsilon > 0$. We define the sets

$$D_n = \{h \in K : \ h(x_k) \leq l + \varepsilon, \ k > n\}, \quad n = 1, 2, \dots$$

Clearly, each D_n is w^*-closed and convex, and $B \subset \bigcup_{n=1}^{\infty} D_n$. Moreover, we have $\limsup_k f(x_k) \leq l + \varepsilon$ for any $f \in \bigcup_{n=1}^{\infty} D_n$. The assumed condition (i) implies that the union $\bigcup_{n=1}^{\infty} D_n$ is norm-dense in K. This fact together with the boundedness of the fixed sequence (x_k) easily leads us to

$$\sup_{g\in K}(\limsup_k g(x_k)) \leq l + \varepsilon.$$

Since the fixed $\varepsilon > 0$ is arbitrary (ii) follows.

(ii)⇒(iii) For a w^*-compact convex subset $K \subset X^*$ and $c \in R$ we define

$$K^c = \{x \in X : \ \max x(K) \leq c\}, \quad K^{cc} = \{f \in X^* : \ \sup f(K^c) \leq c\}.$$

The following well-known Fact (which is an easy consequence of a separation theorem) will be used in the proof..

Fact.

(1) If $c > 0$ *then* $K^{cc} = \{\lambda g :\ 0 \le \lambda \le 1,\ g \in K\}$.
(2) If $c < 0$ *and* $0 \notin K$ *then* $K^{cc} = \{\lambda g :\ 1 \le \lambda,\ g \in K\}$.
In all cases $\mathrm{int}K^c = \{x \in X :\ \max x(K) < c\} \neq \emptyset$.

Also we will use the following trivial observation. Let a functional f separates two sets A and B, i.e. $\inf f(A) \ge \alpha \ge \sup f(B)$ with $\alpha \neq 0$. Then by passing to a multiple of f we can get instead of α in the inequality above any real number β with $\alpha\beta > 0$.

Put

$$C = \mathrm{co}\{x_k\},\ \ a = \inf_{\sum \lambda_i = 1, \lambda_i \ge 0} (\sup_{g \in K} g(\sum \lambda_i x_i)) = \inf_{x \in C} \sup_{g \in K} g(x).$$

In view of (iii) we need to prove that $\sup_{f \in K} \limsup_k f(x_k) \ge a$. Assume to the contrary that $\sup_{f \in K} \limsup_k f(x_k) < a$. Let $b = \sup_{f \in K} \limsup_k f(x_k)$, if $a \le 0$, and if $a > 0$ then let b be any number such that $b \ge \sup_{f \in K} \limsup_k f(x_k)$ and $a > (a+b)/2 > 0$. We always have

$$b \ge \sup_{f \in K} \limsup_k f(x_k). \tag{2.8}$$

Next if $a \le 0$ then $b < 0$, and hence $0 \notin K$. If $a > 0$ then from what is said above follows that $(a+b)/2 > 0$. Put $c = (a+b)/2$. Clearly $c \neq 0$ in both cases. It follows from Fact that

$$\mathrm{int}K^c = \{x \in X :\ \max x(K) < c\} \neq \emptyset.$$

Put

$$\delta = \frac{a-b}{2\sup_{g \in K} ||g||},\quad C_1 = \mathrm{cl}\{C + \delta B_X\}.$$

An easy calculation shows that

$$\inf_{y \in C_1} \max_{g \in K} g(y) \ge c.$$

Indeed, for any $y = x + z,\ x \in C,\ ||z|| \le \delta$, we have

$$\max_{g \in K} g(y) \ge \max_{g \in K} g(x) - \delta ||g|| = \max_{g \in K} g(x) - \frac{a-b}{2\sup_{g \in K} ||g||} ||g|| \ge a - \frac{a-b}{2} = c.$$

Therefore $C_1 \cap \mathrm{int}K^c = \emptyset$, and by a separation theorem there is $t_1 \in X^*$ with

$$\inf t_1(C_1) \ge c \ge \sup t_1(K^c), \tag{2.9}$$

where we used subsequently that $0 \in \mathrm{int}C_1$ and the observation after Fact. From the right hand side in inequality (2.9) we obtain $t_1 \in K^{cc}$. By Fact

$$t_1 = \lambda t,\ \ \lambda \ge 1,\ \ t \in K,\ \ \text{if}\ \ c < 0,$$

$$t_1 = \lambda t,\ \ 0 \le \lambda \le 1,\ \ t \in K,\ \ \text{if}\ \ c > 0,$$

(recall that $c \neq 0$.) From the left-side inequality in (2.9) we deduce (in both cases: $c < 0$ and $c > 0$) that $\inf t(C) \ge c$. Since $c > b$ we obtain that $\limsup_k t(x_k) \ge c > b$, contradicting (2.8). The proof is complete.

(iii)$\Rightarrow$(i). We shall do it by contradiction. Let us fix an increasing sequence D_n of w^* closed and convex subsets of K such that $B \subset \bigcup_{n=1}^{\infty} D_n$. Let us proceed by contradiction and assume that there is $z_0^* \in K$ such that $z_0^* \notin \overline{\cup_{n=1}^{\infty} D_n}^{||\cdot||}$. Fix $\delta > 0$ such that

$$B[z_0^*, \delta] \cap D_n = \emptyset,\ \text{for every}\ n \in \mathbb{N}.$$

The separation theorem in (X^*, w^*) applied to the w^*-compact sets $B[0,\delta]$ and $D_n - z_0^*$ provides us with $x_n \in X$, $\|x_n\| = 1$, and $\alpha_n \in \mathbb{R}$ such that

$$\inf_{v^* \in B[0,\delta]} x_n(v^*) > \alpha_n > \sup_{y^* \in D_n} x_n(y^*) - x_n(z_0^*).$$

We have

$$-\delta = \inf_{v^* \in B[0,\delta]} x_n(v^*),$$

and consequently we have produced a sequence $(x_n)_n$ in B_X such that for each $n \in \mathbb{N}$ we have

$$x_n(z_0^*) - \delta > x_n(y^*) \text{ for every } y^* \in D_n. \tag{2.10}$$

Fix $x^{**} \in B_{X^{**}}$ a w^*-cluster point of the sequence $(x_n)_n$ and let $(x_{n_k})_k$ be a subsequence of $(x_n)_n$ such that $x^{**}(z_0^*) = \lim_k x_{n_k}(z_0^*)$. We can and do assume that

$$x_{n_k}(z_0^*) > x^{**}(z_0^*) - \frac{\delta}{2}, \text{ for every } k \in \mathbb{N}. \tag{2.11}$$

Since $B \subset \cup_n D_n$ and $(D_n)_n$ is increasing, given $b^* \in B$ there exists $k_0 \in \mathbb{N}$ such that $b^* \in D_{n_k}$ for every $k \geq k_0$. Inequality (2.10) implies now that

$$x^{**}(z_0^*) - \delta \geq \limsup_k x_{n_k}(b^*) \text{ for every } b^* \in B. \tag{2.12}$$

On the other hand, inequality (2.11) implies that

$$w(z_0^*) > x^{**}(z_0^*) - \frac{\delta}{2}, \text{ for every } w \in \operatorname{co}\{x_{n_k} : k \in \mathbb{N}\}. \tag{2.13}$$

Now (iii) applies to conclude

$$x^{**}(z_0^*) - \delta \overset{(2.12)}{\geq} \sup_{b^* \in B} \limsup_k x_{n_k}(b^*) \overset{\text{(iii)}}{\geq} \inf\big\{\sup_K w : w \in \operatorname{co}\{x_{n_k} : k \in \mathbb{N}\}\big\} \geq$$
$$\geq \inf\big\{w(z_0^*) : w \in \operatorname{co}\{x_{n_k} : k \in \mathbb{N}\}\big\} \overset{(2.13)}{>} x^{**}(z_0^*) - \frac{\delta}{2},$$

¿From the inequalities above we obtain $0 \geq \delta$ which is a contradiction that finishes the proof.

(iii)⇒(iv). Observe that (2.6) follows from (i) (take D_n being the w^* closed convex hull of of $B,\ n = 1, 2, ...$), and (2.7) follows trivially from (iii).

(iv)⇒(iii). From (2.6) follows that

$$\inf_{\sum \lambda_i = 1, \lambda_i \geq 0} (\sup_{g \in K} g(\sum \lambda_i x_i)) = \inf_{\sum \lambda_i = 1, \lambda_i \geq 0} (\sup_{g \in B} g(\sum \lambda_i x_i)),$$

and the proof is over. □

The following corollary strengthens Simons' inequality and (I)-property of boundaries.

Corollary 2.3. *Let X be a Banach space, $K \subset X^*$ be a w^*-compact and convex subset with a boundary $B_1 \subset K$. Then any subset $B \subset B_1$ with $\overline{\operatorname{co} B}^{\|\cdot\|} \supset B_1$, enjoys all the properties (i)-(iv) of Theorem 2.2.*

Proof. The inclusions $B \subset B_1 \subset \overline{\operatorname{co} B}^{\|\cdot\|}$ imply that for any bounded sequence (x_k) in X the following equality holds:

$$\sup_{f \in B} (\limsup_k f(x_k)) = \sup_{g \in B_1} (\limsup_k g(x_k)).$$

Since B_1 is a boundary of K, it follows from Simons' Lemma 2.1 that

$$\sup_{f\in B_1}(\limsup_k f(x_k)) = \sup_{g\in K}(\limsup_k g(x_k)).$$

Therefore it follows

$$\sup_{f\in B}(\limsup_k f(x_k)) = \sup_{g\in K}(\limsup_k g(x_k)),$$

i.e. B satisfies (ii) in Theorem 2.2. Hence B enjoys all the equivalent properties from Theorem 2.2. The proof is complete. □

3. Finitely self predictable sets

Let us denote by $\mathcal{F}_A$ the family of finite subsets of a given set A.

Definition 3.1. *Let X be a Banach space and $C \subset X^*$. We say that C is finitely-self-predictable (FSP in short) if there is a map*

$$\xi : \mathcal{F}_X \to \mathcal{F}_{\operatorname{co} C}$$

from the family of all finite subsets of X into the family of all finite subsets of $\operatorname{co} C$ such that for any increasing sequence σ_n in $\mathcal{F}_X,\ n = 1, 2, ...,$ with

$$E = \left[\,\cup_{n=1}^{\infty} \sigma_n\right], \quad D = \cup_{n=1}^{\infty}\xi(\sigma_n),$$

we have

$$C|_E \subset \overline{\operatorname{co}(D|_E)}^{\|\cdot\|}.$$

Remark 3.2. If C is separable then it is FSP. Observe also that if C is FSP then, clearly $\overline{C}$ is FSP. So we always assume without loss of generality that C is closed.

The following proposition shows that if we allow for the sets $\xi(\sigma)$ in Definition 3.1 to be countable, we get an equivalent definition.

Proposition 3.3. *If there is a map ξ_1 defined on $\mathcal{F}_X$ with $\xi_1(\sigma)$ countable for each $\sigma \in \mathcal{F}_X$ and satisfying the conditions in Definition 3.1, then there exits a map ξ defined on $\mathcal{F}_X$ with $\xi(\sigma)$ finite for each $\sigma \in \mathcal{F}_X$ and satisfying the conditions in Definition 3.1.*

Proof. Assume that $\xi_1(\sigma)$ is countable. To prove the proposition it is enough to construct a map $\xi(\sigma)$ with $|\xi(\sigma)| < \infty$, and such that for any increasing sequence σ_n in $\mathcal{F}_X$ we have

$$\bigcup_{n=1}^{\infty} \xi_1(\sigma_n) \subset \bigcup_{n=1}^{\infty} \xi(\sigma_n). \tag{3.1}$$

Let $\xi_1(\sigma) = \{f_i\}_{i=1}^{\infty}$. Define $P_n\big(\xi_1(\sigma)\big) = \{f_i\}_{i=1}^{n}$. Next we define $\xi(\sigma)$ as follows. Let $|\sigma| = m < \infty$, and let A be the family of all non-empty subsets of σ (including σ itself). Define

$$\xi(\sigma) = \bigcup_{\nu\in A} P_m\big(\xi_1(\nu)\big) \tag{3.2}$$

To prove (3.1) assume that $f \in \cup_{n=1}^{\infty}\xi_1(\sigma_n)$. Then there are n_0 and m_0 such that $f \in P_{m_0}\big(\xi_1(\sigma_{n_0})\big)$. Since $|\sigma_n| \to \infty,\ n \to \infty$, it follows that there is n_1 with $|\sigma_{n_1}| > m_0$ and $n_1 > n_0$. It follows from the definition (3.2) that $f \in \xi(\sigma_{n_1})$. The proof is complete. □

Theorem 3.4. *Let X be a Banach space. Assume that $K \subset X^*$ is a w^*-compact convex set with boundary $B \subset K$. If B is FSP then B has (S).*

Proof. Put $C = \operatorname{co} B$ and assume to the contrary that there is $f_0 \in K \setminus \overline{C}^{\|\cdot\|}$. By the separation theorem there is $F_0 \in S_{X^{**}}$ with

$$\sup F_0(C) = \sup F_0(\overline{C}^{\|\cdot\|}) = \alpha < F_0(f_0). \tag{3.3}$$

By Goldstein's theorem we find $x_1 \in S_X$ with

$$|(F_0 - x_1)(f_0)| < 1.$$

Let us write $\xi(\{x_1\}) = \{h_{1j}\}_{j=1}^{p_1} \subset C$ and let us use Goldstein's theorem again to find $x_2 \in S_X$ with

$$|(F_0 - x_2)(f_0)| < 1/2, \;\; |(F_0 - x_2)(h_{1j})| < 1/2, j = 1, 2, \cdots, p_1.$$

Proceeding by recurrence we assume that $x_1, x_2, \ldots, x_n \in S_X$ have been constructed and we let $\xi(\{x_1, x_2, \ldots, x_n\}) = \{h_{n\,j}\}_{j=1}^{p_n} \subset C$. By Goldstein's theorem we find $x_{n+1} \in S_X$ with

$$|(F_0 - x_{n+1})(f_0)| < 1/(n+1),$$
$$|(F_0 - x_{n+1})(h_{ij(i)})| < 1/(n+1) \text{ for } i = 1, 2, \ldots, n \text{ and } j(i) = 1, 2, \ldots, p_i.$$

If we define

$$E = [\{x_i\}_{i=1}^{\infty}],\; D = \cup_n \xi(\{x_1, ..., x_n\}) = \{h_{ij(i)} : i \in \mathbb{N} \text{ and } j(i) = 1, 2, \ldots, p_i\}$$

we have the following properties:

(α) For each $h \in D$ there is $l \in \mathbb{N}$ such that $|(F_0 - x_m)(h)| < 1/m$ for every $m > l$.
(β) $|(F_0 - x_m)(f_0)| < 1/m$ for every $m \in \mathbb{N}$.

Since B is FSP it follows that $B|_E \subset \overline{\operatorname{co}(D|_E)}^{\|\cdot\|}$, and, in particular, $B|_E$ is separable. On the other hand, $B|_E$ is a boundary of $K|_E$ and therefore (see [30, 14, 13]) $\overline{\operatorname{co}(B|_E)}^{\|\cdot\|} = K|_E$. By taking into account that $D \subset C$ finally we can write

$$\overline{C|_E}^{\|\cdot\|} = \overline{\operatorname{co}(D|_E)}^{\|\cdot\|} = K|_E \tag{3.4}$$

Let $G \in B_{E^{**}}$ be any w^*−limit point of $\{x_i\} \subset S_E \subset S_{E^{**}}$. Then by ($\alpha$) we have that

$$G(h|_E) = F_0(h), \text{ for every } h \in D, \tag{3.5}$$

and by (β) we obtain

$$G(f_0|_E) = F_0(f_0). \tag{3.6}$$

By taking into account consequently (3.4), (3.5), (3.4), and (3.6), we conclude that

$$\sup G(K|_E) = \sup G\big(\overline{\operatorname{co}(D|_E)}\big)^{\|\cdot\|} = \sup G(D|_E) =$$
$$\sup F_0(D) \leq \sup F_0(C) = \alpha < F_0(f_0) = G(f_0|_E),$$

contradicting $f_0 \in K$. The proof is complete. □

The following example shows that the FSP of B plays a crucial role in Theorem 3.4, *i.e.*: if we substitute FSP by the weaker condition $B|_E$ is separable, for any separable subspace $E \subset X$, then the conclusion B has (S) may not be true.

Example 3.5. Let $X = C\big([0, \omega_1]\big)$ with the supremum norm, $K = B_{X^*}$, and $B = \big\{ \pm \delta_x : x \in [0, \omega_1)\big\}$. It is known that for any $x \in X$ there is an ordinal $\alpha < \omega_1$ such that x restricted to $[\alpha, \omega_1]$ is constant. It follows that B is a boundary of K. Since X is an Asplund space it follows that $B|_E$ is separable, for any separable subspace $E \subset X$. However, B has no (S). Indeed, for every measure $\mu \in \overline{\operatorname{co}\big\{ \pm \delta_x : x \in [0, \omega_1)\big\}}$ we have that indeed $\|\delta_{\omega_1} - \mu\| \geq 1$: notice that there exists a non-negative continuous function with values 0 at ω_1 and 1 on the support of the measure μ.

Lemma 3.6. *Let X be a Banach space. If there exists a boundary $B \subset B_{X^*}$ which is FSP, then X is an Asplund space.*

Proof. Let ξ be a map saying that B is FSP, see Definition 3.1. Let $E \subset X$ be a closed separable subspace and let $D = \{e_1, e_2, \ldots, e_n, \ldots\}$ a dense subset of E. If we write $\sigma_n := \{e_1, e_2, \ldots, e_n\}$ and $D = \cup_n \xi(\sigma_n)$ then

$$B|_E \subset \overline{\mathrm{co}(D|_E)}.$$

Hence $B|_E$ is a separable boundary for B_{E^*}. Thus E^* is separable, [30, 14, 13], and X is Asplund. □

To prove our main Theorem 3.9 we need the following lemmata.

Lemma 3.7. *Let X be a Banach space. If there exists a boundary $B_1 \subset B_{X^*}$ which is FSP, then any other boundary $B \subset B_{X^*}$ with property (S) is FSP.*

Proof. Let ξ_1 be a map saying that B_1 is FSP. We construct a mapping ξ in the following way. Let σ be a finite subset of X and $\xi_1(\sigma) = \{f_i\}_{i=1}^m$. Using that B has property (S) we can find a countable subset $\{h_j\} \subset B$ with $\xi_1(\sigma) \subset \overline{\mathrm{co}\{h_j\}}$. Let us define $\xi(\sigma) = \{h_j\}$. We obviously have

$$\xi_1(\sigma) \subset \overline{\mathrm{co}\,\xi(\sigma)}. \tag{3.7}$$

We claim that for any increasing sequence σ_n of finite subsets of X if we write $E = \left[\cup_{n=1}^{\infty} \sigma_n\right]$ and $D = \cup_{n=1}^{\infty} \xi(\sigma_n)$, then we have

$$B|_E \subset \overline{\mathrm{co}(D|_E)}. \tag{3.8}$$

Indeed, first we note that since the boundary B_1 is FSP, then $B_1|_E$ is a separable boundary of B_{E^*} and hence it has property (S), *i.e.*,

$$B_{E^*} = \overline{\mathrm{co}(B_1|_E)}. \tag{3.9}$$

Next by using (3.9), FSP property of B_1 and (3.7) we obtain that

$$B|_E \subset B_{E^*} = \overline{\mathrm{co}(B_1|_E)} \subset \overline{\mathrm{co}(\cup_{n=1}^{\infty} \xi_1(\sigma_n)|_E)} \subset \overline{\mathrm{co}(D|_E)},$$

which proves (3.8). Now we can apply Proposition 3.3 to finish the proof. □

Lemma 3.8. *Let $f_0 : X \to S_{X^*}$ be a first Baire class map such that for any $x \in X$ we have $f_0(x)(x) = ||x||$. Let $f_n : X \to B_{X^*}$, $n = 1, 2, \ldots$, be a sequence of continuous maps with $\lim_n f_n(x) = f_0(x)$, for any $x \in X$. If $B_1 := \overline{\cup_{n=0}^{\infty} f_n(X)}$ then B_1 is a FSP boundary.*

Proof. For any finite subset σ of X put $E_\sigma = [\sigma]$. Let A_σ be a $\frac{1}{|\sigma|}-$ net in S_{E_σ}. Put

$$\xi_1(\sigma) = \bigcup_{x \in A_\sigma} \{f_n(x)\}_{n=1}^{\infty} \subset B_1.$$

Let σ_n be an increasing sequence of finite subsets of X, $E = \left[\cup_{n=1}^{\infty} \sigma_n\right]$ and $D = \cup_{n=1}^{\infty} \xi_1(\sigma_n)$. We claim that

$$f_0(S_E) \subset \overline{D} \tag{3.10}$$

Indeed, fix $\varepsilon > 0$ and $x_0 \in S_E$. Let m be such that $||f_0(x_0) - f_m(x_0)|| < \varepsilon/2$. Since f_m is continuous it follows that there is a $\delta > 0$ such that for any $x \in X$ with $||x - x_0|| < \delta$ we have $||f_m(x) - f_m(x_0)|| < \varepsilon/2$. Take n so large that $1/|\sigma_n| < \delta/2$ and that for some

$y \in S_{E_{\sigma_n}}$ we have $||y - x_0|| < \delta/2$. Pick $x \in A_{\sigma_n}$ with $||x - y|| < 1/|\sigma_n| < \delta/2$. An easy calculation shows that $||f_m(x) - f_0(x_0)|| < \varepsilon$ and therefore our claim has been proved.

From (3.10) we obtain that $f_0(S_E)|_E$ is a separable boundary of B_{E^*} and consequently $f_0(S_E)|_E$ has property (S), *i.e.*, $\overline{\text{co}(f_0(S_E)|_E)} = B_{E^*}$. Again from (3.10) follows that

$$B_1|_E \subset B_{E^*} = \overline{\text{co}(f_0(S_E)|_E)} \subset \overline{\text{co}(D|_E)}.$$

Now we can apply Lemma 3.3 to finish the proof. $\square$

A wide class of FSP boundaries is provided by $\sigma-$fragmentable selectors of the duality mapping $J : X \to 2^{B_{X^*}}$ that sends each $x \in X$ to the set

$$J(x) := \{x^* \in B_{X^*} : x^*(x) = \|x\|\},$$

see Corollary 4.4 in Section 4.

Theorem 3.9. *Let X be an Asplund space and B be a boundary of X. Then B has (S) if and only if B is FSP.*

Proof. If B is FSP then by Theorem 3.4 B has (S). Conversely we now prove that every strong boundary is FSP. Since X is an Asplund space it follows that the duality mapping $J : X \to 2^{B_{X^*}}$ has a first Baire class selector

$$f_0 : X \to S_{X^*},$$

see for instance Theorem I.4.2 in [5]. The selector f_0 has associated a sequence $f_n : X \to B_{X^*}$, $n = 1, 2, \ldots$ of continuous maps with $\lim_n f_n(x) = f_0(x)$, for any $x \in X$. The boundary $B_1 := \overline{\cup_{n=0}^{\infty} f_n(X)}$ is a FSP boundary by Lemma 3.8 and Lemma 3.7 that finishes the proof. $\square$

The following theorem gives a characterization of Asplund spaces in terms of boundaries.

Theorem 3.10. *The following statements for a Banach space X are equivalent:*

(i) *X is an Asplund space;*
(ii) *X admits an FSP boundary;*
(iii) *Any boundary B with (S) is FSP.*

Proof. Implication (i)⇒(ii) follows from Theorem 3.9 for, say $B = S_{X^*}$. For the implication (ii)⇒(iii) apply Lemma 3.7. Finally, to prove (iii)⇒(i) we take $B = B_{X^*}$; clearly B has property (S), and by (iii) B is FSP, and (i) follows. The proof is complete. $\square$

4. $\sigma-$FRAGMENTED MAPS

We shall deal in this section with boundaries constructed with σ-fragmentable maps. This class of maps is wide enough as to include all Borel measurable maps between complete metric spaces: σ-fragmentable maps were introduced in [17] and they have been extensively studied in [20], [26] and [2]. Let us introduce them with the following property, see [2, Section 2] for a complete characterization:

Definition 4.1. *A map $f : T \to E$ is σ-fragmented if, and only if, f is the pointwise limit of a sequence of maps $f_n : T \to E$, such that for every $n \in \mathbb{N}$ there are sets $\{T_m^n, m = 1, 2, \cdots\}$ with $T = \cup_{m=1}^{\infty} T_m^n$ and such that for every $m \in \mathbb{N}$ and every closed subset $F \subset T_m^n$ the restriction $f_n|_F$ has at least a point of continuity,* An important property of σ-fragmented maps between metric spaces is that they send separable subsets of the domain space into separable subsets of the range space in the precise way described in the theorem below:

Theorem 4.2. [26, Theorem 2.14] *Let (T, d) and (E, ρ) be metric spaces and $f : T \to E$ a σ-fragmented map. Then, for every $t \in T$ there exists a countable set $W_t \subset T$ such that*

$$f(t) \in \overline{\bigcup \{f(W_{t_n}) : \, n = 1, 2, \dots\}}^{\rho}$$

whenever $\{t_n\}$ is a sequence converging to t in (T, d). In particular, $f(S)$ is separable whenever S is a separable subset of T.

Whereas the above result has been used in [26] as an important tool for renorming in Banach spaces we will use it here as the key result to prove Theorem 4.3. We stress that it has been known for a long time that Borel maps from a complete metric space into a metric space send separable subsets of the domain into separable subsets of the range, see for instance [33, Theorem 4.3.8]. It should be noted that σ-fragmented maps are not necessarily Borel measurable though: for instance, every map between metric spaces with separable range is σ-fragmented. Let us remark that a map with domain a metric space and with values in a normed space is Baire one if, and only if, it is σ-fragmented and the preimage of open sets are $\mathcal{F}_\sigma$ sets, see [20, Chapter 2] and [15].

We can prove now a localized version of one of the main results in [2].

Theorem 4.3. *Let X be a Banach space. Assume that $K \subset X^*$ is a w^*-compact convex set and $f : X \to K$ is a $\sigma-$fragmented selector for the attaining map*

$$F_K(x) := \{k \in K : k(x) = \sup \{g(x) : g \in K\}\}.$$

Then

$$\overline{\operatorname{co} f(X)}^{\|\cdot\|} = K.$$

Proof. Let us prove that $f(X)$ is an FSP boundary of K and the result here will follow from Theorem 3.4.

By Theorem 4.2 there is a map ϕ from X into the family of all countable subsets of $f(X)$ such that

$$f(x) \in \overline{\cup_n \phi(x_n)},$$

whenever $x = \lim_n x_n, \ x, x_n \in X$.

For any finite subset σ of S_X put $E_\sigma = [\sigma]$ and select A_σ as a $\frac{1}{|\sigma|}-$ net in the finite dimensional sphere S_{E_σ}. Now we define the map

$$\psi(\sigma) = \cup_{x \in A_\sigma} \phi(x) \subset f(X).$$

Let σ_n be an increasing sequence of finite subsets of X. Put $E = \left[\, \cup_{n=1}^{\infty} \sigma_n \right]$ and $D = \cup_{n=1}^{\infty} \psi(\sigma_n)$. We will prove now that

$$f(S_E) \subset \overline{D}. \tag{4.1}$$

Fix $x_0 \in S_E$. Since the sequence σ_n is increasing there are points $x_p \in S_{E_{\sigma_{n_p}}}, p = 1, 2, \dots$ with $n_1 < n_2 < \dots < n_p < \dots$ and $\lim_p \|x_p - x_0\| = 0$. Let us choose for every p an element $y_p \in A_{\sigma_{n_p}}$ such that $\|x_p - y_p\| \leq \frac{1}{|\sigma_{n_p}|}$. We consequently have $\lim_p \|x_0 - y_p\| = 0$ and thus $f(x_0) \in \overline{\cup_{p=1}^{\infty} \phi(y_p)}$, so $f(x_0) \in \overline{\cup_{n=1}^{\infty} \psi(\sigma_n)}$ and the proof for (4.1) is finished.

By our hypothesis the set $(f(S_E))|_E$ is a separable boundary of $K|_E$. Since separable boundaries are strong, see Theorem I.2 in [14],[30] or [13], we have that $K|_E \subset \overline{\text{co}(f(S_E)|_E}$, thus $f(X)|_E \subset \overline{\text{co}(D|_E)}$ which proves the FSP property of $f(X)$ and finishes the proof. □

Corollary 4.4. *Let X be a Banach space and let f be a $\sigma-$fragmented selector for the duality mapping $J : X \to 2^{B_{X^*}}$. Then $f(X)$ is a FSP boundary.*

The following Corollary stresses Remark I. 5.1 in [5]

Corollary 4.5. *Let X be a Banach space and $J : X \to 2^{B_{X^*}}$ the duality mapping. The following statements are equivalent:*

(i) *X is an Asplund space.*
(ii) *J has a Baire one selector.*
(iii) *J has a $\sigma-$fragmented selector.*
(iv) *J has a selector $f : X \to X^*$ such that $f(X)$ is FSP.*

Proof. For the implication (i)⇒(ii) we use that if X is Asplund, then Theorem 8 in [19] provides a Baire one selector for J, see also Theorem I.4.2 in [5]. The implication (ii) ⇒(iii) follows from the fact that every Baire one map is σ-fragmentable, see Corollary 7 in [17]. Finally, (iii) ⇒(iv) follows from Corollary 4.4. Finally, (iv) ⇒(i) follows from Theorem 3.10. □

5. STRENGTHENING THE (I)-PROPERTY AND DESCRIPTIVE BOUNDARIES

Given $x^* \in X^*$, $D \subset X$ and $\varepsilon > 0$ we write

$$V(x^*, D, \varepsilon) := \{y^* \in X^* : |y^*(x) - x^*(x)| \leq \varepsilon, \text{ for every } x \in D\}.$$

Denote by $\mathcal{C}_{B_X}$ the family of countable subsets of B_X. Note that, while the family

$$\{V(x^*, D, \varepsilon)\}_{D \in \mathcal{F}_X, \varepsilon > 0}$$

is a basis of w^*-neighborhoods for x^*, the family

$$\{V(x^*, D, \varepsilon)\}_{D \in \mathcal{C}_{B_X}, \varepsilon > 0}$$

is a basis of neighborhoods for x^* for the locally convex topology γ in X^* of uniform convergence on bounded and countable subsets of X. The topology γ was used in [29] to characterize Asplund spaces X are those for which (X^*, γ) is Lindelöf. Other papers where topology γ has been studied are [2, 3] and [4].

Recall that a topological space Y is Lindelöf if every family of closed subsets of Y with empty intersection contains a countable subcollection with empty intersection.

We start with the next easy lemma.

Lemma 5.1. *Let C be a w^*-compact* (resp. *γ-Lindelöf) subset of X^*. For given $z_0^* \in X^*$ and $\delta > 0$ the following statements are equivalent:*

(i) *$B[z_0^*, \delta] \cap C \neq \emptyset$;*
(ii) *$V(z_0^*, D, \delta) \cap C \neq \emptyset$ for each finite* (resp. *countable) set $D \subset B_X$.*

Proof. We prove the case C being γ-Lindelöf. Since $B[z_0^*, \delta] \subset V(z_0^*, D, \delta)$ it is clear that (i) ⇒(ii). Conversely, if we assume that (ii) holds then the family $\{V(z_0^*, D, \delta) \cap C : D \in \mathcal{C}_{B_X}\}$ is made up of γ-closed subsets of C with the property that for every countable subfamily

$$\{V(z_0^*, D_n, \delta) \cap C : n \in \mathbb{N}, D_n \in \mathcal{C}_{B_X}\},$$

has not empty intersection because

$$\emptyset \neq V(z_0^*, \cup_n D_n, \delta) \cap C \subset \bigcap_n V(z_0^*, D_n, \delta) \cap C.$$

Since, C is γ-Lindelöf we conclude that

$$\emptyset \neq \bigcap_{D \in \mathcal{C}_{B_X}} V(z_0^*, D, \delta) \cap C = B[z_0^*, \delta] \cap C,$$

and the proof of (ii) $\Rightarrow$ (i) is finished in this case.

The proof for the equivalence (i) $\Leftrightarrow$ (ii) when C is w^*-compact is similar to the one we have already given for the case γ-Lindelöf keeping in mind now that in compact spaces every family of closed subsets with the finite intersection property has a non empty intersection. □

As a tool for our subsequent study we need to quote first the following result that have been established in [2].

Proposition 5.2 ([2, Theorem 5.4])**.** *Let X be a Banach space. The following statement are equivalent:*

(i) $\ell^1 \not\subset X$;
(ii) *for every w^*-compact convex subset K of X^* and any boundary B of K we have $K = \overline{\operatorname{co} B}^{\gamma}$;*
(iii) *for every w^*-compact subset K of X^*, $\overline{\operatorname{co} K}^{w^*} = \overline{\operatorname{co} K}^{\gamma}$.*

For Asplund spaces the following strong version of the (I)-formula holds.

Proposition 5.3. *Let X be an Asplund space, K a w^*-compact convex subset of the dual space X^* and $B \subset K$ a boundary of K. Then, for any increasing sequence $\{B_n\}_{n=1}^{\infty}$ of subsets of B with $B = \cup_n B_n$ we have*

$$K = \overline{\cup_{n=1}^{\infty} \overline{\operatorname{co} B_n}^{\gamma}}^{\|\cdot\|}. \tag{SI}$$

Proof. Set $B' := \cup_{n=1}^{\infty} \overline{\operatorname{co} B_n}^{\gamma}$. B' is a convex boundary of K. Thus, Proposition 5.2 applies to yield $K = \overline{B'}^{\gamma}$. On the other hand, since X is Asplund the space (X, γ) is Lindelöf, see [29] and [4]. Therefore B' is γ-Lindelöf too and a straightforward application of Lemma 5.1 gives us $\overline{B'}^{\gamma} = \overline{B'}$ that combined with the equality $K = \overline{B'}^{\gamma}$ finishes the proof. □

The same ideas that we have used in the previous proposition are used in the next one that extends [27, Theorem 2.3].

Proposition 5.4. *Let X be an Asplund space, K a w^*-compact convex subset of the dual space X^* and $B \subset K$ a boundary of K. If B is γ-closed, then B is strong.*

Proof. If B is γ-closed, then for any $n \in \mathbb{N}$ the set B^n is a closed subset of $(X^*, \gamma)^n$ that is γ-Lindelöf, see [29] or [4, Theorem 2.3]; thus $(B, \gamma|_B)^n$ is Lindelöf. Now observe that if $(B, \gamma|_B)^n$ is Lindelöf for every $n \in \mathbb{N}$ then the convex hull $\operatorname{co} B$ is γ-Lindelöf too. Indeed, we notice first that $\operatorname{co} B = \cup_n \operatorname{co}_n B$ where for every $n \in \mathbb{N}$ we have written

$$\operatorname{co}_n B := \left\{ \sum_{k=1}^n \lambda_k b_k : 0 \leq \lambda_k \leq 1, b_k \in B, k = 1, 2, \ldots, n, \sum_{k=1}^n \lambda_k = 1 \right\}$$

If $K_n := \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in [0,1]^n : \sum_{k=1}^n \lambda_k = 1\}$, then K_n is compact with the topology induced by the product topology of $[0,1]^n$ and therefore $K_n \times (B, \gamma|_B)^n$ is a Lindelöf space, [7, Corollary 3.8.10]. All things considered the map

$$\psi_n : K_n \times (B, \gamma|_B)^n \to (X^*, \gamma)$$
$$\big((\lambda_k)_k^n, (b_k)_k^n\big) \quad \to \quad \sum_{k=1}^n \lambda_k b_k$$

is continuous and its image $\mathrm{co}_n\, B$ is therefore γ-Lindelöf. Hence $\mathrm{co}\, B = \cup_n \mathrm{co}_n\, B$ is a γ-Lindelöf convex boundary of K and we finally conclude that

$$K \overset{Prop.5.2}{=} \overline{\mathrm{co}\, B}^{\gamma} \overset{Lem.5.1}{=} \overline{\mathrm{co}\, B}.$$

The proof is over. $\square$

A topological space (T, τ) is said to be K*-analytic* if there is an upper semi-continuous set-valued map $F : \mathbb{N}^{\mathbb{N}} \to 2^T$ such that $F(\sigma)$ is compact for each $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $F(\mathbb{N}^{\mathbb{N}}) := \bigcup\{F(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}\} = T$. Our basic reference for K-analytic spaces is [18].

We use the following conventions: $\mathbb{N}^{(\mathbb{N})}$ is the set of finite sequences of positive integers; if $\alpha = (n_1, n_2, \dots, n_k, \dots) \in \mathbb{N}^{\mathbb{N}}$ and if $k \in \mathbb{N}$, then we write $\alpha|k := (n_1, n_2, ..., n_k) \in \mathbb{N}^{(\mathbb{N})}$. Notice that if $\alpha = (n_1, n_2, \dots, n_k, \dots) \in \mathbb{N}^{\mathbb{N}}$ and for every $k \in \mathbb{N}$ we let

$$[n_1, n_2, \dots, n_k] = \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta|k = (n_1, n_2, \dots, n_k)\}$$

then $\big([n_1, n_2, \dots, n_k]\big)_{k=1}^{\infty}$ is a basis of neighborhoods for α in $\mathbb{N}^{\mathbb{N}}$.

Proposition 5.5. *Let X be a Banach space and $\mathbf{B} : \mathbb{N}^{\mathbb{N}} \to 2^{X^*}$ a w^*-upper semicontinuous map such that $B_\alpha := \mathbf{B}(\alpha)$ is w^*-compact for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then we have*

$$\overline{\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \overline{\mathrm{co}\, B_\alpha}^{w^*}} = \overline{\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \overline{\mathrm{co}\, B_\alpha}^{w^*}}^{\gamma}. \tag{5.1}$$

Proof. Since the set in left hand side of (5.1) is clearly contained in set in the right hand side, to prove the equality (5.1) we only have to prove that if

$$z_0^* \notin \overline{\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \overline{\mathrm{co}\, B_\alpha}^{w^*}}$$

then

$$z_0^* \notin \overline{\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \overline{\mathrm{co}\, B_\alpha}^{w^*}}^{\gamma}.$$

Fix $\delta > 0$ such that

$$B[z_0^*, \delta] \cap \overline{\mathrm{co}\, B_\alpha}^{w^*} = \emptyset, \text{ for every } \alpha \in \mathbb{N}^{\mathbb{N}}.$$

We apply Lemma 5.1 for each w^*-compact set $\overline{\mathrm{co}\, B_\alpha}^{w^*}$ to obtain a finite subset D_α of B_X such that $V(z_0^*, D_\alpha, \delta) \cap \overline{\mathrm{co}\, B_\alpha}^{w^*} = \emptyset$. The separation theorem in (X^*, w^*) applied to the w^*-closed set $V(z_0^*, D_\alpha, \delta)$ and the w^*-compact set $\overline{\mathrm{co}\, B_\alpha}^{w^*}$ provides us with $x_\alpha \in X$, and $\lambda_\alpha > \xi_\alpha$ in $\mathbb{R}$ such that

$$V(z_0^*, D_\alpha, \delta) \subset G^{\lambda_\alpha} := \{y^* \in X^* : x_\alpha(y^*) > \lambda_\alpha\} \tag{5.2}$$

and

$$\overline{\mathrm{co}\, B_\alpha}^{w^*} \subset G_{\xi_\alpha} := \{y^* \in X^* : \xi_\alpha > x_\alpha(y^*)\}.$$

Since G_{ξ_α} is w^*-open and $B_\alpha \subset G_{\xi_\alpha}$, the w^*-upper semicontinuity of $\mathbf{B}$ implies that for some $k_\alpha \in \mathbb{N}$ if we write $\alpha|k_\alpha = (n_1, n_2, \ldots, n_{k_\alpha})$ we have that

$$\begin{aligned}\mathbf{B}\big([n_1, n_2, \ldots, n_{k_\alpha}]\big) := \bigcup\{\mathbf{B}(\beta) : \beta \in \mathbb{N}^{\mathbb{N}}, \beta|k_\alpha = \alpha|k_\alpha\}\big) \subset \\ \subset G_{\xi_\alpha} := \{y^* \in X^* : \xi_\alpha > x_\alpha(y^*)\}\end{aligned}$$

We notice that

$$\overline{\operatorname{co}\mathbf{B}\big([n_1, n_2, \ldots, n_{k_\alpha}]\big)}^{w^*} \subset \{y^* \in X^* : \xi_\alpha \geq x_\alpha(y^*)\}$$

and since $\lambda_\alpha > \xi_\alpha$ the inclusion (5.2) leads us to

$$V(z_0^*, D_\alpha, \delta) \cap \overline{\operatorname{co}\mathbf{B}\big([n_1, n_2, \ldots, n_{k_\alpha}]\big)}^{w^*} = \emptyset, \text{ for every } \alpha \in \mathbb{N}^{\mathbb{N}}. \tag{5.3}$$

Since $\mathbb{N}^{(\mathbb{N})}$ is countable, the family

$$\mathcal{C} := \{\mathbf{B}\big([n_1, n_2, \ldots, n_{k_\alpha}]\big) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$$

is countable too and it can be written as $\mathcal{C} = \{D_n : n \in \mathbb{N}\}$. Now we can rewrite (5.3) in terms of the D_n's in the following way: for every $n \in \mathbb{N}$ there is a finite set $F_n \subset B_X$ such that

$$V(z_0^*, F_n, \delta) \cap \overline{\operatorname{co} D_n}^{w^*} = \emptyset.$$

The latter implies that

$$V(z_0^*, \cup_{n=1}^{\infty} F_n, \delta) \cap \big[\cup_{n=1}^{\infty} \overline{\operatorname{co} D_n}^{w^*}\big] = \emptyset,$$

that implies that

$$z_0^* \notin \overline{\bigcup_{n\in\mathbb{N}} \overline{\operatorname{co} D_n}^{w^*}}^{\gamma}. \tag{5.4}$$

We notice now that for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ we have $B_\alpha \subset \mathbf{B}\big([n_1, n_2, \ldots, n_{k_\alpha}]\big)$, and therefore (5.4) implies that $z_0^* \notin \overline{\bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \overline{\operatorname{co} B_\alpha}^{w^*}}^{\gamma}$, and the proof is over. □

We reach now a main result for us that extends [16, Theorem 3.3].

Theorem 5.6. *Let X be a Banach space. The following statements are equivalent:*

(i) *X does not contain ℓ^1;*
(ii) *for every w^*-compact subset K of X^* any w^*-K-analytic boundary B of K is strong;*
(iii) *for every w^*-compact subset C of X^* we have*

$$\overline{\operatorname{co} C}^{w^*} = \overline{\operatorname{co} C}. \tag{5.5}$$

Proof. The equivalence (i) $\Leftrightarrow$ (iii) is [16, Theorem 3.3]: we explicitly keep this equivalence here for further use. Since, every w^*-compact set C is w^*-K-analytic the implication (ii) $\Rightarrow$ (iii) is easily obtained when bearing in mind that C is a boundary of $K := \overline{\operatorname{co} C}^{w^*}$.

To finish we prove (i) $\Rightarrow$ (ii). Let B be a w^*-K-analytic boundary of K and let $T : \mathbb{N}^{\mathbb{N}} \to 2^B$ a w^*-compact valued upper semicontinuous map such that $B = \cup\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. For each $\alpha \in \mathbb{N}^{\mathbb{N}}$ the set $\{\beta \in \mathbb{N} : \beta \leq \alpha\}$ is compact in $\mathbb{N}^{\mathbb{N}}$ and therefore its image

$$\mathbf{B}(\alpha) := T\big(\{\beta \in \mathbb{N} : \beta \leq \alpha\}\big) = \big\{T(\beta) : \beta \in \mathbb{N}^{\mathbb{N}}, \beta \leq \alpha\big\}$$

is w^*-compact. It is easily seen that $\mathbf{B}$ is w^*-upper semicontinuous and since $T(\alpha) \subset \mathbf{B}(\alpha)$, for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ we obtain that $B = \cup\{\mathbf{B}(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. The definition of $\mathbf{B}$

clearly implies that $\mathbf{B}(\alpha) \subset \mathbf{B}(\beta)$ whenever $\alpha \leq \beta$. Observe that $\operatorname{co} B = \bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \operatorname{co}\mathbf{B}(\alpha)$. This allows us to finally obtain

$$\begin{aligned} K \supset \overline{\operatorname{co} B} = \overline{\bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \overline{\operatorname{co}\mathbf{B}(\alpha)}} &\overset{(iii)}{=} \overline{\bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \overline{\operatorname{co}\mathbf{B}(\alpha)}^{w^*}} \overset{Prop.5.5}{=} \\ = \overline{\bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \overline{\operatorname{co}\mathbf{B}(\alpha)}^{w^*}}^{\gamma} &\supset \overline{\operatorname{co} B}^{\gamma} \overset{Prop.5.2}{=} K. \end{aligned}$$

The proof is finished. $\square$

We stress that Godefroy proved in [14, Theorem III.3] that if $K \subset X^*$ is w^*-compact set and B is a weak-K-analytic boundary then B is strong. Notice that in general the hypothesis of w^*-K-analyticity is weaker than this of weak-K-analyticity: indeed, for every non Asplund space X the dual unit ball B_{X^*} is w^*-K-analytic but it is not weakly-K-analytic: indeed, if B_{X^*} were weakly K-analytic then B_{X^*} would be weakly Lindelöf, that is, X^* would be weakly Lindelöf and [6, Proposition 1.8] applies to conclude X is Asplund.

We need the following lemma in the proof of Proposition 5.8. Although the lemma easily follows from known duality arguments we include a proof to help with the reading of subsequent results.

Lemma 5.7. *Let X be a Banach space, $Y \subset X$ a subspace and let w^* denote the weak* topology in X^{**}. The following properties hold:*

(i) $\overline{B_Y}^{w^*} = B_{X^{**}} \cap \overline{Y}^{w^*}$*;*

(ii) *if Y is separable, $\ell^1 \not\subset Y$ and $D \subset Y$ is bounded, then $\left(\overline{D}^{w^*}, w^*\right)$ is Rosenthal compact.*

Proof. Property i) follows from duality arguments in the dual pair $\langle X^{**}, X^*\rangle$. Consider $B_Y = B_X \cap Y$ as a subset X^{**}. The bipolar theorem [25, §20.8.(5)] and the formulas for the polar of an intersection and a union, [25, §20.8.(8) and §20.8.(9)] allow us to write

$$\begin{aligned} \overline{B_Y}^{w^*} &= (B_Y)^{\circ\circ} = [(B_X \cap Y)^\circ]^\circ = \left[\overline{\operatorname{aco}\left((B_X)^\circ \cup Y^\circ\right)}\right]^\circ = \\ &= (B_X)^{\circ\circ} \cap Y^{\circ\circ} = B_{X^{**}} \cap \overline{Y}^{w^*}. \end{aligned}$$

Let us prove (ii). Write $i : Y \hookrightarrow X$ to denote the inclusion map from Y into X. The bi-adjoint map $i^{**} : Y^{**} \to X^{**}$ is injective and $w^*_{Y^{**}}$-to-w^* continuous. Thus we have

$$D \subset i^{**}(\overline{D}^{w^*_{Y^{**}}}) \subset \overline{D}^{w^*}.$$

Since $i^{**}(\overline{D}^{w^*_{Y^{**}}})$ is w^*-compact we obtain that

$$i^{**}(\overline{D}^{w^*_{Y^{**}}}) = \overline{D}^{w^*}$$

and therefore $i^{**} : (\overline{D}^{w^*_{Y^{**}}}, w^*_{Y^{**}}) \to (\overline{D}^{w^*}, w^*)$ is an homeomorphism. If Y is separable and $\ell^1 \not\subset Y$ then $(\overline{D}^{w^*_{Y^{**}}}, w^*_{Y^{**}})$ is Rosenthal compact, see [28], and therefore $(\overline{D}^{w^*}, w^*)$ is Rosenthal compact too and the proof of (ii) is finished. $\square$

Proposition 5.8. *Let X be a Banach space not containing ℓ^1. The following properties hold:*

(i) *For every countable and bounded set $D \subset X$ its w^*-closure $\overline{D}^{w^*}$ in X^{**} is contained in $(X^*,\gamma)'$ and it is γ-equicontinuous. Furthermore, $\big(\overline{D}^{w^*}, w^*\big)$ is Rosenthal compact.*

(ii) $(X^*,\gamma)' = \{x^{**} \in X^{**} : \lim_n x_n = x^{**}$ *for some* $(x_n)_n \subset X\}$, *where the limits involved are taken in the w^*-topology.*

(iii) *For every norm bounded countable set $D' \subset (X^*,\gamma)'$ there is a bounded and countable set $D \subset X$ such that $\overline{D'}^{w^*} \subset \overline{D}^{w^*}$. Hence, $\overline{D'}^{w^*}$ is a γ-equicontinuous subset of $(X^*,\gamma)'$ and $\big(\overline{D'}^{w^*}, w^*\big)$ is Rosenthal compact.*

Proof. The proof of (i) is as follows. If we consider D as a family of functionals defined on X^* then $D \subset X^{**}$ is γ-equicontinuous. Therefore its pointwise closure in $\mathbb{R}^{X^*}$, that coincides with its w^*-closure in X^{**}, denoted by $\overline{D}^{w^*}$, is γ-equicontinuous again and thus $\overline{D}^{w^*} \subset (X^*,\gamma)' \subset X^{**}$. Since $Y := [D]$ is separable and $\ell^1 \not\subset X$, we conclude that $\ell^1 \not\subset Y$ and therefore statement (ii) in Lemma 5.7 implies that $\big(\overline{D}^{w^*}, w^*\big)$ is Rosenthal compact and the proof of (i) is finished.

Statement (ii) easily follows from statement (i). Let us write

$$W := \{x^{**} \in X^{**} : w^* - \lim_n x_n = x^{**} \text{ for some } (x_n)_n \subset X\}.$$

Note that the sequences $(x_n)_n$ involved in the definition of W are necessarily bounded for the norm. Therefore such a sequence $(x_n)_n \subset (X^*,\gamma)'$ is γ-equicontinuous and its limit $x^{**} = w^* - \lim_n x_n$ is γ-continuous. This explain the inclusion $W \subset (X^*,\gamma)'$. The other way around: we prove now that $(X^*,\gamma)' \subset W$. If $x^{**} \in X^{**}$ is γ-continuous, then the there exist a norm bounded and countable subset $D \subset X$ such that

$$|x^{**}(x^*)| < 1 \text{ for each } x^* \in V(0, D, 1).$$

In other words x^{**} belongs to the absolute bipolar $D^{\circ\circ}$ of D in $(X^*,\gamma)'$. The separation (Bipolar) theorem, [25, §20.8.(5)], implies that $x^{**} \in \overline{\operatorname{aco} D}^{w^*}$, where $\operatorname{aco} D$ stands for the absolutely convex hull of D. Hence $x^{**} \in \overline{\operatorname{aco}_{\mathbb{Q}} D}^{w^*}$. Since $\operatorname{aco}_{\mathbb{Q}} D \subset X$ is countable and bounded, statement (i) applies to tell us that the space $\big(\overline{\operatorname{aco}_{\mathbb{Q}} D}^{w^*}, w^*\big)$ is Rosenthal compact, and in particular it is an angelic space, see [1]. Thus x^{**} is the w^*-limit of a sequence in $\operatorname{aco}_{\mathbb{Q}} D$. The latter says that $(X^*,\gamma)' \subset W$ and the proof for (ii) is finished.

We prove (iii). Take $D' \subset (X^*,\gamma)'$ countable and norm bounded. We can and do assume that $D' \subset B_{X^{**}}$. Statement (ii) ensures us of the existence of $F \subset X$ countable such that $D' \subset \overline{F}^{w^*}$ –we do not assume that F is bounded. If we write $Y := [F]$ then Y is separable and statement (i) in Lemma 5.7 says that $\overline{B_Y}^{w^*} = B_{X^{**}} \cap \overline{Y}^{w^*}$. Take $D \subset B_Y$ countable and norm dense: the inclusion $D' \subset \overline{B_Y}^{w^*}$ implies that $D' \subset \overline{D}^{w^*}$ and the proof of the first part of (iii) is finished. Once this is done statement (i) applied to $\overline{D}^{w^*}$ gives us the second part of (iii) and the proof is completed. $\square$

Our previous work allows us to prove that the $\sup - \limsup$ property (see the equality (SLS) in the introduction) can be extended to more general functions. This result appears as one of the ultimate forms of the vintage Rainwater's theorem on sequential convergence on extreme points, see Theorem 3.60 in [9].

Theorem 5.9. *Let X be a Banach space not containing ℓ^1, K a w^*–compact convex subset of the dual space X^* and $B \subset K$ a boundary of K. Let $(z_n)_n$ be a bounded* *sequence in* $(X^*, \gamma)'$ *then*

$$\sup_{b^* \in B} \{\limsup_{n\to\infty} z_n(b^*)\} = \sup_{x^* \in K} \{\limsup_{n\to\infty} z_n(x^*)\}. \tag{5.6}$$

Proof. Write l for the left hand side in (5.6). Since for any two sequences in $\mathbb{R}$ we have $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$, we easily obtain that

$$l = \sup_{b^* \in \operatorname{co} B} \{\limsup_{n\to\infty} z_n(b^*)\}. \tag{5.7}$$

On one hand (iii) in Proposition 5.8 says that the set $\{z_n : n \in \mathbb{N}\}$ is a γ-equicontinuous subset of $(X^*, \gamma)'$. On the other hand, Proposition 5.2 gives us the equality $K = \overline{\operatorname{co} B}^{\gamma}$. Fix $\varepsilon > 0$. Given any $x^* \in K$ there is $b^* \in \operatorname{co} B$ such that $|z_n(x^*) - z_n(b^*)| < \varepsilon$ for every $n \in \mathbb{N}$. The latter together with (5.7) imply that $\limsup_{n\to\infty} z_n(x^*) \le l + \varepsilon$. Since $x^* \in K$ is arbitrary we conclude that

$$\sup_{x^* \in K} \{\limsup_{n\to\infty} z_n(x^*)\} \le l + \varepsilon,$$

for every ε. Hence equality (5.6) holds and the proof is over. □

Remark 5.10. Note that if the Banach space X has the property that for every w^*-compact convex subset of the dual space X^* and any boundary B of K the thesis of Proposition 5.9 holds, then X does not contain ℓ^1. Indeed, equality (5.6) implies in particular that for every $x^{**} \in (X^*, \gamma)'$ we have

$$\sup_{x^* \in B} x^{**}(x^*) = \sup_{x^* \in K} x^{**}(x^*).$$

The latter says that $K = \overline{\operatorname{co} B}^{\gamma}$ and now Proposition 5.2 applies to give us that X cannot contain ℓ^1. □

We finish the paper with the following question that appears in [14, Question V.1] that seems to be still open.

Question 5.11. Let X be a separable Banach space with $\ell^1 \not\subset X$ and $\mathcal{E}$ the set of w^*-exposed points of B_{X^*}. Is it true that $B_{X^*} = \overline{\operatorname{co} \mathcal{E}}^{\|\cdot\|}$.

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