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A Sequential Property of Set-Valued Maps*

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1. INTRODUCTION

Let X and Y be Hausdorff topological spaces and $F\colon X \to Y$ a set-valued map.

A problem of continuing interest in analysis has been to study under what conditions, for a given point $x \in X$, we can replace Fx by another set Cx in order to have upper semicontinuity in x together with some compactness property in Cx. For instance, if we are dealing with selection problems for F it is useful to have some compact subset Cx with $Cx \subset Fx$ at a first stage, see [12]. If we are dealing with extension problems for the range space $F(X)$, it would be useful to have some compact subset Cx with $Fx \subset Cx$, see [3]. On the other hand, if we are looking for extension problems in the domain space $X \subset S$, it would be useful to have some compact subset Cs, for $s \in \bar{X}$, with $\overline{F(N \cap X)} \cap Cs \neq \varnothing$ for any neighbourhood N of s in S, see [17].

It is surprising that all these problems that have been studied by many different people have a common underlying structure. Our main objective in this paper is to reveal this common structure, and to study conditions to ensure the compactness of the sets Cx in each case.

In Section 2 we deal with decreasing sequences of sets and we study the compactness of their sets of cluster points. For this purpose we use some "sequential cluster sets" that have been previously used by Hansell, Jayne, Labuda, and Rogers [12], and by the authors in [3].

In Section 3 we apply these results to obtain the theorems on boundaries of upper semicontinuous set-valued maps stated in [12, 16].

In Section 4 we apply these techniques to the problem of extending the range space of a set-valued map obtaining an upper semicontinuous compact set-valued map. See our previous paper [3].

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In Section 5 we specialize these results to the particular case of describing countably determined spaces. The informal idea is that when we have big enough compact sets in a topological space, it should itself be countably determined. Our results improve those previously obtained by Mercourakis [18], even for Banach spaces with the weak topology.

In the last section we give a characterization of cosmic spaces through the property of having a family of compact sets covering it in a particular way. Our results here also improve those of Mercourakis [19].

All the topological spaces we shall use here are assumed to be Hausdorff spaces. Standard references for notation and concepts are [10, 15].

A subset M of a topological space X is said to be countably compact (or relatively countably compact) if every sequence in M has a cluster point in M (or in X, respectively).

A net $\{x_i : i \in I, \geqslant\}$ in a set X is said to be eventually in a subset $A \subset X$ if there is some index i_A in I such that

$$\{x_i : i \in I,\ i \geqslant i_A\} \subset A$$

and it is said that it is frequently in A if for every $j \in I$ there is some $i_j > j$ such that $x_{i_j} \in A$.

2. On Decreasing Sequences of Sets

Let X be a Hausdorff topological space and $\mathscr{A}$ decreasing sequence of non void subsets

$$\mathscr{A} = \{A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots\}$$

in X. If $\{x_i : i \in I, \geqslant\}$ is a net in X, we shall say that it is eventually in the sequence $\mathscr{A}$ if it is eventually in every A_n.

The following definition is a natural extension of the concept of relatively countably compact subset of X:

Definition 1. $\mathscr{A} = (A_n)$ is said to be relatively countably compact if every sequence which is eventually in $\mathscr{A}$ has a cluster point in X. If the cluster point can be chosen in some fixed subset S of X, $\mathscr{A}$ is said to be relatively countably compact in S. The boundary of $\mathscr{A}$ is the set of cluster points of $\mathscr{A}$:

$$B(\mathscr{A}) := \bigcap_{n=1}^{\infty} \overline{A_n}.$$

It is our intention here to give descriptions of the boundary $B(\mathscr{A})$ and to study its compactness behaviour for a given decreasing sequence of sets $\mathscr{A}$ in X.

The extension of the concept of relatively compact subset of X is the following:

DEFINITION 2. $\mathscr{A} = (A_n)$ is said to be relatively compact if every net which is eventually in $\mathscr{A}$ has a cluster point in X. When the cluster point can be chosen in some fixed subsets S of X, $\mathscr{A}$ is said to be relatively compact in S.

To take our discussion further, we shall need the "sequential cluster set"

$$C(\mathscr{A}, S) = \{y \in S : \exists (y_n) \text{ eventually in } \mathscr{A} \text{ and } y_n \rightsquigarrow y\},$$

where the symbol $\rightsquigarrow$ means that y is a cluster point of the sequence (y_n).

PROPOSITION 1. *If $\mathscr{A} = (A_n)$ is a relatively countably compact sequence in S and X is a regular space, then*

$$\overline{C(\mathscr{A}, S)} = \bigcap_{n=1}^{\infty} \overline{A_n}.$$

Proof. The inclusion $\overline{C(\mathscr{A}, S)} \subset \bigcap_{n=1}^{\infty} \overline{A_n}$ is always true. On the other hand, take any point x in $\bigcap_{n=1}^{\infty} \overline{A_n}$ and U any closed neighbourhood of x. For every positive integer p, there is some point y_p in $A_p \cap U$, and so the sequence $\{y_p := p = 1, 2, ...\}$ has a cluster point y in $C(\mathscr{A}, S)$. As U is closed, we have

$$y \in U \cap C(\mathscr{A}, S) \neq \varnothing$$

and therefore $x \in \overline{C(\mathscr{A}, S)}$, because the reasoning holds for every closed neighbourhood of x in X. ▌

$C(\mathscr{A}, S)$ has some kind of "upper semicontinuity":

PROPOSITION 2. *If $\mathscr{A} = (A_n)$ is a relatively countably compact sequence in S, then for every open set U of X that contains the set $C(\mathscr{A}, S)$ there is some positive integer p such that $A_p \subset U$.*

Proof. Let us suppose that the assertion does not hold. For every positive integer p, we could find a point

$$x_p \in A_p \setminus U.$$

The sequence $\{x_p : p = 1, 2, ...\}$ is eventually in $\mathscr{A}$ and it has a cluster point x in $C(\mathscr{A}, S) \subset U$. On the other hand

$$\{x_p : p = 1, 2, ...\} \subset X \backslash U$$

and thus

$$x \in \overline{\{x_p : p = 1, 2, ...\}} \subset X \backslash U.$$

This contradiction finishes the proof. ∎

Remark 1. If $C(\mathscr{A}, S)$ is a relatively compact set, then $\overline{C(\mathscr{A}, S)} = \bigcap_{n=1}^{\infty} \overline{A_n}$ without assuming any regularity on X. Indeed, it is enough to apply Proposition 2 to open disjoint neighbourhoods of $\overline{C(\mathscr{A}, S)}$ and any $x \in X \backslash \overline{C(\mathscr{A}, S)}$.

Let us now begin with the study of the compactness behaviour of $C(\mathscr{A}, X)$.

MAIN LEMMA. *If $\mathscr{A} = (A_n)$ is a relatively countably compact sequence, then the following conditions are equivalent*:

(1) *For every sequence (x_n) which is eventually in $\mathscr{A}$ the closure $\overline{\{x_n : n = 1, 2, ...\}}$ is countably compact.*

(2) *$C(\mathscr{A}, X)$ is countably compact.*

Proof. (1) ⇒ (2) Let (y_j) be a sequence in $C(\mathscr{A}, X)$. For every positive integer j, there is a sequence (x_n^j) in X eventually in $\mathscr{A}$, such that y_j is a cluster point of (x_n^j). There is an increasing sequence

$$n_1^j < n_2^j < \cdots < n_p^j \cdots, \qquad \text{with} \quad n_1^j = 1$$

of positive integers such that

$$x_n^j \in A_k \text{ whenever } n_k^j \leqslant n < n_{k+1}^j, \qquad k = 1, 2, ...$$

Let (z_n) be equal to the sequence

$$\Big\{x_2^1, x_2^1, ..., x_{n_2^1-1}^1, x_{n_2^1}^1, ..., x_{n_3^1-1}^1, x_{n_2^2}^2, ..., x_{n_3^2-1}^2, x_{n_3^1}^1, ..., x_{n_4^1-1}^1,$$
$$x_{n_3^2}^2, ..., x_{n_4^2-1}^2, x_{n_3^3}^3, ..., x_{n_4^3-1}^3, x_{n_4^1}^1, ...\Big\}.$$

It is quite clear that the sequence $\{z_n : n = 1, 2, ...\}$ is eventually in $\mathscr{A}$, therefore its closure is countably compact in X. But.

$$y_j \in \overline{\{z_n := 1, 2, ...\}}, \qquad j = 1, 2...$$

because

$$\{x_n^j : n \geqslant n_j^j\} \subset \{z_n : n = 1, 2, ...\}.$$

Therefore, (y_j) has a cluster point y in X. Moreover, $y \in C(\mathscr{A}, X)$ because it is also a cluster point of the sequence (z_n) which is eventually in $\mathscr{A}$. Indeed, if U is an open neighbourhood of y and p is any positive integer, there is some integer $j_p > p$ such that $y_{j_p} \in U$, and so for some $m > p$ it follows that $z_m \in U$, because U is an open set and y_{j_p} is a cluster point of the sequence (z_n).

(2) ⇒ (1) Let $\{x_n\}$ be a sequence which is eventually in $\mathscr{A}$. We have

$$\overline{\{x_n : n = 1, 2, ...\}} = \{x_n : n = 1, 2, ...\} \cup \bigcap_{n=1}^{\infty} \overline{\{x_n : p \geqslant n\}}$$

$$\subset \{x_n : n = 1, 2, ...\} \cup C(\mathscr{A}, X)$$

from where it easily follows that $\overline{\{x_n : n = 1, 2, ...\}}$ is countably compact in X. Indeed, if (y_n) is a sequence in $\overline{\{x_n : n = 1, 2, ...\}}$, and the set $\{y_n : n = 1, 2, ...\}$ is infinite, then either it is frequently in $C(\mathscr{A}, X)$ or it is contained in $\{x_n : n = 1, 2, ...\}$ after removing a finite number of points. In any case, some subsequence of (y_n) has a cluster point, because of the relatively countably compactness either of $C(\mathscr{A}, X)$ or the sequence $\mathscr{A}$. ∎

Remark 2. It follows from the proof that (1) ⇒ (2) that any cluster point of a sequence in $C(\mathscr{A}, X)$, for a relatively countably compact sequence $\mathscr{A}$, is also a point of $C(\mathscr{A}, X)$.

COROLLARY 1. *If $\mathscr{A}$ is relatively countably compact sequence in S and the closure of every sequence which is eventually in $\mathscr{A}$ is countably compact in X, then $C(\mathscr{A}, S)$ is relatively countably compact.*

Proof. It is enough to observe that $C(\mathscr{A}, S) = C(\mathscr{A}, X) \cap S$. ∎

To obtain countably compactness of $C(\mathscr{A}, S)$ we need something more.

COROLLARY 2. *If $\mathscr{A}$ is a relatively countably compact sequence in S, then the following conditions are equivalent:*

(1) *For every sequence $\{x_n\}$ which is eventually in $\mathscr{A}$ the set $\overline{\{x_n : n = 1, 2, ...\}} \cap S$ is countably compact in X*

(2) *$C(\mathscr{A}, S)$ is countably compact.*

Proof. This is exactly the same as the proof of the Main Lemma, if we use the fact that the sequences are in S. ∎

For spaces in which the relatively countably compact sets are relatively compact much more can be said. One obvious class of spaces with this property is provided by the dual Banach spaces endowed with their weak* topologies, or more generally by the locally convex spaces which are quasi-complete for the Mackey topology. It is easily seen that all Dieudonné complete spaces have this property [12].

COROLLARY 3. *Let X be a space in which the relatively countably compact subsets are relatively compact. If $\mathscr{A} = (A_n)$ is a relatively countably compact sequence in S, then it is a relatively compact sequence in $\bar{S}$, and $C(\mathscr{A}, S)$ is a relatively compact subset of X. Further, the sequence of closures*

$$\bar{\mathscr{A}} = \{\bar{A}_n : n = 1, 2, \ldots\}$$

is a relatively compact sequence in $\bar{S}$ when X is assumed to be regular.

Proof. By Corollary 1 we know that $C(\mathscr{A}, S)$ is relatively compact in X. Therefore the boundary

$$B(\mathscr{A}) = \bigcap_{n=1}^{\infty} \overline{A_n} = \overline{C(\mathscr{A}, S)} = \bigcap_{n=1} \overline{A_n} \cap \bar{S}$$

is a compact subset of X (Remark 1). If $\{x_1, i \in I, \geqslant\}$ is a net which is eventually in $\mathscr{A}$, it has a cluster point in $B(\mathscr{A})$, because the family of subsets

$$\{F_i = \overline{\{x_j : j \in I, j \geqslant i\}} \cap B(\mathscr{A}), i \in I\}$$

is a decreasing family of non-empty closed subsets of the compact set $B(\mathscr{A})$. Therefore

$$\bigcap \{F_i : i \in I\} \cap B(\mathscr{A}) \neq \varnothing$$

which implies the relatively compactness of $\mathscr{A}$ in $\bar{S}$.

Finally if (x_n) is a sequence in the regular space X, which is eventually in $\bar{\mathscr{A}}$, then (x_n) has a cluster point in the compact set $\overline{C(\mathscr{A}, S)}$. Otherwise, there is an open neighbourhood U of $\overline{C(\mathscr{A}, S)}$ such that (x_n) is eventually in $X \backslash \bar{U}$, and so Proposition 2 gives a contradiction, because there is some positive integer p such that $A_p \subset U$. ∎

COROLLARY 4. *Let X be a space in which the relatively countably compact subsets are relatively compact and with the property:*

> *Given a relatively compact subset K of X and $x \in \bar{K}$, there is a countable subset D of K such that $x \in \bar{D}$.*

If $\mathscr{A} = (A_n)$ is a relatively countably compact sequence then $C(\mathscr{A}, X)$ is a compact subset of X.

Proof. The main lemma says that $C(\mathscr{A}, X)$ is countably compact. The remark that follows it implies, under our conditions, that $C(\mathscr{A}, X)$ is closed. Indeed, if $x \in \overline{C(\mathscr{A}, X)} \setminus C(\mathscr{A}, X)$, then there is a sequence $(x_n) \subset C(\mathscr{A}, X)$ such that $x \in \overline{\{x_n : n = 1, 2, ...\}}$. But $x \neq x_n$ for $n = 1, 2, ...$, therefore, x is a cluster point of (x_n), and so a point of $C(\mathscr{A}, X)$. Consequently, $C(\mathscr{A}, X)$ is closed. ∎

Let us recall that a topological space X is said to be angelic if the closure of each relatively countably compact subset A of X is compact, and consists precisely of the limits of sequences from A. The class of angelic spaces is large, as has been shown in [2, 3, 11, 20, 21], and it contains many locally convex spaces in their weak topologies.

COROLLARY 5. *Let X be an angelic space. If $\mathscr{A} = (A_n)$ is a relatively countably compact sequence in the set S of X, then it is a relatively compact sequence in S and $C(\mathscr{A}, S)$ is compact in X with*

$$C(\mathscr{A}, S) = \bigcap_{n=1}^{\infty} \overline{A_n} \subset S.$$

Proof. It will be enough to prove that $C(\mathscr{A}, X) \subset S$. If $x \in C(\mathscr{A}, X)$, then the angelic property gives us a sequence (x_n) that is eventually in $\mathscr{A}$ and converges to x. This sequence also has a cluster point in S, and so $x \in S$. ∎

3. ON BOUNDARIES OF UPPER SEMICONTINUOUS MULTIVALUED MAPS

A set-valued map F from a topological space X into a topological space Y is said to be upper semicontinuous at a point x_0 of X if, for every open set U in Y containing $F(x_0)$, there is a neighbourhood N of x_0 with $F(N) \subset U$. The function F is said to be upper semicontinuous on X, if F is upper semicontinuous at each point of X, or equivalently, if $\{x \in X : F(x) \subset U\}$ is an open set in X whenever U is an open set in Y.

Let $F: X \to Y$ be a set-valued map. Following I. Labuda [16] a *cap* (of upper semicontinuity) of F at x_0 is a set K in Y such that the map F is upper semicontinuous when K replaces $F(x_0)$, i.e., when the map

$$F(x) = \begin{cases} K & \text{for} \quad x = x_0 \\ F(x) & \text{otherwise} \end{cases}$$

is upper semicontinuous at x_0. The outer part at x_0 is the map

$$x \to F(x) \backslash F(x_0)$$

and a set K in Y is a *peak* of F at x_0 if K is a cap of the outer part of F at x_0 and K is contained in $F(x_0)$ [16, 12]. In other words, a set $K \subset F(x_0)$ is a peak of F at x_0 if for every open set V containing K there exists a neighbourhood N of x_0 such that $F(N) \backslash F(x_0) \subset V$.

Let us note that if F admits a peak at x_0, then it is automatically upper semicontinuous at x_0, and, if F is upper semicontinuous at x_0, then $F(x_0)$ itself is a trivial peak for F at x_0. We are concerned now with the problem of selecting a compact peak of F at x_0. The following striking theorem was stated by Choquet [4, p. 70]:

A multivalued map F between metric spaces that is upper semicontinuous at x_0 admits a compact peak at x_0.

Here the compact peak positively appears as a consequence of upper semicontinuity alone.

In 1977, Dolecki introduced in [6, 7] the set

$$\text{Frac } F(x_0) = \cap \{\overline{F(U) \backslash F(x_0)} : U \text{ is a neighbourhood of } x_0\}$$

which he called the "active frontier" of F at x_0, and he showed that $\text{Frac } F(x_0)$ is compact when Y is a metric space and x_0 has a countable neighbourhood basis; Dolecki and Rolewicz showed in [9] under the same assumption that $\text{Frac } F(x_0) \subset F(x_0)$ and finally, Dolecki and Lechicki proved in [8] that $\text{Frac } F(x_0)$ has Choquet's property. Thus

THEOREM (Choquet–Dolecki). *Let x_0 be a first countable point in X and Y a metric space. If F is a multivalued map from X into Y that is upper semicontinuous at x_0, then* $\text{Frac } F(x_0)$ *is compact, and moreover, it is the smallest peak of F at x_0.*

Recently the notion of the active frontier has been successfully applied in a series of papers by Jayne and Rogers [13, 14], Hansell, Jayne, Rogers, and Labuda [12, 16]. In the last papers the compactness of the active frontier, as well as the compactness of the set $K(x_0) = \text{Frac } (x_0) \cap F(x_0)$ are obtained under weak assumptions on X, Y, and F. Previous work by Jayne and Rogers [13, 14] obtains powerful selection results for upper semicontinuous set-valued maps in cases when this boundary is compact.

It is our aim here to show how to obtain all the former results when we use our main lemma on decreasing sequences of sets stated in the last paragraph.

In what follows x_0 will be a point in the topological space X with a countable basis of neighbourhoods $\{N_i : i \in \mathbb{N}\}$, and F a set-valued map from X into Y. We write $\mathscr{F}$ to denote the decreasing sequence

$$F_p = F(N_p) \setminus F(x_0), \qquad p = 1, 2, \ldots.$$

LEMMA 1. *If F is upper semicontinuous at x_0, then the sequence $\mathscr{F} = (F_n)$ is relatively countably compact in $F(x_0)$.*

Proof. If (y_n) is a sequence which is eventually in $\mathscr{F}$, then it satisfies the contitions of lemma 3 in [12], an so (y_n) has a cluster point in $F(x_0)$. ∎

Applying our results in the first section we have

THEOREM 1. (Labuda [16]). *If F is upper semicontinuous at the first countable point x_0 of X and Y is a space in which the relatively countably compact subsets are relatively compact, then the set $C(\mathscr{F}, F(x_0))$ is a relatively compact peak of F at x_0.*

Proof. Our Corollary 1 says that $C(\mathscr{F}, F(x_0))$ is relatively countably compact and therefore relatively compact. Our Proposition 2 says that $C(\mathscr{F}, F(x_0))$ is a peak of F at x_0. ∎

Let us observe that we already know that

$$\overline{C(\mathscr{F}, F(x_0))} = \text{Frac}\, F(x_0) \subset \overline{F(x_0)}.$$

Therefore, in order to obtain a compact peak of F at x_0 we need something more to control the closure of the set $C(\mathscr{F}, F(x_0))$ inside $F(x_0)$. A natural condition is that $F(x_0)$ be closed in the G_δ-topology whenever Y is a regular space [12, 16]. In this case $\text{Frac}\, F(x_0) \subset F(x_0)$ and therefore $\text{Frac}\, F(x_0) = \overline{C(\mathscr{F}, F(x_0))}$ is a compact peak of F at x_0.

More restrictions on Y give the following,

THEOREM 2. (Hansell–Jayne–Labuda–Rogers [12]). *If F is upper semicontinuous at the first countable point x_0 of X and Y is an angelic space, then the set $C(\mathscr{F}, F(x_0))$ is a compact peak of F at x_0 and*

$$C(\mathscr{F}, F(x_0)) = \text{Frac}\, F(x_0) \subset F(x_0).$$

Proof. It is enough to apply Corollary 5 to reach the conclusion. ∎

For more applications and discussion of these results see [12, 16].

4. Generating Upper Semicontinuous Set-Valued Maps

In what follows x_0 will be a first countable point in the topological space X and $\{N_i : i \in \mathbb{N}\}$ will be a basis of neighbourhoods of x_0 in X. Let F be a set-valued map from X into a topological space Y. Let us write $\mathscr{F}$ to denote the sequence

$$F_p = F(N_p), \qquad p = 1, 2, \ldots.$$

We are going to apply our previous results to study the problem of constructing a compact cap of the map F at x_0 verifying $F(x_0) \subset K$.

We shall say that F is *countably subcontinuous* at x_0 when the decreasing sequence $\mathscr{F}$ is relatively countably compact [17]. Of course in that case, $F(x_0)$ is relatively countably compact.

THEOREM 3. *If the set-valued map F is countably subcontinuous at x_0, then the following are equivalent*:

(i) *$C(\mathscr{F}, Y)$ is countably compact cap of $\mathscr{F}$ at x_0.*

(ii) *For every sequence (y_n) eventually in $\mathscr{F}$, the closure $\overline{\{y_n : n = 1, 2, \ldots\}}$ is countably compact.*

Proof. $C(\mathscr{F}, Y)$ is always a cap of F at x_0 after Proposition 2. Thus, the former equivalence is none other than our main lemma for this particular situation. ∎

COROLLARY 3.1. *Let X be a first countable topological space and Y a regular space in which the relatively countably compact subsets are relatively compact. If F is a set-valued mapping which is countably subcontinuous at each point of X, then there exists an upper semicontinuous compact set-valued map $\tilde{F}$ from X into Y such that*

$$Fx \subset \tilde{F}x = \bigcap \{\overline{F(U)} : U \text{ neighbourhood of } x\}.$$

Proof. It is enough to apply the last theorem at each point x of X to obtain a countably compact cap Cx of F at x. The closure of this cap is the compact cap we are looking for. Indeed $\overline{Cx}$ is none other than the boundary.

$$\bigcap \{\overline{F(U)} : U \text{ is a neighbourhood of } x\}.$$

The upper semicontinuity of the map $\tilde{F}$ easily follows from the compactness of $\tilde{F}x$ together with the regularity of Y. ∎

A web-compact topological space Y is defined as the closure of the image of a set-valued mapping $F: X \to Y$ which is countably subcontinuous at each point of a separable and metrizable space X [20]. If in Y the relatively countably compact subsets are relatively compact, the last corollary says that Y has a dense countably determined subspace. This result was proved by the authors in [3] in order to obtain the fact that compact subsets of $C_p(X)$ are Gul'ko compact spaces.

Note. Let us mention here that the problem of extending a given upper semicontinuous set-valued map F defined on a subset A of X to some point $x \in X \setminus A$ has been recently solved by A. Lechicki and S. Levi [17] using the same techniques. During the preparation of this paper we have received a preprint [17] of their work, where, among other results, details of the former problem can be found.

5. Countably Determined Structures

If Y is a countably determined topological space, there is an upper semicontinuous compact set-valued mapping T from a separable metric space M onto Y. The upper semicontinuity of T ensures that $T(K)$ is compact whenever K is a compact subset of M. Therefore we have a mapping from the lattice of the compact sets of M, $\mathscr{K}(M)$, into the lattice of the compact sets of Y, $\mathscr{K}(Y)$, namely

$$K_T: \mathscr{K}(M) \to \mathscr{K}(Y)$$

defined by $K_T(A) := T(A)$, $A \in \mathscr{K}(M)$. Obviously, this mapping verifies that $K_T(A_1) \subset K_T(A_2)$ if $A_1 \subset A_2$ and $Y = \bigcup \{K_T(A): A \in \mathscr{K}(M)\}$.

In general, a family of compact sets $\{Y_K: K \in \mathscr{K}(M)\}$ *in* Y *verifying that* $Y_{K_1} \subset Y_{K_2}$ *whenever* $K_1 \subset K_2$ *will be called a partially ordered family of compact sets based on* M.

In general, it is not true that a partially ordered family of compact sets in Y, based on M and covering Y gives a countably determined structure on the space Y [20, 23].

In what follows we shall consider the Hausdorff metric d^H on $\mathscr{K}(M)$:

$$d^H(A, B) = \sup\{d(a, B), d(A, b): a \in A, b \in B\},$$

where d is the metric of the space M. $(\mathscr{K}(M), d^H)$ is a separable metric space [5].

THEOREM 4. *Let* Y *be a regular topological space. The following are equivalent*:

(i) *Y is countably determined.*

(ii) *Y is Lindelöf and there is a partially ordered family of compact sets in Y, based on a separable metric space M,* $\{Y_K : K \in \mathscr{K}(M)\}$, *that covers Y, i.e.,* $Y = \bigcup\{Y_K : K \in \mathscr{K}(M)\}$.

(iii) *Y is Dieudonné complete and there is a partially ordered family of compact sets in Y, based on a separable metric space M,* $\{Y_K : K \in \mathscr{K}(M)\}$, *that covers Y.*

(iv) *The relatively countably compact subsets in Y are relatively compact and there is a partially ordered family of compact sets in Y, based on a separable metric space M,* $\{Y_K : K \in \mathscr{K}(M)\}$, *that covers Y.*

Proof. (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are obvious.

(iv) ⇒ (i) The mapping F from the metric space $(\mathscr{K}(M), d^H)$ onto Y defined by $F(K) := Y_K$, $K \in \mathscr{K}(M)$, verifies the conditions of our Corollary 3.2. Indeed, if $\{K_n : n = 1, 2, ...\}$ is a sequence in $\mathscr{K}(M)$ that d^H-converges to K in $\mathscr{K}(M)$, then the set

$$Z = \bigcup \{K_n : n = 1, 2, ...\} \cup K$$

is a compact subset of M, and so

$$\bigcup \{F(K_n) : n = 1, 2, ...\} \subset Y_Z$$

and the conditions of the corollary hold. So there is an upper semi-continuous compact set-valued mapping from the separable metric space $(\mathscr{K}(M), d^H)$ onto the space Y which is thus countably determined. ∎

Let us remark that in the work by Mercourakis [18] the above structure has been applied to the weak topology of a Banach space. Nevertheless our Theorem 4 for a Banach space E with the weak topology says more than Mercourakis result in one direction:

COROLLARY 4.1. *Let Y be a Banach space. The following are equivalent:*

(i) *Y is weakly countably determined.*

(ii) *There is a partially ordered family of weakly compact subsets in Y,* $\{Y_K : K \in \mathscr{K}(M)\}$, *based on a separable metric space M that is total in Y.*

Proof. It is enough to apply that the weakly relatively countably compact subsets of Y are weakly relatively compact, together with the fact that, if some dense subspace of Y is weakly countably determined, then the same is true for the whole space Y [24]. ∎

For other characterizations with the same spirit see [22].

In the case of K-analytic structures the space M can be chosen to be equal to $\mathbb{N}^{\mathbb{N}}$ with the product topology of discrete spaces. In that case we have a fundamental family of compact sets of $\mathbb{N}^{\mathbb{N}}$ given by

$$A_\alpha = \{\beta = \mathbb{N}^{\mathbb{N}} : \beta \leqslant \alpha\}, \qquad \alpha \in \mathbb{N}^{\mathbb{N}},$$

where for two sequences $\alpha = (a_n)$ and $\beta = (b_n)$ the symbol $\leqslant$ means

$$\beta \leqslant \alpha \Leftrightarrow b_n \leqslant a_n, \qquad n = 1, 2, \ldots.$$

For our purpose, to give a partially ordered family of compact subsets based on $\mathbb{N}^{\mathbb{N}}$ is equivalent to giving a family $\{Y_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact subsets of Y such that

$$Y_\alpha \subset Y_\beta \qquad \text{whenever} \quad \alpha \leqslant \beta \text{ in } \mathbb{N}^{\mathbb{N}}.$$

THEOREM 5. *Let Y be a regular topological space. The following are equivalent*:

(i) *Y is K-analytic.*

(ii) *Y is Lindelöf and there is a partially ordered family of compact sets $\{Y_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ which covers Y.*

(iii) *Y is Dieudonné complete and there is a partially ordered family of compact sets $\{Y_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ which covers Y.*

(iv) *The relatively countably compact subsets in Y are relatively compact and there is a partially ordered family of compact sets $\{Y_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ which covers Y.*

Previous results of this kind can be found in [1].

6. Cosmic Topological Spaces

Let us recall that a topological space Y is said to be a cosmic space if it is the continuous image of a separable metric space. For any regular cosmic space Y the complement of the diagonal in the product $Y \times Y$ is a Lindelöf space, and so Y is submetrizable, that is, there is a metrizable topology on Y coarser than the original one.

We are going to apply our former results to give conditions to assure that a given submetrizable space is going to be a cosmic space.

THEOREM 6. *Let Y be a submetrizable topological space. The following conditions are equivalent*:

(i) *Y is a cosmic space.*

(ii) *There is a partially ordered family of compact sets,* $\{Y_K: K \in \mathscr{K}(M)\}$, *based on a metrizable and separable space M such that* $Y = \bigcup\{Y_K: K \in \mathscr{K}(M)\}$.

Proof. The only non-trivial implication is (ii) $\Rightarrow$ (i). Let us consider the product space $S := (\mathscr{K}(M), d^H) \times (Y, d')$, where d' is a submetric for Y. Let us consider

$$\Sigma = \{(K, y) \in S : y \in Y_K\}$$

and the onto mapping $\varphi: \Sigma \to Y$ defined by $\varphi(K, y) = y$. It is quite clear that φ is continuous because d metrizes every compact subset of Y.

Obviously $\varphi(\Sigma) = Y$. Moreover, Σ is a separable subspace of the separable metric space S. Indeed, the space $(\mathscr{K}(M), d^H)$ is separable, and (Y, d') is a Lindelöf space, according to Theorem 4 and consequently, is also separable. ∎

In the analytic case the last result reads as follows [2, Theorem 14]:

THEOREM 7. *Let Y be a submetrizable topological space. The following conditions are equivalent*:

(i) *Y is an analytic space.*

(ii) *There is a partially ordered family of compact sets* $\{Y_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *which covers Y.*

Let us remark here that Theorem 7 for metric spaces has recently been proved by Mercourakis [19]. Another application in the same spirit is the following result.

THEOREM 8. *Let Y be a countably determined topological space. If* $\mathfrak{T}$ *is any coarser topology with a basis of cardinality* α, *then there is a dense subset of Y with cardinality less than or equal to* α.

Proof. Let M be a separable metric space and $T: M \to Y$ an upper semicontinuous compact set-valued map such that $Y = \bigcup\{Tx: x \in M\}$. Let us consider the product space $M \times Y[\mathfrak{T}]$ and its subspace $\Sigma = \{(x, y) : y \in Tx\}$ together with the onto mapping $\varphi: \Sigma \to Y$ given by $\varphi(x, y) = y$. The upper semicontinuity of T implies that φ is continuous, thus we obtain the result because Σ has a basis of cardinality α. ∎

This result was stated by Talagrand [24, Théorème 2.4] in the case in which the topology $\mathfrak{T}$ is regular.

Note. Some different applications of the results given here, in the framework of locally convex spaces, will appear elsewhere.

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