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# Strictly convex norms and topology

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Abstract

We introduce a new topological property called $(*)$ and the corresponding class of topological spaces, which includes spaces with $G_\delta$-diagonals and Gruenhage spaces. Using $(*)$, we characterize those Banach spaces which admit equivalent strictly convex norms, and give an internal topological characterization of those scattered compact spaces $K$, for which the dual Banach space $C(K)^*$ admits an equivalent strictly convex dual norm. We establish some relationships between $(*)$ and other topological concepts, and the position of several well-known examples in this context. For instance, we show that $C(\mathcal{K})^*$ admits an equivalent strictly convex dual norm, where $\mathcal{K}$ is Kunen's compact $S$-space. Also, under additional axioms, we provide examples of compact scattered non-Gruenhage spaces of cardinality $\aleph_1$ having $(*)$.

## 1. *Introduction*

All Banach spaces considered in this paper are real and, unless explicitly stated otherwise, all topological spaces are Hausdorff. Throughout this paper, we shall be defining new norms on existing Banach spaces. These new norms will always be equivalent to the given canonical norms. Banach space notation and terminology is standard throughout.

A norm $\|\cdot\|$ on a Banach space $X$ is said to be *strictly convex* (or rotund) if, given $x, y \in X$ satisfying $\|x\| = \|y\| = \|\frac{1}{2}(x+y)\|$, we have $x = y$ (see [**5**, p. 404]). Geometrically, this means that the unit sphere $S_X$ of $X$ in this norm has no non-trivial line segments, or, equivalently, every element of $S_X$ is an extreme point of the unit ball $B_X$.

Clearly, there are many Banach spaces whose natural norms are not strictly convex. However, by appealing to the linear and topological properties of a given space, it is often possible to define a new norm that is strictly convex. Changing the norm in this way is often called *renorming*. In certain cases, we would like the new norm to possess, in addition, some form of lower semicontinuity. For instance, we may wish for a norm on a dual space $X^*$ to be $w^*$-lower semicontinuous, so that it is the dual of some norm on $X$. Alternatively, we may like a norm on a $C(K)$-space to be lower semicontinuous with respect to the topology of pointwise convergence. Such additional requirements can make norms much more difficult to construct, but they do bestow certain benefits. For example, if $X^*$ can be endowed with a strictly convex dual norm, then the predual norm on $X$ is automatically Gâteaux smooth, by virtue of Šmulyan's Lemma; cf. [**8**, Theorem I.1.4].

Despite the natural and intuitive nature of strict convexity, the question of whether a Banach space may be given such a norm turns out to be rather difficult to answer in general. A number of mathematicians have sought to establish more easily verifiable sufficient conditions and necessary conditions for a space to admit a strictly convex norm. Before outlining this paper, we mention some of the contributions to this collective endeavour. Specialists will realize that

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it is possible to endow many (but not all) of the spaces below with norms sporting stronger properties than strict convexity, but we prefer not to dwell on such properties here. For a fuller discussion, we refer the reader to [**8**, **37**].

In [**5**, Theorem 9], it is shown that every separable Banach space admits a strictly convex norm. By following Clarkson's proof, Day showed that if a Banach space $X$ is separable, then $X^*$ admits a strictly convex dual norm [**6**, Theorem 4]. If $\Gamma$ is a set, then $c_0(\Gamma)$ admits a strictly convex norm [**6**, Theorem 10] (see also [**8**, Definition II.7.2]). On the other hand, if $\Gamma$ is uncountable, then the space $\ell^c_\infty(\Gamma)$ of countably supported bounded functions $x : \Gamma \longrightarrow \mathbb{R}$ with the supremum norm is simply too big to admit a strictly convex norm [**6**, Theorem 8] (see [**8**, Theorem II.7.12]).

Amir and Lindenstrauss showed that if $X$ is weakly compactly generated (WCG), then both $X$ and $X^*$ admit a strictly convex norm and a strictly convex dual norm, respectively [**2**, Theorem 3]. These results rely on the fact that if a bounded linear map $T : X \longrightarrow Y$ is injective and $Y$ admits a strictly convex norm, then so does $X$. If $X$ is WCG, then we can find such maps on both $X$ and $X^*$, where $Y = c_0(\Gamma)$ for some $\Gamma$. Then [**6**, Theorem 10] can be applied.

At the time, such a 'linear transfer' into some $c_0(\Gamma)$ was the only way of showing that spaces admitted strictly convex norms. Moreover, $\ell^c_\infty(\Gamma)$, $\Gamma$ uncountable, was the 'smallest' space known not to admit a strictly convex norm. In [**7**], the authors construct an increasing transfinite sequence $(X_\alpha)_{1\leqslant\alpha<\omega_1}$ of spaces of Baire-1 functions on $[0, 1]$, all admitting strictly convex norms, and none admitting a bounded linear injective map into any $c_0(\Gamma)$, provided $\alpha \geqslant 2$. Moreover, by refining Day's argument [**6**, Theorem 8], they showed that the union $Y = \bigcup_{\alpha<\omega_1} X_\alpha$ does not admit a strictly convex norm, and that there is no bounded linear injective map from $\ell^c_\infty([0, 1])$ into $Y$.

The fact that the dual of every WCG space admits strictly convex dual norm, with a necessarily Gâteaux smooth predual norm, prompted Lindenstrauss to conjecture that if $X$ admits a Gâteaux smooth norm, then it must embed as a subspace of some WCG space [**20**]. Mercourakis provided a negative answer to this conjecture by showing that if $X$ is a *weakly countably determined* (WCD) space, then both $X$ and $X^*$ admit strictly convex norms [**21**, Theorems 4.6 and 4.8], by virtue of linear transfers (although not into $c_0(\Gamma)$ in general).

Papers such as [**7**, **21**] suggest that there is no simple way of determining whether or not a general Banach space may be equipped with a strictly convex norm, in terms of its linear topological structure. Since then, the problem of classifying Banach spaces admitting strictly convex norms has been approached from a more topological perspective, and particular attention has been paid to strictly convex dual norms and $C(K)$-spaces. Any Banach space $X$ embeds isometrically into $C(B_{X^*}, w^*)$, and this fact enables certain results about $C(K)$-spaces to be generalized to all Banach spaces, by phrasing them in terms of the topological structure of $(B_{X^*}, w^*)$.

For example, if $X^*$ admits a strictly convex *dual* norm, then $(B_{X^*}, w^*)$ is fragmentable [**30**, Theorem 1.1]. We can say that a topological space is *fragmentable* if it admits, for each $n \in \mathbb{N}$, an increasing well-ordered family of open subsets $(U_\xi)_{\xi<\lambda_n}$, with the property that given distinct points $x$ and $y$, we can find some $n_0$ and $\xi < \lambda_{n_0}$ such that $\{x, y\} \cap U_\xi$ is a singleton [**29**, Theorem 1.9]. The idea of point separation features throughout this paper. Indeed, the notion of strict convexity can be viewed as a form of point separation.

The necessity condition above is far from sufficient however. The class of fragmentable spaces is very large and includes, for instance, all scattered spaces. Recall that a topological space is *scattered* if every non-empty subspace admits a relatively isolated point. In the year before [**21**] appeared, Talagrand showed that the space $C(\omega_1 + 1)^*$ does not admit a strictly convex dual norm [**39**, Théorème 3], where $\omega_1$ is the first uncountable ordinal considered in its (scattered) order topology. On the other hand, the dual unit ball $(B_{C(\omega_1+1)^*}, w^*)$ is fragmentable [**29**, Theorem 3.1].

The next significant sufficiency condition we mention requires a definition.

DEFINITION 1.1. A compact space $K$ is *descriptive* if it admits a *$\sigma$-isolated network*, that is to say, a family $\mathscr{N} = \bigcup_{n=1}^{\infty} \mathscr{N}_n$ of subsets of $K$, satisfying the following conditions:

(i) $N \cap \overline{\bigcup \mathscr{N}_n \setminus \{N\}}$ is empty whenever $N \in \mathscr{N}_n$ and $n \in \mathbb{N}$;

(ii) if $x \in U \subseteq K$, where $U$ is open, then there exists $n \in \mathbb{N}$ and $N \in \mathscr{N}_n$ such that $x \in N \subseteq U$.

This topological covering property arose out of the theory of 'generalized metric spaces' [**11**]. The class of descriptive compact spaces is large. For example, if $X$ is WCD, then $(B_{X^*}, w^*)$ is descriptive [**28**, Corollary 2.4; **38**, Théorème 3.6]. In [**28**, Theorem 3.3], Raja showed that if $K$ is *descriptive*, then $C(K)^*$ admits a strictly convex dual norm. This result can be adapted to give a sufficient condition which applies to a wide class of dual Banach spaces [**24**, Theorem 1.3], including duals of WCD spaces. We remark that a compact scattered space $K$ is descriptive if and only if it is *$\sigma$-discrete*, that is, $K = \bigcup_{n=1} D_n$, where each $D_n$ is discrete in its relative topology. This fact follows from [**28**, Lemma 2.2].

Despite these advances, there is a very large gap between the class of descriptive spaces and $\omega_1 + 1$ and the more general class of fragmentable spaces. Some years prior to the publication of Raja [**28**], Haydon constructed some strictly convex dual norms on spaces of the form $C(K)^*$, where the $K$ are 1-point compactifications of certain trees in their interval topologies [**15**, Theorem 7.1]. It turns out that some of these spaces are not descriptive, so Haydon's sufficient condition is not covered by Raja's umbrella.

In [**33**, Theorem 6], the second author generalized Haydon's result by characterizing those trees for which the associated spaces $C(K)^*$ admit strictly convex dual norms. Later, in [**34**], this order-theoretic characterization was reproved in internal, topological terms. To state this result, we need another definition.

DEFINITION 1.2. A compact space $K$ is called *Gruenhage* if there exists a sequence $(\mathscr{U}_n)_{n=1}^{\infty}$ of families of open subsets of $K$, and sets $R_n$, $n \geqslant 1$, with the property that

(i) if $x, y \in K$ are distinct, then there exist $n \in \mathbb{N}$ and $U \in \mathscr{U}_n$, such that $\{x, y\} \cap U$ is a singleton and

(ii) $U \cap V = R_n$ whenever $U, V \in \mathscr{U}_n$ are distinct.

This definition is equivalent to the original [**12**, p. 372] (see [**34**, Proposition 2]). Every descriptive compact space is Gruenhage [**34**, Corollary 4].

THEOREM 1.3 [**34**, Theorems 7 and 16]. *Let $K$ be compact. Then the following statements hold.*

(i) *If $K$ is Gruenhage, then $C(K)^*$ admits a strictly convex dual lattice norm.*

(ii) *If $K$ is the 1-point compactification of a tree and $C(K)^*$ admits a strictly convex dual norm, then $K$ is Gruenhage.*

Theorem 1.3(i) can be adapted to give a sufficient condition [**34**, Corollary 10] which applies to a class of dual Banach spaces even wider than that covered by Oncina and Raja [**24**, Theorem 1.3]. There are other instances of necessity besides Theorem 1.3(ii). For instance, if the Banach space $X$ has an (uncountable) unconditional basis, then $X^*$ admits a strictly convex dual norm if and only if $(B_{X^*}, w^*)$ is Gruenhage (equivalently, if $(B_{X^*}, w^*)$ is descriptive)

[**36**, Theorem 6]. Despite some courageous attempts, it was not possible to prove the converse implication of Theorem 1.3(i). Many of the results of this paper are the product of efforts to resolve this difficulty.

This paper is organized as follows. In Section 2, we introduce a generalization of Gruenhage's property, labelled $(*)$ (Definition 2.6), and use it to give a characterization of Banach spaces that admit a strictly convex norm satisfying some additional lower semicontinuity property (Theorem 2.7). This characterization attempts to topologize as much as possible the geometric condition of strict convexity. In Section 3, we use $(*)$ to find an analogue of Theorem 1.3 that applies to all scattered compact spaces (Theorem 3.1). This class is significant in Banach space theory because $C(K)$ is an *Asplund* space if and only if $K$ is scattered. In doing so, we show that $(*)$ comes close to providing a complete topological characterization of those $K$, for which $C(K)^*$ admits a strictly convex dual norm. In Section 4, we establish some of the topological properties of $(*)$ and its position in the wider context of covering properties, and provide some examples of scattered compact spaces, some of which having $(*)$ and others not. In particular, we give examples of scattered non-Gruenhage compact spaces having $(*)$ (Example 2). Thus, Theorem 3.1 does not follow from previous results such as Theorem 1.3. Along the way, we answer an open question concerning Kunen's compact $S$-space $\mathcal{K}$: specifically, we show that $\mathcal{K}$ is Gruenhage (Proposition 4.7). In several cases, including Example 2, we shall assume extra principles independent of the usual axioms of set theory. Finally, in Section 5, we present some open problems stemming from this study.

## 2. *A characterization of strict convexity in Banach spaces*

In this section, we provide a general characterization of strictly convex renormings in Banach spaces. Throughout this section, $X$ will be a Banach space (and occasionally a general topological space) and $F \subseteq X^*$ a norming subspace. Recall that $\sigma(X, F)$ denotes the coarsest topology on $X$ with respect to which every element of $F$ is continuous. We begin by presenting a useful folklore result, together with a brief sketch proof.

PROPOSITION 2.1. *Let $F \subseteq X^*$ be a norming subspace. Suppose that there exists a sequence of $\sigma(X, F)$-lower semicontinuous convex functions $\varphi_n : X \longrightarrow [0, \infty)$ such that given distinct $x, y \in X$, we can find $n \in \mathbb{N}$ satisfying*

$$\varphi_n(\tfrac{1}{2}(x+y)) < \max\{\varphi(x), \varphi(y)\}. \tag{2.1}$$

*Then $X$ admits a $\sigma(X, F)$-lower semicontinuous strictly convex norm $|||\cdot|||$. Instead, if $X$ is a Banach lattice, (2.1) holds whenever $x, y \in X_+$ are distinct, and*

$$\varphi_n(x) \leqslant \varphi_n(y),$$

*whenever $|x| \leqslant |y|$ and $n \in \mathbb{N}$, then $|||\cdot|||$ is a $\sigma(X, F)$-lower semicontinuous strictly convex lattice norm.*

*Proof.* Let $\|\cdot\|$ denote the original norm on $X$. We define a new norm by

$$|||x|||^2 = \sum_{n,q} c_{n,q}\|x\|_{n,q}^2,$$

where $\|\cdot\|_{n,q}$ is the Minkowski functional of

$$C_{n,q} = \{x \in X \,:\, \varphi_n(x)^2 + \varphi_n(-x)^2 \leqslant q\},$$

whenever $q$ is a rational number satisfying $q > 2\varphi_n(0)^2$, and where the constants $c_{n,q} > 0$ are chosen to ensure the uniform convergence of the sum on bounded sets. By a standard convexity argument (cf. [8, Fact II.2.3]), it can be shown that if $|||x||| = |||y||| = \frac{1}{2}|||x+y|||$, then $\varphi_n(x) = \varphi_n(y) = \varphi_n(\frac{1}{2}(x+y))$ for all $n$, whence $x = y$ by hypothesis. If we adopt the lattice hypotheses instead, then clearly $|||\cdot|||$ is also a lattice norm, and strictly convex on $X_+$. To see that the strict convexity extends to all of $X$, let $x, y \in X$ and suppose that $|||x||| = |||y||| = \frac{1}{2}|||x+y|||$. Then $\frac{1}{2}|||\,|x|+|y|\,||| = |||x|||$ as well, so strict convexity on $X_+$ yields $|x| = |y|$. If we set $w = \frac{1}{2}(x+y)$, then repeating the above gives us $|x| = |w|$. A simple lattice argument (for example, [34, p. 749]) leads us to conclude that $x = y$. □

Our characterization adopts several ideas from [25, 26]. Recall that if $A$ is a subset of a locally convex space, then an open *slice* $U$ of $A$ is the intersection of $A$ with an open half-space of $X$. The following proposition will be our main tool.

Proposition 2.2. *Let $A$ be a bounded subset of $X$ and $\mathscr{U}$ be a family of non-empty $\sigma(X,F)$-open slices of $A$. Then there exists a $\sigma(X,F)$-lower semicontinuous 1-Lipschitz convex function $\varphi$ with the property that whenever $x, y \in A$, $\{x,y\} \cap \bigcup \mathscr{U}$ is non-empty and*

$$\varphi(x) = \varphi(y) = \varphi(\tfrac{1}{2}(x+y)),$$

*we have $x, y \in U$ for some $U \in \mathscr{U}$.*

Proposition 2.2 is an immediate corollary of the next result, dubbed the 'Slice Localization Theorem'.

Theorem 2.3 [26, Theorem 3]. *Let $A$ be a bounded subset of $X$ and $\mathscr{U}$ be a family of non-empty $\sigma(X,F)$-open slices of $A$. Then there is an equivalent $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|$ such that, for every sequence $(x_n)_{n=1}^{\infty} \subseteq X$ and $x \in A \cap \bigcup \mathscr{U}$, if*

$$2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \longrightarrow 0,$$

*then there is a sequence of slices $(U_n)_{n=1}^{\infty} \subseteq \mathscr{U}$ and $n_0 \in \mathbb{N}$ such that the following conditions are satisfied*:

(i) *$x, x_n \in U_n$ whenever $n \geqslant n_0$ and $x_n \in A$;*
(ii) *for every $\delta > 0$ there is some $n_\delta \in \mathbb{N}$ such that*

$$x, x_n \in \overline{(\mathrm{conv}(A \cap U_n) + \delta B_X)}^{\sigma(X,F)},$$

*for all $n \geqslant n_\delta$.*

The Slice Localization Theorem can be used to simplify the proofs of network characterizations of Banach spaces that admit locally uniformly rotund norms. To prove Proposition 2.2, all we need to do is apply Theorem 2.3 with $x_n = y$ for all $n$. However, there is a more transparent proof of this proposition which we provide for completeness.

Of key importance to the proof is the concept of $F$-distance, introduced in [25]. Let $D \subseteq X$ be a non-empty, convex bounded subset. Given $\xi \in X^{**}$, define

$$\|\xi\|_F = \sup\{\xi(f) \,:\, f \in B_{X^*} \cap F\}. \tag{2.2}$$

It is clear that $\|\cdot\|_F$ is $\sigma(X^{**},F)$-lower semicontinuous ($\sigma(X^{**},F)$ being the only generally non-Hausdorff topology mentioned in this paper). Now set

$$\varphi(x) = \inf\{\|x - d\|_F \,:\, d \in \overline{D}^{\sigma(X^{**},X^*)}\}.$$

DEFINITION 2.4. Given a non-empty, convex bounded subset $D \subseteq X$, we call $\varphi(x)$ the *$F$-distance* from $x \in X$ to $D$.

We pass to the bidual of $X$ in order to control the lower semicontinuity properties of $\varphi$. The notion of $F$-distance has a number of useful properties which we list in the next lemma.

LEMMA 2.5. *Let $\varphi(x)$ be the $F$-distance from $x \in X$ to $D$. Then the following conditions are satisfied*:

(i) *$\varphi$ is convex and* 1*-Lipschitz*;
(ii) *$\varphi$ is $\sigma(X,F)$-lower semicontinuous*;
(iii) $\overline{D}^{\sigma(X,F)} = \varphi^{-1}(0)$.

Properties (i) and (ii) are proved in [**25**, Proposition 2.1] and the third is a straightforward exercise involving the Hahn–Banach separation theorem. Now we can give our alternative proof of Proposition 2.2.

*Proof of Proposition* 2.2. For each $U \in \mathscr{U}$ and $x \in X$, define $\varphi_U(x)$ to be the $F$-distance from $x$ to $(\operatorname{conv} A) \setminus U$. Since $A$ is bounded, we can define another convex, $\sigma(X,F)$-lower semicontinuous, 1-Lipschitz function by

$$\varphi(x) = \sup\{\varphi_U(x) \,:\, U \in \mathscr{U}\}.$$

Let $x, y \in A$ with $\{x,y\} \cap \bigcup \mathscr{U}$ non-empty and suppose that

$$\varphi(x) = \varphi(y) = \varphi(\tfrac{1}{2}(x+y)).$$

Without loss of generality, we can assume that $x \in U$ for some $U \in \mathscr{U}$. Since $U \cap \overline{(\operatorname{conv} A) \setminus U}^{\sigma(X,F)}$ is empty, we have $\varphi(x) \geqslant \varphi_U(x) > 0$ by Lemma 2.5, part (iii). Pick $\varepsilon > 0$ such that $\varphi(x) > 5\varepsilon^2$ and choose $V \in \mathscr{U}$ with the property that

$$\varphi(\tfrac{1}{2}(x+y))^2 < \varphi_V(\tfrac{1}{2}(x+y))^2 + \varepsilon^2.$$

We have

$$\begin{aligned} 0 &= \tfrac{1}{2}(\varphi(x)^2 + \varphi(y)^2) - \varphi(\tfrac{1}{2}(x+y))^2 \\ &> \tfrac{1}{2}(\varphi_V(x)^2 + \varphi_V(y)^2) - \varphi_V(\tfrac{1}{2}(x+y))^2 - \varepsilon^2 \\ &\geqslant \tfrac{1}{2}(\varphi_V(x)^2 + \varphi_V(y)^2) - \tfrac{1}{4}(\varphi_V(x) + \varphi_V(y))^2 - \varepsilon^2 \\ &= \tfrac{1}{4}(\varphi_V(x) - \varphi_V(y))^2 - \varepsilon^2, \end{aligned}$$

thus

$$|\varphi_V(x) - \varphi_V(y)| < 2\varepsilon. \tag{2.3}$$

Since $\varphi_V$ is convex, we have $\max\{\varphi_V(x), \varphi_V(y)\} \geqslant \varphi_V(\tfrac{1}{2}(x+y))$. Together with (2.3), this implies

$$\begin{aligned} \min\{\varphi_V(x), \varphi_V(y)\} &\geqslant \max\{\varphi_V(x), \varphi_V(y)\} - 2\varepsilon \\ &\geqslant \varphi_V(\tfrac{1}{2}(x+y)) - 2\varepsilon \\ &\geqslant (\varphi(\tfrac{1}{2}(x+y)) - \varepsilon^2)^{1/2} - 2\varepsilon \\ &> 0. \end{aligned}$$

Therefore, $\varphi_V(x), \varphi_V(y) > 0$. Since $x, y \in A$, we get $x, y \in V$. □

Proposition 2.2 motivates the introduction of the central topological concept featuring in this paper.

Definition 2.6. We say that a topological space $X$ has $(*)$ if there exists a sequence $(\mathscr{U}_n)_{n=1}^\infty$ of families of open subsets of $X$, with the property that, given any $x, y \in X$, there exists $n \in \mathbb{N}$ such that the following properties are satisfied:
(i) $\{x, y\} \cap \bigcup \mathscr{U}_n$ is non-empty;
(ii) $\{x, y\} \cap U$ is at most a singleton for all $U \in \mathscr{U}_n$.

Any sequence $(\mathscr{U}_n)_{n=1}^\infty$ satisfying the conditions of Definition 2.6 will be called a $(*)$-*sequence* for $X$. In addition, if $X$ is locally convex and $A \subseteq X$, then we say $A$ has $(*)$ *with slices* if $A$ admits a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$, with the property that every element of $\bigcup_{n=1}^\infty \mathscr{U}_n$ is an open slice of $A$.

Remark 1. It will be convenient to note that if $A \subseteq X$, then to say that $(A, \sigma(X, F))$ has $(*)$ with slices is equivalent to there being a family of subsets $G_n \subseteq (S_{X^*} \cap F) \times \mathbb{R}$, $n \in \mathbb{N}$ such that, given distinct $x, y \in A$, we have $n \in \mathbb{N}$ satisfying
(a) $\max\{f(x), f(y)\} > \lambda$ for some $(f, \lambda) \in G_n$ and
(b) $\min\{g(x), g(y)\} \leqslant \mu$ for every $(g, \mu) \in G_n$.

Our characterization follows.

Theorem 2.7. *Let $F \subseteq X^*$ be a 1-norming subspace. Then the following are equivalent*:
(i) *$X$ admits a $\sigma(X, F)$-lower semicontinuous strictly convex norm*;
(ii) *$(X, \sigma(X, F))$ has $(*)$ with slices*;
(iii) *$(S_X, \sigma(X, F))$ has $(*)$ with slices*;
(iv) *there is a sequence of subsets $(X_n)_{n=1}^\infty$ of $X$, such that*

$$\{(x, y) \in X^2 \,:\, x \neq y\} \subseteq \bigcup_{n=1}^\infty X_n^2,$$

*and where each $(X_n, \sigma(X, F))$ has $(*)$ with slices.*

*Proof.* (i) $\Rightarrow$ (ii): let $\|\cdot\|$ be a $\sigma(X, F)$-lower semicontinuous strictly convex norm on $X$. Then $F$ is also 1-norming for $\|\cdot\|$. Let

$$G_q = (S_{(X, \|\cdot\|)^*} \cap F) \times \{q\},$$

for each rational number $q > 0$. We verify that $(X, \sigma(X, F))$ has $(*)$ by showing that the $G_q$ satisfy (a) and (b) of Remark 1. Given distinct $x, y \in X$, assume that $\|x\| \leqslant \|y\|$. The strict convexity of $\|\cdot\|$ tells us that $\|\frac{1}{2}(x + y)\| < \|y\|$. Let rational $q$ satisfy $\|\frac{1}{2}(x + y)\| < q < \|y\|$. Since $F$ is 1-norming for $\|\cdot\|$, we know that $f(y) > q$ for a pair $(f, q) \in G_q$, giving (a). Now suppose $g(y) > q$ for some $(g, q) \in G_q$. Then certainly $g(x) \leqslant q$; else we would have

$$q < \tfrac{1}{2} g(x + y) \leqslant \tfrac{1}{2}\|x + y\|,$$

which does not make any sense. This shows that (b) is also satisfied.

Condition (ii) $\Rightarrow$ (iii) is trivial because $(*)$ with slices is inherited by subsets. Condition (iii) $\Rightarrow$ (ii): if $(S_X, \sigma(X, F))$ has $(*)$ with slices, then we take sets $G_n$, $n \in \mathbb{N}$ that satisfy (a) and (b) of Remark 1. We can assume that $G_n \subseteq (S_{X^*} \cap F) \times (-1, 1)$ for every $n$. Given rational

$q, r > 0$, set

$$H_q = (S_{X^*} \cap F) \times \{q\} \quad \text{and} \quad L_{n,q,r} = \{(f, q(\lambda + r)) : (f, \lambda) \in G_n\}.$$

We claim that the $H_q$ and $L_{n,q,r}$ verify that $(X, \sigma(X, F))$ has $(*)$, using Remark 1.

Let $x, y \in X$ be distinct, with $\|x\| \leqslant \|y\|$. If $\|x\| < \|y\|$, then we choose rational $q$ to satisfy $\|x\| < q < \|y\|$. Since $F$ is 1-norming, it is easy to check that (a) and (b) are fulfilled by $H_q$. Now suppose $\|x\| = \|y\|$. We know that, with respect to $x/\|x\|$ and $y/\|y\|$, (a) and (b) are satisfied by some $G_n$. Without loss of generality, assume $f(x) > \|x\|\lambda$, where $(f, \lambda) \in G_n$. Our argument depends on the sign of $\lambda$. If $\lambda \geqslant 0$, then choose rational $q, r > 0$ satisfying

$$f(x) > \|x\|(\lambda + r) \quad \text{and} \quad \frac{\|x\|}{1 + r} < q < \|x\|.$$

The constants have been arranged to ensure

$$\mu(\|x\| - q) < \|x\| - q < qr \quad \text{whenever } |\mu| < 1. \tag{2.4}$$

We have $f(x) > \|x\|(\lambda + r) > q(\lambda + r)$. Now suppose that $g(x) > q(\mu + r)$, where $(g, \mu) \in G_n$. Then

$$g(x) > q(\mu + r) > \|x\|\mu$$

by equation (2.4). This means $g(x/\|x\|) > \mu$, whence $g(y/\|y\|) \leqslant \mu$ by (b), giving $g(y) < q(\mu + r)$. In summary, we have shown that (a) and (b) of Remark 1 are fulfilled by $L_{n,q,r}$. If instead $\lambda < 0$, then we choose $r < -\lambda$ as above and ensure that $q$ satisfies

$$\|x\| < q < \frac{\|x\|}{1 - r}.$$

By arguing similarly, we get what we want.

Condition (ii) $\Rightarrow$ (iv) follows easily by setting $X_n = X$. We finish by proving (iv) $\Rightarrow$ (i). By taking intersections with $mB_X$, $m \in \mathbb{N}$, and re-indexing if necessary, we can assume that each $X_n$ is bounded. Let each $X_n$ have a $(*)$-sequence $(\mathscr{U}_{n,m})_{m=1}^{\infty}$, where each element of $\bigcup_{m=1}^{\infty} \mathscr{U}_{n,m}$ is a (non-empty) $\sigma(X, F)$-open slice of $X_n$. Let $\varphi_{n,m}$ denote the convex function constructed by applying Proposition 2.2 to $X_n$ and the family $\mathscr{U}_{n,m}$. We have ensured that if $x, y \in X$ are distinct, then we can find $n$ and $m$ such that $\varphi_{n,m}(\frac{1}{2}(x + y)) < \max\{\varphi_{n,m}(x), \varphi_{n,m}(y)\}$. The rest follows from Proposition 2.1. □

Note that Theorem 2.7(i), (ii) and (iv) are also equivalent when $F$ is simply a norming subspace, rather than a 1-norming subspace. The reliance on slices in the statement of Theorem 2.7 is necessary in general.

EXAMPLE 1. Let $K$ be the product $\{0, 1\}^{\omega_1}$ endowed with the lexicographic order topology. According to [**16**, Example 1], $C(K)$ admits a Kadec norm $\|\cdot\|$ but no strictly convex norm. By the definition of Kadec norms, the weak topology agrees with the norm topology on $S_{(C(K),\|\cdot\|)}$. In particular, $(S_{(C(K),\|\cdot\|)}, w)$ is metrizable, meaning that it has a $\sigma$-discrete base and thus has $(*)$ as well. However, since $\|\cdot\|$ cannot be strictly convex, Theorem 2.7 implies that $(S_{(C(K),\|\cdot\|)}, w)$ does not have $(*)$ *with slices*.

Another characterization of strictly convex renormings, given in terms of certain linear topological decompositions of the squares $X^2$ or $S_X^2$, can be found in [**22**, Theorem 1.2]. The advantage of Theorem 2.7 is that there is not such an explicit dependence on squares, which are generally harder to manage.

We conclude this section by giving a sufficient condition for constructing strictly convex norms. Theorem 2.9 can be applied to many spaces of significance to the theory, such as the

Mercourakis spaces $c_1(\Sigma' \times \Gamma)$ (see [**8**, Section VI.6]), Dashiell–Lindenstrauss spaces and spaces of the form $C(K)^*$, where $K$ is Gruenhage. The idea, which goes back to the classical norm of Day for $c_0(\Gamma)$ (see [**6**, Theorem 10]), is to 'glue together' strictly convex norms on finite-dimensional spaces (which are readily available) to obtain strictly convex norms on larger spaces. Elements of Theorem 2.9 can be found in [**10**, Theorem 5]. Before giving the theorem, we state a simple fact.

FACT 2.8. *Let $\xi : [0,1] \longrightarrow \mathbb{R}$ be a continuous function satisfying $\xi(0)\xi(1) < 0$, and suppose that $\xi_+$ and $\xi_-$ are convex. Then, for every $\lambda \in (0,1)$, we have*

$$\xi_\pm(\lambda) < \lambda\xi_\pm(1) + (1-\lambda)\xi_\pm(0).$$

*Proof.* Take $a \in (0,1)$ satisfying $\xi(a) = 0$, by the continuity of $\xi$. Assume without loss of generality that $\xi(0) > 0$, and let $\lambda \in (0,1)$. If $\lambda \leqslant a$, then setting $\mu = \lambda/a$ gives

$$\xi_+(\lambda) \leqslant (1-\mu)\xi_+(0) + \mu\xi_+(a) = (1-\mu)\xi_+(0) < (1-\lambda)\xi_+(0)$$

and

$$\xi_-(\lambda) \leqslant (1-\mu)\xi_-(0) + \mu\xi_-(a) = 0 < \lambda\xi_-(1).$$

We get similar inequalities if $\lambda > a$. □

Clearly, if $\xi$ is linear, then $\xi_\pm$ are convex. The same is true if $\xi$ is positive and convex.

THEOREM 2.9. *Let $\Theta_n : X \longrightarrow \ell_\infty(\Gamma_n)$ be a sequence of maps such that both functions $x \mapsto \Theta_{n,\pm}(x)(\gamma)$ are $\sigma(X,F)$-lower semicontinuous and convex for every $\gamma \in \Gamma_n$ and $n \in \mathbb{N}$.*

*Let us assume in addition that, for all distinct $x, y \in X$, there are $\lambda \in (0,1)$, $n \in \mathbb{N}$ and a finite set $A \subseteq \Gamma_n$, such that*

$$\Theta_n(x)\restriction_A \neq \Theta_n(y)\restriction_A \tag{2.5}$$

*and*

$$|\Theta_n(z)(\alpha)| > |\Theta_n(z)(\gamma)| \quad \text{whenever } \alpha \in A \text{ and } \gamma \in \Gamma \setminus A, \tag{2.6}$$

*where $z = \lambda x + (1-\lambda)y$. Then $X$ admits a $\sigma(X,F)$-lower semicontinuous strictly convex norm $|||\cdot|||$.*

*Instead, if $X$ is a Banach lattice, $\Theta_{n,\pm}(x) \leqslant \Theta_{n,\pm}(y)$ whenever $|x| \leqslant |y|$ and equations (2.5) and (2.6) apply to distinct $x, y \in X_+$, then $|||\cdot|||$ is a $\sigma(X,F)$-lower semicontinuous strictly convex lattice norm.*

*Proof.* Since $\Theta_{n,\pm}(\cdot)(\gamma)$ are both convex and $\sigma(X,F)$-lower semicontinuous, the same is true of $|\Theta_n(\cdot)(\gamma)|$. Define

$$\Theta_{n,0}(x)(\gamma) = \Theta_n^2(x)(\gamma) \quad \text{and} \quad \Theta_{n,\pm 1}(x)(\gamma) = \Theta_{n,\pm}(x)(\gamma).$$

If $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$, $u \in \ell_\infty(\Gamma)$ and $A \subseteq \Gamma$ is finite, set

$$\varphi_A(u) = \sum_{\gamma \in A} u(\gamma),$$

and put $\varphi_{A,n,i} = \varphi_A \circ \Theta_{n,i}$ for every $n \in \mathbb{N}$ and $i \in \{-1, 0, 1\}$. Certainly, each $\varphi_{A,n,i}$ is $\sigma(X,F)$-lower semicontinuous, non-negative and convex. Finally, let

$$\psi_{m,n,i}(x) = \sup\{\varphi_{A,n,i}(x) \,:\, A \subseteq \Gamma_n \text{ has cardinality } m\}.$$

To complete the proof, we shall show that, for every distinct pair $x, y \in X$, there are $m, n \in \mathbb{N}$ and $i \in \{-1, 0, 1\}$ such that

$$\psi_{m,n,i}(\tfrac{1}{2}(x+y)) < \max\{\psi_{m,n,i}(x),\, \psi_{m,n,i}(y)\} \tag{2.7}$$

holds. Then we can appeal to Proposition 2.1.

Take $\lambda \in (0,1)$, $n \in \mathbb{N}$ and $A \subseteq \Gamma_n$ satisfying (2.5) and (2.6). We consider two cases. First suppose that $\Theta_n(x)(\beta)\Theta_n(y)(\beta) < 0$ for some $\beta \in A$. From (2.6) we know that $\Theta_n(z)(\beta) \neq 0$. Assume for now that $\Theta_n(z)(\beta) > 0$ and define the non-empty set

$$B = \{\alpha \in A \,:\, \Theta_{n,+}(z)(\alpha) > 0\},$$

so that

$$\Theta_{n,+}(z)(\alpha) > \Theta_{n,+}(z)(\gamma),$$

for every $\alpha \in B$ and $\gamma \in \Gamma \setminus B$. Therefore, $\psi_{n,m,1}(z) = \sum_{\alpha \in B} \Theta_{n,+}(z)(\alpha)$, where $m$ is the cardinality of $B$. Set $\xi(t) = \Theta_n(tx + (1-t)y)(\beta)$ for $t \in \mathbb{R}$. As $\xi_\pm$ are convex on $\mathbb{R}$, by hypothesis, they are also continuous. After applying Fact 2.8, we get

$$\Theta_{n,+}(z)(\beta) < \lambda\Theta_{n,+}(x)(\beta) + (1-\lambda)\Theta_{n,+}(y)(\beta),$$

whence

$$\psi_{n,m,1}(z) < \lambda\psi_{n,m,1}(x) + (1-\lambda)\psi_{n,m,1}(y)$$

from which (2.7) quickly follows for $i = 1$, by convexity. If $\Theta_n(z)(\beta) < 0$, then we argue similarly with $i = -1$.

Let us now consider the case

$$\Theta_n(x)(\alpha)\Theta_n(y)(\alpha) \geqslant 0, \tag{2.8}$$

for all $\alpha \in A$. Let $m \in \mathbb{N}$ be the cardinality of $A$. Since $t \mapsto t^2$ is strictly convex, from condition (2.5) we have

$$\begin{aligned}\sum_{\alpha\in A}(\lambda\Theta_n(x)(\alpha) + (1-\lambda)\Theta_n(y)(\alpha))^2 &< \sum_{\alpha\in A}\lambda(\Theta_n(x)(\alpha))^2 + (1-\lambda)(\Theta_n(y)(\alpha))^2\\ &= \lambda\varphi_{A,n,0}(x) + (1-\lambda)\varphi_{A,n,0}(y)\\ &\leqslant \lambda\psi_{m,n,0}(x) + (1-\lambda)\psi_{m,n,0}(y)\\ &\leqslant \max\{\psi_{m,n,0}(x), \psi_{m,n,0}(y)\}.\end{aligned}$$

Given the convexity of $|\Theta_n(\cdot)(\alpha)|$ and equation (2.8), we obtain

$$|\Theta_n(z)(\alpha)| = |\Theta_n(\lambda x + (1-\lambda)y)(\alpha)| \leqslant |\lambda\Theta_n(x)(\alpha) + (1-\lambda)\Theta_n(y)(\alpha)|.$$

This and condition (2.6) imply

$$\psi_{m,n,0}(z) = \varphi_{A,n,0}(z) \leqslant \sum_{\alpha\in A}(\lambda\Theta_n(x)(\alpha) + (1-\lambda)\Theta_n(y)(\alpha))^2.$$

Combining these inequalities, we see that

$$\psi_{m,n,0}(z) < \max\{\psi_{m,n,0}(x), \psi_{m,n,0}(y)\}$$

from which (2.7) follows for $i = 0$, again by convexity. If we adopt the lattice assumptions instead, then each $\psi_{n,m,i}$ satisfies the lattice assumptions in Proposition 2.1. □

In the first corollary below is a sufficient condition of 'Mercourakis type', which is formally more general than similar conditions given in the literature (for example, [**22**, Corollary 2.7]).

Corollary 2.10. *Let $X$ be a subspace or sublattice of $\ell_\infty(\Gamma)$ and suppose that there are subsets $\Gamma_n \subseteq \Gamma$, $n \in \mathbb{N}$, with the property that, given $x \in X$ and $\alpha \in \operatorname{supp} x$, we can find $n$*

*such that* $\alpha \in \Gamma_n$ *and*

$$\{\gamma \in \Gamma_n : |x(\gamma)| \geqslant |x(\alpha)|\}$$

*is finite. Then* $X$ *admits a pointwise lower semicontinuous strictly convex norm or lattice norm, respectively.*

*Proof.* Let $P_n(x)(\gamma) = |x(\gamma)|$ whenever $\gamma \in \Gamma_n$ and $n \in \mathbb{N}$. The coordinate maps are positive and convex. We show that $P_n$ satisfies conditions (2.5) and (2.6) of Theorem 2.9. Given distinct $x, y \in X$, take $n \in \mathbb{N}$ and $\beta \in \Gamma_n$ such that $x(\beta) \neq y(\beta)$. Then there is $\lambda \in (0,1)$ such that $\lambda x(\beta) + (1-\lambda)y(\beta)$ is non-zero. Set $z = \lambda x + (1-\lambda)y$ and take $n \in \mathbb{N}$ such that

$$A = \{\alpha \in \Gamma_n \,:\, |z(\alpha)| \geqslant |z(\beta)|\}$$

is finite. Evidently $\beta \in A$, so $P_n(x)\restriction_A \neq P_n(y)\restriction_A$, and

$$|P_n(z)(\alpha)| \geqslant |z(\beta)| > |P_n(z)(\gamma)|,$$

whenever $\alpha \in A$ and $\gamma \in \Gamma_n \setminus A$. ☐

Corollary 2.11 [**34**, Theorem 7]. *If* $K$ *is Gruenhage, then* $C(K)^*$ *admits a strictly convex dual lattice norm.*

*Proof.* If $K$ is Gruenhage, then (cf. [**34**, Lemma 6]) we can find sequences $(\mathscr{U}_n)_{n=1}^\infty$ and $(R_n)_{n=1}^\infty$ as in Definition 1.2, with the further property that if $\mu \in C(K)^*$ and $\mu(U) = 0$ for all $U \in \mathscr{U}_n$, $n \in \mathbb{N}$, then $\mu = 0$. Let $\Gamma_n = \mathscr{U}_n$ and define

$$\Theta_n(\mu)(U) = |\mu|(U), \qquad U \in \mathscr{U}_n.$$

Since $|\lambda\mu + (1-\lambda)\nu| \leqslant \lambda|\mu| + (1-\lambda)|\nu|$ whenever $\lambda \in [0,1]$, the coordinate maps $\Theta_n(\cdot)(U)$ are positive and convex. If $\mu, \nu \in C(K)^*$ are positive and distinct, then there exist $n \in \mathbb{N}$ and $U \in \mathscr{U}_n$ such that $\mu(U) \neq \nu(U)$. If we set $\tau = \frac{1}{2}(\mu + \nu)$, then we have $\tau(U) > \tau(R_n)$. By considering Definition 1.2(ii), we see that for any $r > \tau(R_n)$, there are only finitely many $V \in \mathscr{U}_n$ satisfying $\tau(V) \geqslant r$. Therefore, conditions (2.5) and (2.6) of Theorem 2.9 apply to positive elements of $C(K)^*$. Now we are able to apply Theorem 2.9. ☐

Dashiell–Lindenstrauss spaces can be shown to have strictly convex lattice norms in a similar way.

## 3. *Strictly convex dual norms on* $C(K)^*$

Evidently, Theorem 2.7 relies on geometric assumptions, in the sense that only sets having $(*)$ with *slices* are considered. According to Example 1, it is not always possible to remove the reliance on slices and deal instead with open sets having no special geometric properties. However, we can live without slices in an important special case. We devote this section to proving the next result.

Theorem 3.1. *Let* $K$ *be a scattered compact space. Then* $C(K)^*$ *admits a strictly convex dual* (*lattice*) *norm if and only if* $K$ *has* $(*)$.

Recall that any compact space $K$ embeds naturally into $(C(K)^*, w^*)$ by identifying points $t \in K$ with their Dirac measures $\delta_t$. It follows therefore from Theorem 3.1 that if $K$ is scattered and $(C(K)^*, w^*)$ has $(*)$ (without slices), then $(C(K)^*, w^*)$ has $(*)$ with slices. One implication of Theorem 3.1 may be proved easily.

PROPOSITION 3.2. *If $C(K)^*$ admits a strictly convex dual norm, then $K$ has $(*)$.*

*Proof.* By Theorem 2.7, if $C(K)^*$ admits a strictly convex dual norm, then $(C(K)^*, w^*)$ has $(*)$, whence $K$ has $(*)$ by the natural embedding. □

To prove the converse implication, we need to refine our $(*)$-sequences so that they satisfy some additional properties. Assume that a topological space $X$ admits a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$. Given any finite sequence of natural numbers $\sigma = (n_1, \ldots, n_k)$, we define the family

$$\mathscr{U}_\sigma = \left\{ \bigcap_{i=1}^{k} U_i \,:\, U_i \in \mathscr{U}_{n_i} \text{ for all } i \leqslant k \right\}.$$

Let us also set $C_n = \bigcup \mathscr{U}_n$ and $C_\sigma = \bigcup \mathscr{U}_\sigma$.

LEMMA 3.3. *Assume that $F \subseteq X$ is a finite subset such that, for all $n$, either $F \cap C_n = \varnothing$ or $F \subseteq C_n$. Then there exists $\sigma = (n_1, \ldots, n_k)$ such that $F \subseteq C_\sigma$ and, moreover, $F \cap V$ is at most a singleton for all $V \in \mathscr{U}_\sigma$.*

*Proof.* Enumerate the set of doubletons $\{x, y\} \subseteq F$ as $\{x_1, y_1\}, \ldots, \{x_k, y_k\}$. For every $i$ there exists $n_i$ such that $\{x_i, y_i\} \cap C_{n_i}$ is non-empty and $\{x_i, y_i\} \cap V$ is at most a singleton for all $V \in \mathscr{U}_{n_i}$. By hypothesis, we have $F \subseteq C_{n_i}$ for all $i$. Put $\sigma = (n_1, \ldots, n_k)$. If $x \in F$, since $F \subseteq C_{n_i}$ for all $i$, let $U_i \in \mathscr{U}_{n_i}$ so that $x \in \bigcap_{i=1}^k U_i \in \mathscr{U}_\sigma$. Therefore, $F \subseteq C_\sigma$. Given $V = \bigcap_{i=1}^k V_i \in \mathscr{U}_\sigma$ and distinct $x, y \in F$, we have some $i$ such that $\{x, y\} \cap W$ is at most a singleton for all $W \in \mathscr{U}_{n_i}$. In particular, $\{x, y\} \cap V \subseteq \{x, y\} \cap V_i$ is at most a singleton. This proves that $F \cap V$ is at most a singleton for any $V \in \mathscr{U}_\sigma$. □

Bearing in mind the $\mathscr{U}_\sigma$ and Lemma 3.3, and by adding new singleton families if necessary, if $X$ has $(*)$ then we can assume that there exists a $(*)$-sequence with additional properties, which we list in the next lemma.

LEMMA 3.4. *If $X$ has $(*)$, then it admits a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$ with the following properties.*

(i) *We have $X = C_1$.*

(ii) *Given $n_1, \ldots, n_k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that*

$$\mathscr{U}_m = \left\{ \bigcap_{i=1}^{k} U_i \,:\, U_i \in \mathscr{U}_{n_i} \text{ for all } i \leqslant k \right\}.$$

(iii) *If $F$ is a finite subset of $X$ such that, for each $n \in \mathbb{N}$, either $F \subseteq C_n$ or $F \cap C_n$ is empty, then there exists $m \in \mathbb{N}$ with the following two properties:*

(a) *$F \subseteq C_m$;*

(b) *$F \cap V$ is at most a singleton for all $V \in \mathscr{U}_m$.*

Armed with these enhanced $(*)$-sequences, we can deliver the proof of Theorem 3.1. We ask that our compact spaces be scattered because the proof relies on the assumption that all measures in $C(K)^*$ are atomic.

*Proof of Theorem* 3.1. One implication was proved in Proposition 3.2. Now assume that $K$ is scattered and let $(\mathscr{U}_n)_{n=1}^\infty$ be a $(*)$-sequence for $K$ satisfying the properties of Lemma 3.4. Given $n \geqslant 1$, $k \geqslant 0$ and finite $L \subseteq \mathbb{N}$, define the seminorm

$$\|\mu\|_{n,k,L} = \sup\left\{ |\mu|\left(\bigcup_{i\in L} C_i \cup \bigcup \mathscr{F}\right) \,:\, \mathscr{F} \subseteq \mathscr{U}_n \text{ and } \operatorname{card}\mathscr{F} = k \right\}.$$

We show that these seminorms satisfy the requirements of Proposition 2.1. To this end, suppose that $\mu$ and $\nu$ are positive, and that

$$\|\mu\|_{n,k,L} = \|\nu\|_{n,k,L} = \tfrac{1}{2}\|\mu+\nu\|_{n,k,L}, \tag{3.1}$$

for all $n$, $k$ and $L$. For a contradiction, we shall suppose also that $\mu \neq \nu$. Since

$$\|\mu\|_{1,0,\{n\}} = \mu(C_n),$$

we have $\mu(C_n) = \nu(C_n) = \frac{1}{2}(\mu+\nu)(C_n)$ for all $n$, by (3.1). By Lemma 3.4(i) and (ii), and the inclusion–exclusion principle, if $I \subseteq \mathbb{N}$, then we know that

$$\mu(C_{I,n}) = \nu(C_{I,n}) = \tfrac{1}{2}(\mu+\nu)(C_{I,n}),$$

where

$$C_{I,n} = \bigcap_{i\leqslant n, i\in I} C_i \setminus \bigcup_{i\leqslant n, i\notin I} C_i.$$

By monotone convergence, it follows that

$$\mu(C_I) = \nu(C_I) = \tfrac{1}{2}(\mu+\nu)(C_I),$$

where

$$C_I = \bigcap_{i\in I} C_i \setminus \bigcup_{i\in\mathbb{N}\setminus I} C_i.$$

Now $K$ is the disjoint union of the $C_I$, where $I$ ranges over non-empty subsets of $\mathbb{N}$, and since $\mu \neq \nu$ are atomic, we can find non-empty $I \subseteq \mathbb{N}$ such that $\mu\restriction_{C_I} \neq \nu\restriction_{C_I}$. We fix this $I$ from now on. Take a countable set $A \subseteq C_I$ such that we can write

$$\mu\restriction_{C_I} = \sum_{t\in A} a_t\delta_t \quad \text{and} \quad \nu\restriction_{C_I} = \sum_{t\in A} b_t\delta_t,$$

for some numbers $a_t, b_t \geqslant 0$. Let

$$p = \max\{\max\{a_t, b_t\} \,:\, t \in A, a_t \neq b_t\},$$

$$q = \max(\{a_t \,:\, a_t < p\} \cup \{b_t \,:\, b_t < p\}),$$

and define the finite, possibly empty, set

$$F = \{t \in A \,:\, a_t = b_t \geqslant p\},$$

and let $k = \operatorname{card} F$. Take finite $G \subseteq A$ such that

$$\sum_{t\in A\setminus G} a_t, \quad \sum_{t\in A\setminus G} b_t < \frac{1}{4}(p-q), \tag{3.2}$$

and $n$ large enough so that

$$\mu(C_{I,n} \setminus C_I), \nu(C_{I,n} \setminus C_I) < \tfrac{1}{4}(p-q). \tag{3.3}$$

Let $H = \{1,\ldots,n\} \cap I$ and $L = \{1,\ldots,n\} \setminus I$. By Lemma 3.4(iii), we can find $m \in \mathbb{N}$ such that $G \subseteq C_m$ and $G \cap V$ is at most a singleton for all $V \in \mathscr{U}_m$. Since $C_I \subseteq \bigcap_{i\in H} C_i$, we can and do assume that $C_m \subseteq \bigcap_{i\in H} C_i$, by Lemma 3.4(ii).

It is by considering the seminorm $\|\cdot\|_{m,k+1,L}$ that we reach our contradiction. Let $u \in A$ such that $a_u \neq b_u$ and $\max\{a_u, b_u\} = p$. Clearly, $u \notin F$. Also, $F \cup \{u\} \subseteq G$. Indeed, if $t \in A \setminus G$, then $a_t, b_t < \frac{1}{4}(p-q) < p$. Without loss of generality, assume that $a_u < b_u = p$. Since $F \cup \{u\} \subseteq C_m$, it is possible to find $\mathscr{G} \subseteq \mathscr{U}_m$ of cardinality $k+1$, such that $F \cup \{u\} \subseteq \bigcup \mathscr{G}$.

By considering $\|\cdot\|_{1,0,L}$ and (3.1), we know that $\mu(\bigcup_{i\in L} C_i) = \nu(\bigcup_{i\in L} C_i)$. We shall denote this common quantity by $c$. We estimate

$$\begin{aligned}\|\nu\|_{m,k+1,L} &\geqslant \nu\left(\bigcup_{i\in L} C_i \cup \bigcup \mathscr{G}\right)\\ &\geqslant \nu\left(\bigcup_{i\in L} C_i\right) + \sum_{t\in F\cup\{u\}} b_t \quad \text{as } (F\cup\{u\}) \cap \bigcup_{i\in L} C_i = \varnothing\\ &\geqslant c + p + \sum_{t\in F} b_t = c + p + \sum_{t\in F} a_t. \end{aligned} \tag{3.4}$$

By (3.1) and the definition of the seminorms, let $\mathscr{H} \subseteq \mathscr{U}_m$ of cardinality $k+1$ be chosen in such a way that

$$\frac{1}{2}(\mu+\nu)\left(\bigcup_{i\in L} C_i \cup \bigcup \mathscr{H}\right) > \|\nu\|_{m,k+1,L} - \frac{1}{4}(p-q).$$

We claim that $a_t \geqslant p$ whenever $t \in \bigcup \mathscr{H} \cap G$. In order to see this, first of all we claim that if $J \subseteq A$ has cardinality at most $k$, then

$$\sum_{t\in J} a_t \leqslant \sum_{t\in F} a_t. \tag{3.5}$$

Indeed, we have $\operatorname{card} F \setminus J \geqslant \operatorname{card} J \setminus F$, since $\operatorname{card} J \leqslant k = \operatorname{card} F$. If $t \in J \setminus F$, then either $a_t < p$ or $a_t \neq b_t$, which means $a_t \leqslant p$ by maximality of $p$. Therefore,

$$\begin{aligned}\sum_{t\in F} a_t - \sum_{t\in J} a_t &= \sum_{t\in F\setminus J} a_t - \sum_{t\in J\setminus F} a_t\\ &\geqslant p(\operatorname{card} F \setminus J) - p(\operatorname{card} J \setminus F) \geqslant 0.\end{aligned}$$

This completes the proof of the claim.

Now we can show that $a_t \geqslant p$ whenever $t \in \bigcup \mathscr{H} \cap G$. If not, then $a_s < p$ for some $s \in \bigcup \mathscr{H} \cap G$, meaning $a_s \leqslant q$. Observe that

$$\bigcup_{i\in L} C_i \cup \bigcup \mathscr{H} \subseteq \left(\bigcup \mathscr{H} \cap G\right) \cup \left(\bigcup \mathscr{H} \cap C_I \setminus G\right) \cup (C_{I,n} \setminus C_I) \cup \bigcup_{i\in L} C_i. \tag{3.6}$$

To see this, it helps to note that

$$\bigcup \mathscr{H} \setminus \bigcup_{i\in L} C_i \subseteq C_m \setminus \bigcup_{i\in L} C_i \subseteq \bigcap_{i\in H} C_i \setminus \bigcup_{i\in L} C_i = C_{I,n}.$$

By the choice of $m$, $\operatorname{card} \bigcup \mathscr{H} \cap G \leqslant k+1$. Hence,

$$\begin{aligned}\mu\left(\bigcup_{i\in L} C_i \cup \bigcup \mathscr{H}\right) &\leqslant \sum_{t\in F} a_t + a_s + \frac{1}{4}(p-q) + \frac{1}{4}(p-q) + c \quad \text{by (3.2), (3.3), (3.5) and (3.6)}\\ &\leqslant \sum_{t\in F} a_t + q + \frac{1}{2}(p-q) + c \quad \text{since } a_s \leqslant q\\ &\leqslant \|\nu\|_{m,k+1,L} - \frac{1}{2}(p-q) \quad \text{by (3.4)}.\end{aligned}$$

However, this means

$$\frac{1}{2}(\mu+\nu)\left(\bigcup_{i\in L} C_i \cup \bigcup \mathscr{H}\right) \leqslant \frac{1}{2}\|\nu\|_{m,k+1,L} - \frac{1}{4}(p-q) + \frac{1}{2}\|\nu\|_{m,k+1,L},$$

which contradicts the choice of $\mathscr{H}$. Therefore, $a_t \geqslant p$ whenever $t \in \bigcup \mathscr{H} \cap G$. By a similar argument applied to the $b_t$, we have $b_t \geqslant p$ whenever $t \in \bigcup \mathscr{H} \cap G$. Hence, we know that $a_t = b_t$ for $t \in \bigcup \mathscr{H} \cap G$, lest we contradict the maximality of $p$. It follows that $\bigcup \mathscr{H} \cap G \subseteq F$. However, this forces

$$\begin{aligned}\mu\left(\bigcup_{i\in L} C_i \cup \bigcup \mathscr{H}\right) &\leqslant \sum_{t\in F} a_t + \frac{1}{4}(p-q) + \frac{1}{4}(p-q) + c \quad \text{by (3.2), (3.3) and (3.6)}\\ &< \|\nu\|_{m,k+1,L} - \frac{1}{2}(p-q).\end{aligned}$$

Just as above, this contradicts the choice of $\mathscr{H}$. □

Remark 2. Most of Theorem 3.1 follows from Theorem 2.7. Starting with a $(*)$-sequence from Lemma 3.4, we can show directly that $(C(K)^*, w^*)$ has $(*)$ with slices. For $n, k \in \mathbb{N}$, finite $L \subseteq \mathbb{N}$ and rational $q > 0$, define $\mathscr{V}_{n,k,L,q,+}$ to be the family of all $w^*$-open sets

$$\left\{\mu \in C(K)^* \,:\, \mu_+\left(\bigcup_{i\in L} C_i \cup \bigcup \mathscr{F}\right) > q\right\},$$

where $\mathscr{F} \subseteq \mathscr{U}_n$ has cardinality $k$. Define $\mathscr{V}_{n,k,L,q,-}$ accordingly. By using essentially the same method as that presented above, it can be shown that the $\mathscr{V}_{n,k,L,q,\pm}$ form a $(*)$-sequence. Moreover, if $V \in \mathscr{V}_{n,k,L,q,\pm}$, then $C(K)^* \setminus V$ is convex. By the Hahn–Banach Theorem, each such $V$ can be written as a union of $w^*$-open half-spaces. Therefore, we can write down a $(*)$-sequence for $(C(K)^*, w^*)$, the elements of which being families of half-spaces. What we lose here is the fact that the norm in Theorem 3.1 is a lattice norm, which is why we give the proof as is.

## 4. *Topological properties of* $(*)$ *and examples*

In this section, we explore the properties of $(*)$ and see how it compares with related concepts in the literature. In particular, under the continuum hypothesis (CH) or the axiom $\mathfrak{b} = \aleph_1$, we provide examples of compact scattered non-Gruenhage spaces having $(*)$. This means that Theorem 3.1 does not follow from existing results such as Theorem 1.3.

A topological space $X$ is said to have a *$G_\delta$-diagonal* if its diagonal

$$\{(x,x) \,:\, x \in X\}$$

is a $G_\delta$ set in $X^2$. This concept has been studied extensively in general metrization theory; see, for example, [**11**, Section 2]. It is easy to show that $X$ has a $G_\delta$-diagonal if and only if there is a sequence $(\mathscr{G}_n)_{n=1}^\infty$ of open covers of $X$ such that, given $x, y \in X$, there exists $n$ with the property that $\{x, y\} \cap U$ is at most a singleton for all $U \in \mathscr{G}_n$ (see [**11**, Theorem 2.2]). Equivalently, if we consider the 'stars'

$$\operatorname{st}(x,n) = \bigcup\{U \in \mathscr{G}_n \,:\, x \in U\},$$

then $\bigcap_{n=1}^\infty \operatorname{st}(x,n) = \{x\}$ for every $x \in X$. In keeping with previous notation, we call such a sequence a *$G_\delta$-diagonal sequence*. Compact spaces with $G_\delta$-diagonals are metrizable (cf. [**11**, Theorem 2.13]), so $(*)$ is evidently a strict generalization of the $G_\delta$-diagonal property. In some cases, it is possible to reduce problems about $(*)$ to the $G_\delta$-diagonal case; see Theorem 4.3 and Proposition 4.12, and also the partitioning of $K$ into the $C_I$ in the proof of Theorem 3.1.

Next, we compare $(*)$ with Gruenhage's property.

PROPOSITION 4.1. *If $X$ is Gruenhage, then it has $(*)$.*

*Proof.* If $X$ is Gruenhage, then let $(\mathscr{U}_n)_{n=1}^\infty$ and $R_n$ be as in Definition 1.2. Let $\mathscr{V}_n = \{R_n\}$ for each $n$. Given distinct $x, y \in X$, there exist $n$ and $U \in \mathscr{U}_n$, such that $\{x,y\} \cap U$ is a singleton. If $x \in R_n$, then $y \notin R_n$ and it is true that $\{x,y\} \cap U = \{x\}$ for every $U \in \mathscr{V}_n$, because $\mathscr{V}_n$ is a singleton; likewise if $y \in R_n$. So we assume now that $x, y \notin R_n$. Now it is true that $\{x,y\} \cap V$ is at most a singleton for every $V \in \mathscr{U}_n$, since if $y \in V$, then $V \neq U$, and if $x \in V$, then $x \in U \cap V = R_n$. □

There are an abundance of compact spaces that are Gruenhage, but non-descriptive and so quite far from being metrizable; see [**34**, Corollary 17] or Theorem 4.6 and subsequent remarks, below. In Example 2, we show that under additional axioms there exist compact, scattered non-Gruenhage spaces of cardinality $\aleph_1$ having $(*)$. Now we see that $(*)$ implies fragmentability.

PROPOSITION 4.2. *If $X$ has $(*)$, then $X$ is fragmentable.*

*Proof.* Let $X$ have a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$. We well order each $\mathscr{U}_n$ as $(U_\xi^n)_{\xi<\lambda_n}$. Now define $V_\alpha^n = \bigcup_{\xi \leqslant \alpha} U_\xi^n$ for $\alpha < \lambda_n$. We claim that, given distinct $x, y \in X$, there exist $n$ and $\alpha < \lambda_n$ such that $\{x,y\} \cap V_\alpha^n$ is a singleton. As explained in Section 1, this is enough to give fragmentability. Indeed, take $n \in \mathbb{N}$ with the properties given in Definition 2.6, and pick the least $\alpha < \lambda_n$ such that $\{x,y\} \cap U_\alpha^n$ is a singleton. Then $\{x,y\} \cap U_\xi^n$ must be empty for all $\xi < \alpha$, thus

$$\{x,y\} \cap V_\alpha^n = \{x,y\} \cap U_\alpha^n$$

is a singleton. □

Theorem 4.3 is a generalization of a result of Chaber (cf. [**11**, Theorem 2.14]), which states that countably compact spaces with $G_\delta$-diagonals are compact (and thus metrizable). It allows us to glean a few more topological consequences of the $(*)$ property. As preparation, fix an open cover $\mathscr{V}$ of a countably compact (non-empty) space $X$. Suppose that $X$ has a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$, with $C_n = \bigcup \mathscr{U}_n$ for each $n$. Define

$$\mathscr{A}_X = \left\{ I \subseteq \mathbb{N} : X \setminus \bigcup_{n \in I} C_n \neq \varnothing \right\}.$$

Clearly, $\mathscr{A}_X$ is a hereditary family of subsets of $\mathbb{N}$. Moreover, it is compact in the pointwise topology. Indeed, if $J \notin \mathscr{A}_X$, then by the countable compactness of $X$, we can find finite $G \subseteq J$ such that $G \notin \mathscr{A}_X$. It follows that $\mathbb{P}(\mathbb{N}) \setminus \mathscr{A}_X$ is open. Furthermore, $\varnothing \in \mathscr{A}_X$ because $X$ is non-empty, so $\mathscr{A}_X$ is also non-empty. From these facts, we deduce that $\mathscr{A}_X$ admits an element that is maximal with respect to inclusion.

THEOREM 4.3. *If $X$ is countably compact and has $(*)$, then $X$ is compact.*

*Proof.* Fix an open cover $\mathscr{V}$ of $X$ and $(*)$-sequence $(\mathscr{U}_n)_{n=1}^{\infty}$ as above. We define a decreasing transfinite sequence of countably compact subspaces $X_\alpha$ of $X$, together with maximal $M_\alpha \in \mathscr{A}_{X_\alpha}$ and finite $\mathscr{F}_\alpha \subseteq \mathscr{V}$, such that the following conditions are satisfied:

(i) $X_\alpha = X \setminus \bigcup_{\xi<\alpha} \bigcup \mathscr{F}_\xi$;

(ii) $M_\xi \notin \mathscr{A}_{X_\alpha}$ whenever $\xi < \alpha$.

To begin, set $X_0 = X$. Given $X_\alpha$, we take some maximal $M_\alpha \in \mathscr{A}_{X_\alpha}$ and set $Y = X_\alpha \setminus \bigcup_{n\in M_\alpha} C_n$. We claim that $(\mathscr{U}_n)_{n\in\mathbb{N}\setminus M_\alpha}$ is a $G_\delta$-diagonal sequence for $Y$. Indeed, the maximality of $M_\alpha$ implies that $Y \subseteq C_n$ whenever $n \in \mathbb{N} \setminus M_\alpha$. If $x, y \in Y$, then by $(*)$, there exists $n$ such that $\{x, y\} \cap C_n$ is non-empty, and $\{x, y\} \cap U$ is at most a singleton for all $U \in \mathscr{U}_n$. By definition $Y \cap C_k$ is empty whenever $k \in M_\alpha$, so necessarily $n \in \mathbb{N} \setminus M_\alpha$. Our claim is proved.

By Chaber's result, $Y$ is compact. Therefore, there exists a finite set $\mathscr{F}_\alpha \subseteq \mathscr{V}$, such that

$$X_\alpha \setminus \bigcup_{n\in M_\alpha} C_n = Y \subseteq \bigcup \mathscr{F}_\alpha.$$

Define $X_{\alpha+1} = X'_\alpha = X_\alpha \setminus \bigcup \mathscr{F}_\alpha$. We have (i) immediately and (ii) follows because $M_\alpha \notin \mathscr{A}_{X_{\alpha+1}}$ and $\mathscr{A}_{X_{\alpha+1}} \subseteq \mathscr{A}_{X_\alpha}$. If $X_{\alpha+1}$ is empty, then we stop the recursion. If $\lambda$ is a countable limit ordinal and $X_\alpha$ is non-empty for all $\alpha < \lambda$, set $X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha$. Conditions (i) and (ii) follow. By countable compactness, $X_\lambda$ is also non-empty.

This process has to stop at a countable (successor) stage, because $(\mathscr{A}_{X_\alpha})$ is a strictly decreasing family of closed subsets of the separable metric space $\mathbb{P}(\mathbb{N})$. Thus, $X_{\alpha+1}$ is empty for some $\alpha < \omega_1$. By (i) we get

$$X \subseteq \bigcup_{\xi\leqslant\alpha} \bigcup \mathscr{F}_\xi,$$

and so $X$ is covered by $\bigcup_{\xi\leqslant\alpha} \mathscr{F}_\xi$. By a final application of countable compactness, we extract from this a finite subcover. □

The next result generalizes [**24**, Corollary 4.3] from descriptive spaces to spaces with $(*)$.

Corollary 4.4. *If $L$ is locally compact and has $(*)$, then $L \cup \{\infty\}$ is countably tight and sequentially closed subsets of $L \cup \{\infty\}$ are closed.*

*Proof.* The first assertion follows directly from Theorem 4.3 and the second follows from Proposition 4.2 and the fact that compact fragmentable spaces are sequentially compact (see [**9**, Lemma 2.1.1; **29**, Corollary 2.7]). Note that if $L$ is any locally compact space with $(*)$, then its 1-point compactification $L \cup \{\infty\}$ has $(*)$ also. All we need to do is adjoin to any $(*)$-sequence for $L$ the singleton family $\{L\}$, which separates all points in $L$ from $\infty$. □

Concerning stability properties of $(*)$ under mappings, we have the next result.

Proposition 4.5. *If $K$ is a scattered compact space with $(*)$ and $\pi : K \longrightarrow M$ is a continuous surjective map, then $M$ has $(*)$.*

*Proof.* If $K$ has $(*)$, then by Theorem 3.1, $C(K)^*$ admits a strictly convex dual norm $\|\cdot\|$. If we define $T : C(M) \longrightarrow C(K)$ by $T(f) = f \circ \pi$, then it is standard to check that

$$|||\nu||| = \inf\{\|\mu\| \,:\, T^*(\mu) = \nu\}$$

defines a strictly convex dual norm on $C(M)^*$. Therefore, $M$ has $(*)$, again by Theorem 3.1. □

The proof above is concise and straightforward, but also utterly opaque, as it leaves the reader with no idea of how to construct a $(*)$-sequence on $M$ in terms of a $(*)$-sequence on $K$. We outline a second approach to proving Proposition 4.5, which we include because we believe it gives the reader more idea of what is going on. The dual map $S = T^*$ above is a natural extension of $\pi$ if we identify points in $K$ and $M$ with their Dirac measures in $C(K)^*$ and $C(M)^*$, respectively. Set

$$\Sigma = \{\mu \in C(K)^* \,:\, \mu \text{ is positive and } \|\mu\|_1 = 1\}.$$

If $t \in M$ and $\mu \in \Sigma$, then $S(\mu) = t$ if and only if $\operatorname{supp}\mu \subseteq \pi^{-1}(t)$. Given a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$ on $K$ with the properties of Lemma 3.4, together with the unions $C_n$, define the $w^*$-compact and convex sets

$$D_{n,q,L} = \left\{\mu \in \Sigma \,:\, \mu\left(\bigcup_{i\in L} C_i \cup U\right) \leqslant q \text{ for all } U \in \mathscr{U}_n\right\},$$

where $n \in \mathbb{N}$, $q \in (0,1) \cap \mathbb{Q}$ and $L \subseteq \mathbb{N}$ is finite. The $D_{n,q,L}$ should be compared to the seminorms $\|\cdot\|_{n,k,L}$ in the proof of Theorem 3.1. Given distinct $s, t \in M$ and $\mu, \nu \in \Sigma$ in $S^{-1}(s)$ and $S^{-1}(t)$, respectively, by following the proof of Theorem 3.1, we can find $n$, $q$ and $L$ such that $\frac{1}{2}(\mu + \nu) \in D_{n,q,L}$, but $\{\mu, \nu\} \cap D_{n,q,L}$ is at most a singleton. There is less to consider in this case because as the supports of $\mu$ and $\nu$ are necessarily disjoint, the set $F$ in the proof of Theorem 3.1 is empty. This is why we only need to consider individual elements of $\mathscr{U}_n$ in the definition of the $D_{n,q,L}$, rather than finite subsets of $\mathscr{U}_n$ as in the definition of the $\|\cdot\|_{n,k,L}$.

By appealing to compactness and convexity, it is possible to select a finite set $G$ of triples $(n, q, L)$ with the property that if we consider the intersection $D_G = \bigcup_{(n,q,L)\in G} D_{n,q,L}$, then $D_G \cap S^{-1}(\frac{1}{2}(s + t))$ is non-empty, but either $D_G \cap S^{-1}(s)$ is empty or $D_G \cap S^{-1}(t)$ is empty. Equivalently, $\frac{1}{2}(s + t) \in S(D_G)$, but $\{s, t\} \cap S(D_G)$ is at most a singleton. The set $S(D_G)$ is $w^*$-compact and convex, so the complement $C(M)^* \setminus S(D_G)$ can be written as the union of a family $\mathscr{V}_G$ of $w^*$-open half-spaces of $C(M)^*$. From what we know, it can be easily verified that the families $\mathscr{V}_G$, as $G$ ranges over all finite subsets of triples $(n, q, L)$, induce a $(*)$-sequence on $M$.

Now we move on to examples. We are chiefly interested in exploring $(*)$, Gruenhage's property and the gap between them. Given that descriptive spaces are Gruenhage and spaces with $(*)$ are fragmentable, we shall confine our attention to spaces that are fragmentable but non-descriptive.

The first thing to point out is that $(*)$ is not equivalent to fragmentability, because $\omega_1$ is scattered (hence fragmentable), but does not have $(*)$. That $\omega_1$ does not have $(*)$ is clear, either directly from Corollary 4.4 or from Theorem 3.1 and [**39**, Théorème 3], which we mentioned in Section 1. Any locally compact space having $(*)$ necessarily has a countably tight 1-point compactification, but this condition is not sufficient. Hereafter, all of our examples of locally compact spaces without $(*)$ have countably tight 1-point compactifications.

Next, we consider trees. A *tree* $(T, \leqslant)$ is a partially ordered set with the property that given any $t \in T$, its set of predecessors $\{s \in T \,:\, s \leqslant t\}$ is well ordered. The tree order induces a natural locally compact, scattered *interval topology*. To render this topology Hausdorff, we shall only consider trees $T$ with the property that every non-empty totally ordered subset of $T$ has at most one minimal upper bound. An antichain is a subset of $T$, no two distinct elements of which are comparable. For further definitions and discussions about trees, and their role in renorming theory, we refer the reader to [**14**, **15**, **33**, **34**, **37**, **40**].

If $P$ and $Q$ are partially ordered sets, then we say that a map $\rho : P \longrightarrow Q$ is *strictly increasing* if $\rho(x) < \rho(y)$ whenever $x < y$. If such a map exists, then we write $P \preccurlyeq Q$. In [**33**, Definition 5], the second author introduced a totally ordered set $Y$ to address the problem of when $C_0(T)^*$ admits a strictly convex dual norm. We remark of $Y$ that $\mathbb{R} \preccurlyeq Y$, $Y^\alpha \preccurlyeq Y$ for all $\alpha < \omega_1$, where $Y^\alpha$ is ordered lexicographically, and finally $Y$ contains no uncountable, well-ordered subsets

[**33**, Section 4]. By combining Theorem 3.1 with [**34**, Corollary 17], we obtain the next result; see also [**37**, Theorem 26].

THEOREM 4.6. *If $T$ is a tree, then the following are equivalent*:
(i) *$T$ is Gruenhage*;
(ii) *$T$ has* $(*)$;
(iii) *$C_0(T)^*$ admits a strictly convex dual norm*;
(iv) *$T \preccurlyeq Y$.*

Note that the 1-point compactification $T \cup \{\infty\}$ of a tree $T$ is countably tight if and only if $T$ admits no uncountable branches. Indeed, suppose that $T$ admits no uncountable branches. Since each $t \in T$ admits a countable neighbourhood, the only point we need to test is $\infty$. If $\infty \in \bar{A}$ for some uncountable $A \subseteq T$, then by a standard result of Ramsey theory, either $A$ contains an uncountable totally ordered set or a countably infinite antichain $E$. Only the second possibility is valid, whence $\infty \in \bar{E}$. The converse implication follows immediately from the fact that $\omega_1 + 1$ is not countably tight. Thus, we restrict our attention to trees with no uncountable branches.

Given a partially ordered set $P$, we set

$$\sigma P = \{A \subseteq P \,:\, A \text{ is well-ordered}\}.$$

Kurepa introduced this notion and proved the following fact: for all $P$, we have $\sigma P \not\preccurlyeq P$. On the other hand, it is straightforward to show that $\sigma\mathbb{R}^\alpha \preccurlyeq \mathbb{R}^\alpha \times \{0,1\}$ (see [**33**, Proposition 23]). Moreover, it is known that $T$ is descriptive if and only if $T \preccurlyeq \mathbb{Q}$ (see [**33**, Theorem 4]). Therefore, we conclude that $\sigma\mathbb{Q}$ and $\sigma\mathbb{R}^\alpha$, $\alpha < \omega_1$, are all Gruenhage, non-descriptive spaces (see [**34**, p. 752] or [**37**, p. 405]). Instead, if we consider any total order $W$ satisfying $Y \preccurlyeq W$, then $\sigma W \not\preccurlyeq Y$ and so $\sigma W$ does not have $(*)$. In addition, if $W$ does not contain any uncountable well-ordered subsets, then $\sigma W$ is free of uncountable branches.

There is another type of tree without uncountable branches and without $(*)$. A subset $E$ of a tree is a *final part* if $u \in E$ whenever $t \in E$ and $t \leqslant u$. If $E$ is a final part, then we say that $E$ is *dense* if every element of $T$ is comparable with some element of $E$, and $T$ is called *Baire* if every countable intersection of dense final parts (which is itself a final part) are again dense. A subset $E$ is called *ever branching* if, given any $t \in E$, there exist incomparable elements $u, v \in E$ satisfying $t < u, v$. If $T$ admits an ever-branching Baire subtree, then $C_0(T)$ does not admit a Gâteaux norm [**14**, Theorem 2.1]. Therefore, no such tree can have $(*)$. An ever-branching Baire tree without uncountable branches exists; see [**14**, Proposition 3.1; **40**, Lemma 9.12]. Recall that a tree $T$ is called *Suslin* if it contains no uncountable branches or antichains. The existence of Suslin trees is independent of ZFC; see, for example, [**40**, Section 6]. Every Suslin tree contains an ever-branching Baire subtree [**40**, p. 246], so we conclude that no Suslin tree has $(*)$ either.

It is clear from Theorem 4.6 that in order to find examples of non-Gruenhage spaces with $(*)$, we must search further afield. A topological space $X$ is said to be *hereditarily separable* (HS) if every subspace of $X$ is separable. Clearly, the 1-point compactification of a locally compact HS space is countably tight. These spaces are interesting for us because if $K$ is compact, HS and non-metrizable, then it is automatically non-descriptive. This fact is stated in [**24**, Proposition 4.2] but no direct proof is given, so an argument is sketched here for completeness. If $\mathscr{H}$ is an isolated family of subsets of $K$, then $\mathscr{H}$ must be countable, because by hereditary separability there is a countable subset of $\bigcup\mathscr{H}$ which meets every member of $\mathscr{H}$. Therefore, if $K$ is a descriptive compact HS space, then it admits a countable network, whence it is metrizable.

Since we want compact, non-metrizable HS spaces that are also fragmentable, it is necessary to assume extra axioms. A space $X$ is *hereditarily Lindelöf* (HL) if every subspace of $X$ is Lindelöf. If $K$ is compact, fragmentable and HL, then it is metrizable (cf. [**19**, Corollary 9]). Thus, we want HS spaces that are not HL; such objects are called *S-spaces*. We refer the reader to [**31**] for an introduction to $S$-spaces and also the related $L$-spaces. It is known that under MA + ¬CH (where MA stands for Martin's axiom), there are no compact $S$-spaces (cf. [**31**, Theorem 6.4.1]), and in fact it is consistent that there are no $S$-spaces at all (cf. [**31**, Theorem 7.2.1]). Therefore, we must assume extra axioms if we are to find any animals in this particular zoo.

Our treatment of $S$-spaces proceeds as follows. First, we outline two approaches for constructing $S$-spaces by refining existing topologies, and show that these yield Gruenhage spaces. Second, we take advantage of these two approaches to provide examples of compact non-Gruenhage spaces of cardinality $\aleph_1$ having $(*)$ and show, given a further mild assumption, that no object of this kind can exist under MA + ¬CH. Finally, we present a third method of constructing $S$-spaces and show that no space built in this way can have $(*)$.

The spaces developed using the first approach are sometimes called 'Kunen lines', despite the fact that none of them are linearly ordered. Assuming CH, the authors of [**18**] develop a machine that accepts as input a first countable HS space $(X, \rho)$ of cardinality $\aleph_1$, and generates a finer topology $(X, \tau)$, which is locally compact, scattered, HS and non-Lindelöf. In applications, $X$ is usually a subset of $\mathbb{R}$ and $\rho$ is the induced metric topology.

Later, this process was developed to ensure that $(X, \tau)^n$ is HS for all $n \in \mathbb{N}$; see [**23**, Section 7]. The resulting 1-point compactification $\mathcal{K}$ is known to Banach space theorists as 'Kunen's compact $S$-space'. It is not explicitly stated in [**23**, Section 7] that the resulting topology on $X$ refines that of the real line, but the authors believe that it is meant to. If the topology is such a refinement, then necessarily the Euclidean diameters of the $B_k^\alpha$ (which form the building blocks of neighbourhoods of points; see [**23**, (8), p. 1124]) have to tend to 0 as $k \to \infty$. It can be checked that this condition is also sufficient to produce a refinement. We note further that an alternative approach to [**23**, Section 7] is given in [**42**, Theorem 2.4], and there, the fact that the original topology is refined is explicitly stated.

Of course, it is clear that any refinement of a Gruenhage space is again Gruenhage, because we can use exactly the same open sets to separate points. Therefore, assuming the adjustment to the diameters of the $B_k^\alpha$ above, we have the following result.

PROPOSITION 4.7. *The Kunen lines are Gruenhage spaces. In particular, $C(\mathcal{K})^*$ admits a strictly convex dual norm, the predual of which is necessarily Gâteaux smooth.*

The second approach refines topologies as above, but this time using the axiom $\mathfrak{b} = \aleph_1$, where $\mathfrak{b}$ is the minimal cardinality of a subset of Baire's space $\mathbb{N}^{\mathbb{N}}$, which has no upper bound with respect to the partial ordering $<^*$ of eventual dominance ($x <^* y$ if and only if $x(n) < y(n)$ for all but finitely many $n$). Evidently, $\mathfrak{b} \leqslant \mathfrak{c}$ and CH implies $\mathfrak{b} = \aleph_1$. More interestingly, the assertion $(\mathfrak{b} = \aleph_1) + \neg$CH is relatively consistent.

Under $\mathfrak{b} = \aleph_1$, it is shown in [**41**, Theorem 2.5] that the topology of any set of reals of cardinality $\aleph_1$ may be refined to give a locally compact, scattered, non-Lindelöf topology which is HS in its finite powers.

PROPOSITION 4.8. *The spaces of Todorčević in* [**41**, Theorem 2.5] *are Gruenhage.*

Before presenting our third approach to construct $S$-spaces, we give our examples under additional axioms of compact, scattered non-Gruenhage spaces with $(*)$. We shall utilize the methods of Kunen, Todorčević and others, and adopt an idea from [1]. The spaces could also be compared, at some distance, to the split interval.

In fact, we construct locally compact, scattered non-Gruenhage spaces with $G_\delta$-diagonals. The 1-point compactifications of these spaces have $(*)$. For our examples, we shall make use of the following observation about Gruenhage spaces of cardinality no larger than the continuum.

Proposition 4.9 [37, Proposition 2]. *Let $X$ be a topological space with* $\operatorname{card} X \leqslant \mathfrak{c}$. *Then $X$ is Gruenhage if and only if there is a sequence $(U_n)_{n=1}^{\infty}$ of open subsets of $X$ with the property that if $x, y \in X$, then $\{x, y\} \cap U_n$ is a singleton for some $n$.*

The hypotheses of the next theorem isolate the ingredients essential to our examples.

Theorem 4.10. *Suppose that we have a metric $d$ on $\omega_1$ and, for all but countably many $\alpha < \omega_1$, an injective sequence $s_\alpha = (s_{\alpha,n}) \subseteq \alpha$ which converges to $\alpha$ with respect to $d$. Moreover, suppose that whenever $J \subseteq \omega_1$ is uncountable, there exists $\alpha \in J$ such that $s_\alpha \cap J$ is infinite. Then there exists a locally compact, scattered, first countable non-Gruenhage (Hausdorff) space of cardinality $\aleph_1$ with a $G_\delta$-diagonal.*

*Proof.* Fix $\alpha_0$ such that $s_\alpha$ exists as above for $\alpha \geqslant \alpha_0$. Set $X = \omega_1 \times \{\pm 1\}$, let $q : X \longrightarrow \omega_1$ be the natural projection and define $t : X \longrightarrow X$ by $t(\alpha, i) = (\alpha, -i)$. We obtain our topology on $X$ by building increasing topologies $\tau_\alpha$ on the sets $\alpha \times \{\pm 1\}$, $\alpha < \omega_1$, by transfinite induction. Hereafter, diam will denote diameters with respect to $d$. The points $(\alpha, i)$, $i = \pm 1$, will have a countable base of compact open neighbourhoods $U(\alpha, i, n)$, $n \in \mathbb{N}$, such that the following conditions are satisfied:

(i) if $\xi < \alpha$, then $\xi \times \{\pm 1\} \in \tau_\alpha$ and $\tau_\xi$ is the topology on $\xi \times \{\pm 1\}$ induced by $\tau_\alpha$;
(ii) $U(\alpha, i, n) \setminus \{(\alpha, i)\} \subseteq \alpha \times \{\pm 1\}$;
(iii) $\operatorname{diam}(q(U(\alpha, i, n))) < 2^{-n}$;
(iv) $U(\alpha, -i, n) = t(U(\alpha, i, n))$;
(v) $q\restriction_{U(\alpha,i,n)}$ is injective;
(vi) for every $n$, $i$ and $\alpha \geqslant \alpha_0$, the set $(s_\alpha \times \{-i\}) \setminus U(\alpha, i, n)$ is finite.

To take care of limit stages $\alpha$, we set

$$\tau_\alpha = \{U \subseteq \alpha \times \{\pm 1\} : U \cap (\xi \times \{\pm 1\}) \in \tau_\alpha \text{ for all } \xi < \alpha\}.$$

Now assume that $\tau_\alpha$ has been found. We define $\tau_{\alpha+1}$ by constructing neighbourhoods $U(\alpha, i, n)$, $n \in \mathbb{N}$, of the points $(\alpha, i)$, $i = \pm 1$.

If $\alpha < \alpha_0$, then set $U(\alpha, i, n) = \{(\alpha, i)\}$ for $i = \pm 1$ and $n \in \mathbb{N}$. Henceforth, we assume that $\alpha \geqslant \alpha_0$. Since $s_{\alpha,n} \to \alpha$, we can select $l_1 < l_2 < l_3 < \ldots$ to ensure that

(a) $\operatorname{diam}(\{s_{\alpha,m} : m \geqslant l_n\}) < 2^{-n}$ for each $n$.

By considering (iii) applied to $\xi < \alpha$, and (a) above, for every $m$ we can find $k_m$ such that

(b) $q(U(s_{\alpha,m}, -1, k_m)) \cap q(U(s_{\alpha,m'}, -1, k_{m'})) = \varnothing$,

whenever $m \neq m'$ and

(c) $\operatorname{diam}\left(q\left(\bigcup_{m \geqslant l_n} U(s_{\alpha,m}, -1, k_m)\right)\right) < 2^{-n}$,

for every $n$. Finally, define

$$U(\alpha, i, n) = \{(\alpha, i)\} \cup \bigcup_{m \geqslant l_n} U(s_{\alpha,m}, -i, k_m).$$

These neighbourhoods are compact and open. Extend $\tau_\alpha$ to $\tau_{\alpha+1}$ in the obvious way. It is clear that we have (i) and (ii), and that $\tau_{\alpha+1}$ is locally compact. Property (iii) follows from (c) above. That $\tau_{\alpha+1}$ is Hausdorff is a consequence of the inductive hypothesis, (iii), and the fact that $U(\alpha,1,1) \cap U(\alpha,-1,1) = \varnothing$. Properties (iv) and (v) follow from the inductive hypothesis, the definition of $U(\alpha,i,n)$ and (b). Lastly, (vi) holds because

$$\{(s_{\alpha,m},-i) \,:\, m \geqslant l_n\} \subseteq U(\alpha,i,n).$$

This completes the induction. The topology on $X$ is given by

$$\{U \subseteq X \,:\, U \cap (\alpha \times \{\pm 1\}) \in \tau_\alpha \text{ for all } \alpha < \omega_1\}.$$

We show that $X$ is scattered. If $E \subseteq X$ is non-empty, then let $\alpha$ be minimal, subject to $E \cap \{(\alpha,\pm 1)\}$ being non-empty. If $(\alpha,i) \in E$, then by (i) and (ii), $U = (\alpha \times \{\pm 1\}) \cup \{(\alpha,i)\}$ is open, and $E \cap U = \{(\alpha,i)\}$.

Next, we show that $X$ has a $G_\delta$-diagonal. Set

$$\mathscr{G}_n = \{U(\alpha,i,n) \,:\, (\alpha,i) \in X\}.$$

Let $(\alpha,i),(\beta,j) \in X$. If $\alpha \neq \beta$, then pick $n$ such that $d(\alpha,\beta) \geqslant 2^{-n}$. We cannot have $(\beta,j) \in \operatorname{st}((\alpha,i),n)$ because, if so, then $(\alpha,i),(\beta,j) \in U(\gamma,k,n)$ for some $(\gamma,k)$, giving

$$d(\alpha,\beta) \leqslant \operatorname{diam}\left(q(U(\gamma,k,n))\right) < 2^{-n}$$

by (iii). If $\alpha = \beta$ and $i \neq j$, then, by (v), we cannot have $(\alpha,i),(\beta,j) \in U(\gamma,k,n)$ for any $(\gamma,k)$ or $n$. Whatever the case,

$$\bigcap_{n=1}^{\infty} \operatorname{st}((\alpha,i),n) = \{(\alpha,i)\}.$$

This shows that $(\mathscr{G}_n)_{n=1}^\infty$ is a $G_\delta$-diagonal sequence.

Finally, we prove that $X$ is not Gruenhage. Bearing in mind Proposition 4.9, we suppose for a contradiction that there exists a sequence of open subsets $(V_n)_{n=1}^\infty$, with the property that, given $\alpha < \omega_1$, we can find an $n$ that forces

$$\{(\alpha,1),(\alpha,-1)\} \cap V_n$$

to be a singleton. Define

$$J_{n,i} = \{\alpha < \omega_1 \,:\, (\alpha,i) \in V_n \text{ and } (\alpha,-i) \notin V_n\}.$$

By assumption, $\omega_1 = \bigcup_{n,i} J_{n,i}$, so there exist $n$ and $i$ such that $J = J_{n,i}$ is uncountable. Given the hypotheses, the intersection $s_\alpha \cap J$ is infinite for some $\alpha \in J$. Since $\alpha \in J$, we have $(\alpha,i) \in V_n$, so take $m$ satisfying $U(\alpha,i,m) \subseteq V_n$. From (vi) we know that

$$U(\alpha,i,m) \cap ((s_\alpha \cap J) \times \{-i\}) \subseteq V_n \cap (J \times \{-i\})$$

is non-empty. However, this violates the definition of $J$. This contradiction establishes that $X$ is not Gruenhage. □

Now we use axioms to obtain the hypotheses of Theorem 4.10 in two different ways.

Example 2 (CH or $\mathfrak{b} = \aleph_1$). There exist locally compact, scattered, first countable Hausdorff, non-Gruenhage spaces of cardinality $\aleph_1$ with $G_\delta$-diagonals.

*Proof.* Assuming CH, we follow Kunen. Let $(x_\alpha)_{\alpha<\omega_1}$ be a family of distinct real numbers, set $d(\alpha,\beta) = |x_\alpha - x_\beta|$ and let $(A_\xi)_{\xi<\omega_1}$ be an enumeration of all countable subsets of $\omega_1$. Since $(\omega_1,d)$ is separable, there exists $\alpha_0 < \omega_1$ such that $\alpha \in \overline{\alpha_0}^d$ whenever $\alpha \geqslant \alpha_0$. Given such $\alpha$, define $F_\alpha = \{\xi \leqslant \alpha : A_\xi \subseteq \alpha \text{ and } \alpha \in \overline{A_\xi}^d\}$. Since $F_\alpha$ is at most countable, we can find an injective sequence $s_\alpha \subseteq \alpha$ converging to $\alpha$, such that $s_\alpha \cap A_\xi$ is infinite whenever $\xi \in F_\alpha$. Let $J \subseteq \omega_1$ be uncountable. As $(J,d)$ is separable, $A_\xi \subseteq J \subseteq \overline{A_\xi}^d$ for some $\xi < \omega_1$. Because $J$ is uncountable, we can take $\alpha \in J$ large enough to ensure that $\xi \in F_\alpha$, thus $s_\alpha \cap J$ is infinite.

Now assume $\mathfrak{b} = \aleph_1$. With this axiom, we can find a transfinite sequence $(x_\alpha)_{\alpha<\omega_1}$ of increasing sequences in $\mathbb{N}^{\mathbb{N}}$ that is strictly increasing and unbounded in $\mathbb{N}^{\mathbb{N}}$, with respect to $<^*$. Given $\alpha \neq \beta$, define $d(\alpha, \beta) = 2^{-n}$, where $n$ is the least natural number satisfying $x_\alpha(n) \neq x_\beta(n)$, that is, $d$ is induced by the natural metric on $\mathbb{N}^{\mathbb{N}}$. The sequences $s_\alpha$ are obtained by considering the sets $H(b)$ defined in [**41**, Chapter 2]. Given $b = x_\alpha$, the set $H(b) \subseteq \{x_\xi : \xi < \alpha\}$, whenever infinite, converges to $b$ in the natural topology of $\mathbb{N}^{\mathbb{N}}$ (with respect to any enumeration). Moreover, whenever $J \subseteq \omega_1$ is uncountable, $H(b) \cap \{x_\xi : \xi \in J\}$ is infinite for some $\alpha \in J$ (see [**41**, Lemma 2.1]). Clearly, $H(b)$ is finite for at most countably many $b$. □

Together with Theorem 3.1, these examples show that if $C(K)^*$ admits a strictly convex dual norm, then $K$ is not necessarily Gruenhage. This gives a consistent negative solution to [**34**, Problem 14; **37**, Problem 4].

It is possible to make $X$ of Theorem 4.10 HS. Suppose that each $s_\alpha$ above can be partitioned into two sequences $s^j_\alpha$, $j = \pm 1$, with the property that given an uncountable set $J \subseteq \omega_1$, both intersections $s^j_\alpha \cap J$, $j = \pm 1$, are infinite for some $\alpha \in J$. Define

$$U(\alpha, i, n) = \{(\alpha, i)\} \cup \bigcup_{m \geqslant l_n} U(s_{\alpha,m}, j_{\alpha,m} i, k_m),$$

where $j_{\alpha,m} \in \{1, -1\}$ is chosen so that $s_{\alpha,m} \in s^{j_{\alpha,m}}_\alpha$. Then (vi) becomes

(vi)′ for every $n$, $i$, $j$ and $\alpha \geqslant \alpha_0$, the set $(s^j_\alpha \times \{ji\}) \setminus U(\alpha, i, n)$ is finite.

The proof that $X$ is non-Gruenhage follows just as above, by considering $j = -1$. To see that $X$ is HS, suppose that $E \subseteq X$ is non-separable. By transfinite induction, we can find $i$ and an uncountable relatively discrete subspace $F \subseteq E \cap (\omega_1 \times \{i\})$. Now consider $j = 1$.

Assuming CH, we can find these $s^j_\alpha$ by choosing $s_\alpha$ in such a way that $\{n \in \mathbb{N} : s_{\alpha,2n}, s_{\alpha,2n+1} \in A_\xi\}$ is infinite whenever $\xi \in F_\alpha$, where $F_\alpha$ is as in the proof of Example 2. Then set $s^j_{\alpha,n} = s_{\alpha,2n+(j+1)/2}$. The authors suspect that the sets $H(b)$ of Todorčević [**41**, Chapter 2] can be partitioned to obtain the $s^j_\alpha$ when $\mathfrak{b} = \aleph_1$.

There is no hope of constructing spaces like those in Example 2 in ZFC. A space $X$ is called *locally countable* if every point of $X$ admits a countable neighbourhood. For example, trees of height at most $\omega_1$ and 'thin-tall' locally compact spaces are locally countable. It is straightforward to see that a locally compact, locally countable space must be scattered.

Proposition 4.11 (MA + ¬CH). *Suppose that $L$ is a locally compact, locally countable space with $(*)$ and* card $L < \mathfrak{c}$. *Then $L$ is $\sigma$-discrete.*

*Proof.* This follows immediately from Corollary 4.4 and [**3**, Theorem 2.1]. □

We end this section by presenting our third class of $S$-spaces. We shall call a regular, uncountable topological space $X$ an *$O$-space* if every open subset of $X$ is either countable or co-countable. Ostaszewski constructed a locally compact, scattered $O$-space using the clubsuit axiom ♣ [**27**, p. 506]. It is known that ♣ is independent of CH and that ♣ + CH is equivalent to Jensen's axiom ◊ (see [**27**, p. 506; **32**], respectively). It is possible to obtain $O$-spaces by assuming principles strictly weaker than ♣ [**17**, Theorem 2.1]. Unlike the previous constructions, these spaces are built from scratch, rather than by refining an initial space.

Every $O$-space contains an S-subspace. Indeed, if $X$ is an $O$-space, then note that at most one point of $X$ can fail to have a countable open neighbourhood. Thus, we can construct by induction an uncountable subspace $Y = \{x_\alpha : \alpha < \omega_1\}$ such that $\{x_\xi : \xi < \alpha\}$ is open in $Y$ for every $\alpha < \omega_1$. Thus, $Y$ is not Lindelöf. If, for a contradiction, we suppose that $Z \subseteq Y$ is not separable, then, as above, we can construct an uncountable, relatively discrete subspace of $Y$

by induction. However, this cannot exist by the $O$-space property. Therefore, $Y$ is an $S$-space. We can argue similarly to establish that every locally compact $O$-space has a countably tight 1-point compactification.

PROPOSITION 4.12. *If $X$ is an $O$-space, then it does not have* $(*)$.

*Proof.* Suppose that $(\mathscr{U}_n)_{n=1}^{\infty}$ is a $(*)$-sequence for $X$, with $C_n = \bigcup \mathscr{U}_n$ for each $n$. Set

$$J = \{n \in I \,:\, C_n \text{ is uncountable}\}.$$

If $n \in J$, then $X \setminus C_n$ is countable, so

$$E = \bigcup_{n\in J}(X \setminus C_n) \cup \bigcup_{n\in\mathbb{N}\setminus J} C_n$$

is also countable. If we let $A = X \setminus E$, then we see that $A \subseteq C_n$ for all $n \in J$, and $A \cap C_n$ is empty whenever $n \notin J$. For $x \in A$ and $n \in J$, define

$$\mathrm{st}(x, n) = \bigcup\{U \in \mathscr{U}_n \,:\, x \in U\}.$$

Since $(\mathscr{U}_n)_{n=1}^{\infty}$ is assumed to be a $(*)$-sequence for $X$, we have

$$\{x\} = A \cap \bigcap_{n\in J} \mathrm{st}(x, n),$$

for all $x \in A$, that is, $(\mathscr{U}_n)_{n=1}^{\infty}$ induces a $G_\delta$-diagonal sequence on $A$. Given this, it follows that, for each $x \in A$, there exists some $n_x \in J$ such that $\mathrm{st}(x, n_x)$ is countable. Indeed, otherwise,

$$E \cup \bigcup_{n\in J}(X \setminus \mathrm{st}(x, n))$$

is countable, rendering

$$\{x\} = A \cap \bigcap_{n\in J} \mathrm{st}(x, n)$$

uncountable. Since $A$ is uncountable, there exists $n$, which we fix from now on, such that $B = \{x \in A \,:\, n_x = n\}$ is uncountable. Take an enumeration $(x_\alpha)_{\alpha<\omega_1}$ of distinct points in $B$. We find $\alpha_0 < \alpha_1 < \alpha_2 < \ldots < \omega_1$ such that

$$x_{\alpha_\eta} \notin \bigcup_{\xi<\eta} \mathrm{st}(x_{\alpha_\xi}, n),$$

for all $\eta < \omega_1$. Observe that by the symmetry of the sets $\mathrm{st}(x, n)$, we have $x_{\alpha_\xi} \notin \mathrm{st}(x_{\alpha_\eta}, n)$ whenever $\xi \neq \eta$. Therefore, $C = \{x_{\alpha_\xi} \,:\, \xi < \omega_1\}$ is a relatively discrete subspace, which is not permitted by the $O$-space property. □

EXAMPLE 3. Ostaszewski's space [**27**, p. 506] is a locally compact, scattered HS $O$-space. Therefore, it does not have $(*)$.

By refining Ostaszewski's construction, it is possible to use ♣ to build a compact, scattered non-metrizable space $K$, such that $K^n$ is HS for all $n$ (see [**13**, Theorem 4.36]). Moreover, it can be checked that this $K$ is, in addition, an $O$-space. Therefore, unlike $C(\mathcal{K})^*$, the space $C(K)^*$ admits no strictly convex dual norm.

We make a remark about this $C(K)$: the authors do not know if it admits a Gâteaux norm. Since $K$ is separable, $C(K)$ admits a bounded linear, injective map into $c_0$. The authors do not know of any example of an Asplund space with an injective map into a $c_0(\Gamma)$, which does not admit a Gâteaux norm.

## 5. *Problems*

To finish, we present a number of related, unresolved problems. The first problem stems from Theorem 3.1.

PROBLEM 1. If $K$ has $(*)$ and is not scattered, then does $C(K)^*$ admit a strictly convex dual norm?

In fact, we do not even know if $C(L \cup \{\infty\})^*$ admits a strictly convex dual norm whenever $L$ is a locally compact space having a $G_\delta$-diagonal. Proposition 4.5 suggests the next problem.

PROBLEM 2. If $K$ has $(*)$ and is not scattered, and $\pi : K \longrightarrow M$ is a continuous, surjective map, then does $M$ have $(*)$? More generally, if a topological space $X$ has $(*)$ and $f : X \longrightarrow Y$ is a perfect, surjective map, does $Y$ have $(*)$?

It is known that the answer to Problem 2 is positive in the Gruenhage case, including the more general perfect map assertion [**34**, Theorem 23]. It is also known that $G_\delta$-diagonals are not preserved under perfect images. In [**4**, Example 2], an example is given of a locally compact scattered space $L$ having a $G_\delta$-diagonal, and a perfect surjective map $f : L \longrightarrow M$, where the diagonal of $M$ is not a $G_\delta$. However, $L^{(2)}$ is empty, and the same will apply to any perfect image of $L$, so all such images are $\sigma$-discrete and therefore have $(*)$. If Problem 1 has a positive solution, then so will the first part of Problem 2, simply by copying the proof of Proposition 4.5.

For our last problem, we refer the reader to the end of Section 4.

PROBLEM 3. Does $C(K)$ admit a Gâteaux norm, where $K$ is the $O$-space of Ostaszewski [**27**, p. 506] or Hájek, Montesinos, Vanderwerff and Zizler [**13**, Theorem 4.36]?

REMARK 3. Recently, the second-named author gave an example of a scattered, non-Gruenhage compact space having $(*)$, without using extra axioms [**35**].

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