On $\mathcal{T}_p$-locally uniformly rotund norms

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Dedicated to Petar S. Kenderov on the occcasion of his 70th birthday

1 Abstract

Liinear topological characterizations of Banach spaces $E \subset \ell^\infty(\Gamma)$ which admit pointwise locally uniformly rotund norms are obtained. We introduce a new way to construct the norm with families of sliced sets. The topological properties described are related with the theory of generalized metric spaces, in particular with Moore spaces and σ-spaces. A non liner transfer is obtained, Question 6.16 in [28] is answered and some connections with Kenderov's School of Optimization is presented in this paper.

2 Introduction

Renorming theory tries to find isomorphisms for Banach spaces that improve their norms. That means to make the geometrical and topological properties of the unit ball of a given Banach space as close as possible to those of the unit ball of a Hilbert space. The existence of equivalent *good* norms on a particular Banach space depends on its structure and has in turn a deep impact on its geometrical properties. Questions concerning renormings in Banach spaces have been of particular importance to provide smooth functions and tools for optimization theory. An excellent monograph of renorming theory up to 1993 is [5]. In order to have an up-to-date account of the theory we should add [21, 11, 39, 28, 38, 1].

If $(E, \|\cdot\|)$ is a normed space, the norm $\|\cdot\|$ is said to be locally uniformly rotund (**LUR** for short) if

$$\left[\lim_n (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0\right] \Rightarrow \lim_n \|x - x_n\| = 0$$

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for any sequence (x_n) and any x in E. The construction of this kind of norms can be done if we provide enough convex functions on the given space E and we are able to add all of them up with the powerful lemma of Deville, Godefroy and Zizler, see [5, Lemma VII.1.1],[39]. It reads as follows:

Lemma 1 (Deville, Godefroy and Zizler decomposition method). *Let $(E, \|\cdot\|)$ be a normed space, let I be a set and let $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ be families of non-negative convex functions on E which are uniformly bounded on bounded subsets of E. For every $x \in E$, $m \in \mathbb{N}$ and $i \in I$ define*

$$\varphi(x) = \sup\{\varphi_i(x) : i \in I\}, \tag{1}$$
$$\theta_{i,m}(x) = \varphi_i(x)^2 + 2^{-m}\psi_i(x)^2, \tag{2}$$
$$\theta_m(x) = \sup\{\theta_{i,m}(x) : i \in I\}, \tag{3}$$
$$\theta(x) = \|x\|^2 + \sum_{m=1}^{\infty} 2^{-m}(\theta_m(x) + \theta_m(-x)). \tag{4}$$

Then the Minkowski functional of $B = \{x \in E : \theta(x) \leq 1\}$ is an equivalent norm $\|\cdot\|_B$ on E such that if $x_n, x \in E$ satisfy the LUR condition:

$$\lim_n [2\|x_n\|_B^2 + 2\|x\|_B^2 - \|x_n + x\|_B^2] = 0,$$

then there is a sequence (i_n) in I with the properties:

1. $\lim_n \varphi_{i_n}(x) = \lim_n \varphi_{i_n}(x_n) = \lim_n \varphi_{i_n}((x + x_n)/2) = \sup\{\varphi_i(x) : i \in I\}$
2. $\lim_n \left[\frac{1}{2}\psi_{i_n}^2(x_n) + \frac{1}{2}\psi_{i_n}^2(x) - \psi_{i_n}^2\left(\frac{1}{2}(x_n + x)\right)\right] = 0.$

The previous Lemma is the core of the decomposition method for renormings of non separable Banach spaces as described in [5, Chapter VII]. It has been extensively used by R. Haydon in his seminal papers [21, 22] as well as in [23]. Lemma 1 was first introduced by R. Deville and it is based on the construction of an equivalent **LUR** norm on Banach spaces with strong M-basis, [13, Theorem 3.48]. Let us note that if we add lower semicontinuity properties on the involved functions $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ we obtain lower semicontinuity for the new norm $\|\cdot\|_B$.

In order to deal with renorming results valid for the very different weak topologies that appear in the Banach space context we fix, from now on, $(E, \|\cdot\|) \subset \ell^\infty(\Gamma)$ a normed space and we shall deal with the pointwise convergence topology $\mathcal{T}_p$ induced on it; i.e. the product topology of $\mathbb{R}^\Gamma$ induced on E. In order to have $\mathcal{T}_p$-lower semicontinuous renormings we shall work with the linear subspace $F \subset E^*$ generated by the evaluation functionals $\pi_\gamma(x) := x(\gamma)$ for every $x \in E$ and $\gamma \in \Gamma$.

Our approaches for **LUR** renormings are strongly based on the topological concept of *network*. A family of subsets $\mathcal{N}$ in a topological space $(T, \mathcal{T})$ is a network for the topology $\mathcal{T}$

if for every $W \in \mathcal{T}$ and every $x \in W$, there is some $N \in \mathcal{N}$ such that $x \in N \subset W$. A central result for the theory is the following one, see [26, 35, 28]:

Theorem 1 (Slicely Network)**.** *Let E be a normed subspace of $\ell^\infty(\Gamma)$ and $\mathcal{H}$ the family of $\mathcal{T}_p$-open half spaces in E. Then E admits a $\mathcal{T}_p$-lower semicontinuous equivalent* **LUR** *norm if, and only if, there is a sequence (A_n) of subsets of E such that the family of sets*

$$\{A_n \cap H : H \in \mathcal{H}, n \in \mathbb{N}\}$$

is a network for the norm topology in E

The first proof of this result used martingale constructions, [26]. Without martingales a delicate process of convexification of the sets A_n is needed to construct a countable family of seminorms which controls the claim, [35]. Stone's theorem is required if additional information on the structure of the sets A_n is needed, see [26, Chapter 3]. After the slice localization Theorem 3 of [32] the convexification process is not necessary any more. The main construction can be done with the use of Lemma 1 together with Theorem 3 of [32] which seems to be a main tool for the matter. It says that given any family of slices of a bounded set A of a normed space E, it is always possible to construct an equivalent norm such that the **LUR** condition for a sequence (x_n), and a fixed point x in A, implies that the sequence eventually belongs to slices containing the point x. When the slices involved have small diameter, then the sequence is eventually close to x. If the diameter can be made small enough, then the sequence (x_n) converges to x and the new norm is going to be locally uniformly rotund at the point x. In this paper we present an extension of this result to deal with countably many sliced sets which reads as follows:

Theorem 2 (Multiple-slice localization theorem)**.** *Let E be a normed subspace of $\ell^\infty(\Gamma)$. Given sequences $(A_p)_{p=1}^\infty$ and $(\mathcal{H}_n)_{n=1}^\infty$ of bounded subsets A_p of E and families $\mathcal{H}_n$ of $\mathcal{T}_p$-open half spaces respectively, there is an equivalent $\mathcal{T}_p$-lower semicontinuous norm $\|\cdot\|_0$ such that, for every finite selection of pairs of positive integers $(m_1, p_1), (m_2, p_2), \cdots, (m_r, p_r)$, every $x \in \cap_{j=1}^r A_{p_j} \cap \bigcup \mathcal{H}_{m_j}$, and any sequence (x_n) in E with*

$$\lim_n \left(2\|x_n\|_0^2 + 2\|x\|_0^2 - \|x + x_n\|_0^2\right) = 0,$$

it follows that there are sequences of $\mathcal{T}_p$-open half spaces

$$\{(H_n^1, \ldots, H_n^r) \in \mathcal{H}_{m_1} \times \cdots \times \mathcal{H}_{m_r}, n = 1, 2 \ldots\}$$

such that

1. *There is $n_0 \in \mathbb{N}$ with $x \in \cap_{j=1}^r H_n^j$ and $x_n \in H_n^j$ if $x_n \in A_{p_j}$ for any $j = 1, \ldots, r$ whenever $n \geq n_0$ and*
2. *For every $\delta > 0$ there is some n_δ such that*

$$x, x_n \in \overline{\left(\operatorname{co}(\cap_{j=1}^r A_{p_j} \cap H_n^j) + B(0, \delta)\right)}^{\mathcal{T}_p}$$

for all $n \geq n_\delta$.

The main concept we are going to deal with in this paper is the following one:

Definition 1. *If $E \subset \ell^\infty(\Gamma)$ is a normed space we say that its norm $\|\cdot\|_\infty$ is $\mathcal{T}_p$-locally uniformly rotund ($\mathcal{T}_p-$**LUR**) if*

$$\left[\lim_n (2\|x\|_\infty^2 + 2\|x_n\|_\infty^2 - \|x + x_n\|_\infty^2) = 0\right] \Rightarrow \mathcal{T}_p - \lim_n x_n = x$$

for any sequence (x_n) and any x in E.

If Γ is a countably determined topological space and the normed space E consists of continuous functions on Γ, a result of Mercourakis asserts that E admits an equivalent $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p-$**LUR** norm, [24]. Another situation where these happens has been described by Raja for dual spaces $E^* \subset \ell^\infty(B_E)$ when the dual unit ball B_{E^*} is a descriptive compact space with the w*-topology, [37]. It is possible to show that the strictly convex norms constructed by Dashiell and Lindenstrauss in [4] are pontwise LUR subspaces of $\ell^\infty([0,1])$, see [10]. Section 3.3 of [28] gives information of covering properties implied by this kind of norms. A main result was obtained in [27] where it is proved that a normed space E wit a $\sigma(E, E^*)-$**LUR** norm has an equivalent **LUR** norm. In this paper we give an answer to Question 6.16 in [28] presenting a nonlinear transfer result for this kind of norms. As it is said there a good linear topological characterization of the property to have a pointwise-**LUR** renorming is needed to challenge on the problem. Let us summarize in a single theorem the different characterizations we have obtained. Any of them is a main contribution of the present paper. We need to remember the following:

Definition 2. *A family $\mathcal{B} := \{B_i : i \in I\}$ of subsets of $E \subset \ell^\infty(\Gamma)$ is called $\mathcal{T}_p$-slicely isolated (or $\mathcal{T}_p$-slicely relatively discrete) if it is a disjoint family of sets such that for every*

$$x \in \bigcup\{B_i : i \in I\}$$

there exist a $\mathcal{T}_p$-open half space H and $i_0 \in I$ such that

$$H \bigcap \bigcup \{B_i : i \in I, i \neq i_0\} = \emptyset \text{ and } x \in B_{i_0} \cap H.$$

Let us formulate our characterization result for $\mathcal{T}_p-$**LUR** renormings:

Theorem 3 (Main Theorem)**.** *Let Γ be a non void set and E be a subspace of the Banach space $l^\infty(\Gamma)$. The following are equivalent:*

1. *E admits an equivalent $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p$-locally uniformly rotund norm.*
2. *There is a metric ρ on E generating a topology finer then $\mathcal{T}_p$ and a sequence of subsets (A_n) in E such that the family of sets*

$$\{A_n \cap H : H \in \mathcal{H}, n \in \mathbb{N}\},$$

where $\mathcal{H}$ is the family of all $\mathcal{T}_p$-open half spaces in E, is a network for the metric topology associated with ρ on E

3. *The pointwise convergence topology $\mathcal{T}_p$ on E has a σ-slicely isolated network.*

4. *There is a metric d on E generating a topology finer than $\mathcal{T}_p$ and coarser than the norm topology, with a basis $\mathcal{S}$ for the metric topology which also is a network for $\mathcal{T}_p$ that admits a decomposition as*

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$$

where every $\mathcal{S}_n$ is $\mathcal{T}_p$-slicely isolated and norm discrete family of norm open sets.

5. *There are sequences of sets (A_n) in E and of families of $\mathcal{T}_p$-open half spaces $(\mathcal{H}_m)$ such that for every $\mathcal{T}_p$-open half space L and every $x \in L$ there are $n, m \in \mathbb{N}$ such that $x \in A_n \cap H_0$ for some $H_0 \in \mathcal{H}_m$ and*

$$x \in A_n \cap \bigcup \{H : x \in H \in \mathcal{H}_m\} \subset L$$

Let us stress the interaction between metrics and norms in the former theorem and its connection with previous studies by P. Kenderov and W. Moors,[17], when dealing with fragmentable Banach space. Indeed every normed space $E \subset \ell^\infty(\Gamma)$ with an equivalent $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p - \mathbf{LUR}$ norm is going to be fragmentable. Moreover, for every metric ρ satisfying 2 above the Borel subsets for ρ and $\mathcal{T}_p$ are the same.

As a consequence we obtain non-linear transfer results in the last section of the paper. Hopefully they could be applied to solve some of the important questions that remain open in the area.

2.1 Notations

Most of our notation and terminology are standard, otherwise it is either explained here or when needed: unexplained concepts and terminology can be found in our standard references for Banach spaces and renorming [5, 8, 28] and topology [6].

All vector spaces E that we consider in this paper are assumed to be real. Given a subset S of a vector space, we write $\mathrm{co}(S)$ to denote the convex hull of S. If $(E, \|\cdot\|)$ is a normed space then E^* denotes its topological dual. If S is a subset of E^*, then $\sigma(E, S)$ denotes the weakest topology for E that makes each member of S continuous, or equivalently, the topology of pointwise convergence on S. Dually, if S is a subset of E, then $\sigma(E^*, S)$ is the topology for E^* of pointwise convergence on S. In particular $\sigma(E, E^*)$ and $\sigma(E^*, E)$ are the weak (w)and weak* (w^*) topologies respectively. Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x\rangle$ and $x^*(x)$ for the evaluation of x^* at x. If $x \in E$ and $\delta > 0$ we denote by $B(x, \delta)$ (or $B[x, \delta]$) the open (resp. closed) ball centred at x of radius δ: for $x = 0$ and $\delta = 1$ we will simplify our notation and just write $B_E := B[0, 1]$; the unit sphere $\{x \in E : \|x\| = 1\}$ will be denoted by S_E.

3 The tool

Theorem 2 is the tool we use here for the construction of norms. Its origin goes back to the so called slice localization theorem (Theorem 3 in [32]). Here we extend the localization property to countably many sliced sets.

Proof of Theorem 2 For every fixed finite sequence of pairs of positive integers $\sigma = ((m_1, p_1), (m_2, p_2), \cdots, (m_r, p_r))$ we have the following:

CLAIM.- There is an equivalent $\mathcal{T}_p$-lower semicontinuous norm

$$\|\cdot\|_\sigma \leq d_\sigma \|\cdot\|_\infty \tag{5}$$

such that

$$\lim_n \left(2\,\|x_n\|_\sigma^2 + 2\,\|x\|_\sigma^2 - \|x_n + x\|_\sigma^2\right) = 0$$

for x in E with $x \in \cap_{j=1}^r A_{p_j} \cap H_0^j$ for some $H_0^j \in \mathcal{H}_{m_j}, j = 1, 2, \cdots, r$ and any sequence (x_n) in E implies that there are sequences of open half spaces $\left\{H_n^j \in \mathcal{H}_{m_j} : n = 1, 2, ...\right\}$, for $j = 1, 2, \cdots, r$ such that

1. There is $n_0 \in \mathbb{N}$ with $x \in \cap_{j=1}^r H_n^j$ and $x_n \in H_n^j$ if $x_n \in A_{p_j}$ for any $j = 1, \ldots, r$ whenever $n \geq n_0$, and
2. For every $\delta > 0$ there is some n_δ such that
$$x, x_n \in \overline{\left(\mathrm{co}(\cap_{j=1}^r A_{p_j} \cap H_n^j) + B(0, \delta)\right)}^{\mathcal{T}_p}$$
for all $n \geq n_\delta$.

Once our CLAIM is proved the required norm is obtained with the formula:

$$\|x\|_0^2 := \sum_{\sigma \in \mathcal{F}} c_\sigma \|x\|_\sigma^2, \tag{6}$$

where $\mathcal{F}$ denotes the countable family of finite subsets of $\mathbb{N} \times \mathbb{N}$, and constants c_σ are chosen for the uniform sumability of the countable family (6) on bounded sets, which is always possible by (5), then standard convexity arguments (see Fact 2.3, p.45 [5]) and previous explanation finishes the proof . Let us prove our CLAIM and consider $\mathcal{T}_p$-lower semicontinuous and convex functions $\varphi_{H^1,\cdots,H^r}$ and $\psi_{H^1,\cdots,H^r}$ for $H^1 \in \mathcal{H}_{m_1}, \ldots, H^r \in \mathcal{H}_{m_r}$ defined as follows: for H a $\mathcal{T}_p$-open half space and A a bounded subset of E we set

$$\varphi_{H,A} := F - dist(x, \overline{(E \setminus H) \cap co(A)}^{w^*}),$$

where F, as we have said before, denotes the linear span in E^* of the evaluation functionals $\pi_\gamma(x) := x(\gamma)$ for every $\gamma \in \Gamma$, and $F-$ dist denotes the F-distance function as defined in definition 2.2 in [31]; i.e.

$$\varphi_{H,A}(x) = \inf\{\sup\{\langle x - z^{**}, h\rangle : h \in B_{E^*} \cap F\}, z^{**} \in \overline{(E \setminus H) \cap co(A)}^{w^*}\}$$

and define

$$\varphi_{H^1,\cdots H^r} := \sum_{j=1}^{r} \varphi_{H^j,A_{p_j}}.$$

Let us choose a point $a_{H^1,\cdots H^r} \in \cap_{j=1}^r H^j \cap A_{p_j}$ if this set is non empty and the origin otherwise. We set $D_{H^1,\dots,H^r} = \mathrm{co}(\cap_{j=1}^r H^j \cap A_{p_j})$ in the non empty case and $D_{H^1,\dots,H^r} = \{0\}$ otherwise for every $H^1 \in \mathcal{H}_{m_1}, \dots, H^r \in \mathcal{H}_{m_r}$. We define $D^\delta_{H^1,\dots,H^r} := D_{H^1,\dots,JH^r} + B(0,\delta)$ for every $\delta > 0$ and every $H^1 \in \mathcal{H}_{m_1}, \dots, H^r \in \mathcal{H}_{m_r}$. We are going to denote by $p^\delta_{H^1,\dots,H^r}$ the Minkowski functional of the convex body $\overline{D^\delta_{H^1,\dots H^r}}^{\mathcal{T}_p} - a_{H^1,\dots,H^r}$. Then we define the $\mathcal{T}_p$-lower semicontinuous norm $p_{H^1,\dots,H^r}$ by the formula

$$p_{H^1,\dots,H^r}(x)^2 = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} (p^{1/n}_{H^1,\dots,H^r}(x))^2$$

for every $x \in E$. Finally we define the non negative, convex, and $\mathcal{T}_p$-lower semicontinuous function $\psi_{H^1,\dots,H^r}$ as $\psi_{H^1,\dots,H^r}(x)^2 := p_{H^1,\dots,H^r}(x - a_{H^1,\dots,H^r})^2$ for every $x \in E$. We are now in position to apply Lemma 1 to get an equivalent norm $\|\cdot\|_\sigma$ on E such that the condition

$$\lim_n \left(2\,\|x_n\|^2_\sigma + 2\,\|x\|^2_\sigma - \|x_n + x\|^2_\sigma\right) = 0$$

for a sequence $\{x_n : n \in \mathbb{N}\}$ and x in E implies that there exists a sequence of indexes

$$\{(H^1_n, \dots, H^r_n) \in \mathcal{H}_{m_1} \times \cdots \times \mathcal{H}_{m_r} : n = 1, 2, \dots\}$$

such that

$$\lim_n \sum_{j=1}^{r} \varphi_{H^j_n,A_{p_j}}(x) = \lim_n \sum_{j=1}^{r} \varphi_{H^j_n,A_{p_j}}(x_n) = \lim_n \sum_{j=1}^{r} \varphi_{H^j_n,A_{p_j}}((x + x_n)/2) = \tag{7}$$

$$= sup \left\{ \sum_{j=1}^{r} \varphi_{H^j,A_{p_j}}(x) : (H^1, \dots, H^r) \in \mathcal{H}_{m_1} \times \cdots \times \mathcal{H}_{m_r} \right\} \tag{8}$$

and

$$\lim_n [(1/2)\psi^2_{H^1_n,\dots,H^r_n}(x_n) + (1/2)\psi^2_{H^1_n,\dots,H^r_n}(x) - \psi^2_{H^1_n,\dots,H^r_n}((x_n + x)/2)] = 0 \tag{9}$$

If the given point $x \in \bigcap_{j=1}^r A_{p_j} \cap H^j_0$ for some $H^j_0 \in \mathcal{H}_{m_j}, j = 1, 2, \dots, r$ we have a non-void set $\bigcap_{j=1}^r H^j_0 \cap A_{p_j}$ and $\varphi_{H^j_0,A_{p_j}}(x) > 0$ for every $j = 1, \dots, r$. So we have that:

$$sup \left\{ \varphi_{H,A_{p_j}}(x) : H \in \mathcal{H}_{m_j} \right\} > 0,$$

for every $j = 1, \dots, r$. Since

$$sup \left\{ \sum_{j=1}^{r} \varphi_{H^j,A_{p_j}}(x) : (H^1, \dots, H^r) \in \mathcal{H}_{m_1} \times \cdots \times \mathcal{H}_{m_r} \right\} = \sum_{j=1}^{r} sup \left\{ \varphi_{H,A_{p_j}}(x) : H \in \mathcal{H}_{m_j} \right\}$$

(7) provide us with an integer n_0 such that

$$\varphi_{H_n^j,A_{p_j}}(x) > 0, \varphi_{H_n^j,A_{p_j}}(x_n) > 0, \varphi_{H_n^j,A_{p_j}}((x+x_n)/2) > 0$$

whenever $n \geq n_0$, for $j = 1, \ldots, r$, from where our conclusion 1 in the theorem follows. Moreover, by (9) and standard convexity arguments (see Fact 2.3, p.45 [5]) it now follows that for every positive integer q we have that

$$\lim_n [2(p^{1/q}_{H_n^1,\ldots,H_n^r}(x_n - a_{H_n^1,\ldots,H_n^r}))^2 + 2(p^{1/q}_{H_n^1,\ldots,H_n^r}(x - a_{H_n}))^2 - (p^{1/q}_{H_n^1,\ldots,H_n^r}((x_n + x) - 2a_{H_n}))^2] = 0,$$

consequently we arrive to

$$\lim_n [p^{1/q}_{H_n^1,\ldots,H_n^r}(x_n - a_{H_n}) - p^{1/q}_{H_n^1,\ldots,H_n^r}(x - a_{H_n})] = 0, \forall q \in \mathbb{N}.$$

If we fix a positive number δ, open half spaces $H^j \in \mathcal{H}_{m_j}$ and $y \in \cap_{j=1}^r A_{p_j} \cap H^j$ we have that

$$y - a_{H^1,\ldots,H^r} + (y - a_{H^1,\ldots,H^r})\delta\|y - a_{H^1,\ldots,H^r}\|^{-1} \in B(0,\delta) + (y - a_{H^1,\ldots,H^r}) \subset D^\delta_{H^1,\ldots,H^r} - a_{H^1,\ldots,H^r},$$

thus

$$[(1+\delta)\|y - a_{H^1,\ldots,H^r}\|^{-1}](y - a_{H^1,\ldots,H^r}) \in (D^\delta_{H^1,\ldots,H^r} - a_{H^1,\ldots,H^r})$$

and therefore

$$p^\delta_{H^1,\ldots,H^r}(y - a_{H^1,\ldots,H^r}) < [(1 + \delta\|y - a_{H^1,\ldots,H^r}\|^{-1}]^{-1}$$

since $D^\delta_{H^1,\ldots,H^r} - a_{H^1,\ldots,H^r}$ is a norm open set.

Let us choose now the integer q such that $1/q < \delta$ and take an integer $n \geq n_0$ fixed above. We know that $x \in \cap_{j=1}^r A_{p_j} \cap H_n^j$ since $\varphi_{H_n^j,A_{p_j}}(x) > 0$, $j = 1, \ldots, r$ and the given point x belongs to $\cap_{j=1}^r A_{p_j}$. Therefore

$$p^{1/q}_{H_n^1,\ldots,H_n^r}(x - a_{H_n^1,\ldots,H_n^r}) < [(1 + (1/q)\|x - a_{H_n^1,\ldots,H_n^r}\|^{-1}]^{-1},$$

and we can find a number $0 < \xi < 1$ such that

$$p^{1/q}_{H_n^1\ldots,H_n^r}(x - a_{H_n^1,\ldots,H_n^r}) < 1 - \xi,$$

for all $n \geq n_0$, by the boundness of sets (A_n). If we now take the integer n big enough to have

$$p^{1/q}_{H_n^1,\ldots,H_n^r}(x_n - a_{H_n^1,\ldots,H_n^r}) < 1 - \xi,$$

we arrive to the fact that $x_n - a_{H_n^1,\ldots,H_n^r} \in D^\delta_{H_n^1,\ldots,H_n^r} - a_{H_n^1,\ldots,H_n^r}$, and indeed

$$x_n \in \overline{(\text{co}(\cap_{j=1}^r A_{p_j} \cap H_n^j) + B(0,\delta))}^{\mathcal{T}_p},$$

so the proof is over. □

4 Linear topological characterization of $\mathcal{T}_p$-LUR renormings

The first result we present is a characterization in terms of networks, the central topological concept for **LUR** renormings as explained in Section 3.1 of [28]. Our result here corresponds with the equivalence $i) \Leftrightarrow vi)$ of Theorem 3.1, [28], for the locally uniformly rotund case, and with $1 \Leftrightarrow 3$ of Theorem 3 in the Introduction. It was announced in the frame explanation to Question 6.16 of [28]:

Theorem 4 (Network characterization theorem)**.** *A subspace E of the Banach space $l^\infty(\Gamma)$ admits an equivalent $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p$-locally uniformly rotund norm if, and only if, the topology $\mathcal{T}_p$ on E has a σ-slicely isolated network; i.e a network $\mathcal{N}$ that can be written as $\mathcal{N} = \bigcup_{n=1}^\infty \mathcal{N}_n$ where every family $\mathcal{N}_n$ is $\mathcal{T}_p$-slicely isolated.*

Proof. Let us assume we have a σ- slicely isolated network

$$\mathcal{N} = \bigcup \{\mathcal{N}_n : n = 1, 2, ...\}$$

for the pointwise convergence topology on E. For every $n \in \mathbb{N}$ we consider the family $\mathcal{H}_n$ of $\mathcal{T}_p$-open half spaces H such that H meets just one element of the family of sets $\mathcal{N}_n$ together with the union set $A_n := \bigcup \mathcal{N}_n$. If we apply to the families $(\mathcal{H}_n)$ and the bounded sets (A_n) the Theorem 2 we get the equivalent norm $\| \cdot \|_0$. We claim that

$$\lim_n \left(2\|x_n\|_0^2 + 2|\ x\|_0^2 - \|x_n + x\|_0^2\right) = 0 \tag{10}$$

for a given sequence (x_n) and x in E implies that the sequence (x_n) is pointwise convergent to x. Indeed, for every convex and $\mathcal{T}_p$-closed neighbourhood of the origin W we select $\delta > 0$ such that $B(0, \delta) \subset \frac{1}{2}W$, moreover by the network condition of $\mathcal{N}$ we may choose the family $\mathcal{N}_p$ such that $x \in N \in \mathcal{N}_p$ and $N \subset x + \frac{1}{2}W$. Then we have

$$\mathrm{co}(N) + B(0, \delta) \subset x + \frac{1}{2}W + \frac{1}{2}W = x + W. \tag{11}$$

By the thesis of Theorem 2, for $\sigma = (p, p)$, we have a sequence (H_n) of $\mathcal{T}_p$-open half spaces in $\mathcal{H}_p$, with $x \in A_p \cap H_n \subset N$ for $n \geq n_0$, and a positive integer n_δ such that

$$x, x_n \in \overline{(\mathrm{co}(A_p \cap H_n) + B(0, \delta))}^{\mathcal{T}_p} \subset \overline{(\mathrm{co}(N) + B(0, \delta))}^{\mathcal{T}_p} \tag{12}$$

for every $n \geq n_\delta$. Indeed, since the family of half spaces $\mathcal{H}_p$ meets only one element of $\mathcal{N}_p$, every one of the half spaces H_n meets only the set N inside A_p since $x \in N$, thus $H_n \cap A_p \subset N$ and (12) follows. Thus $x_n \in x + W$ for $n \geq n_\delta$, our claim is proved and the proof for this implication is over.

The reverse implication follows the construction already done in [27] for the weak topology of a Banach space with a weakly **LUR** norm. Indeed the Main Lemma 3.19, p.59 of [28] gives the proof as it is showed in the first part of Theorem 3.21, p.61 in [28]. □

Remark 1. *Direct construction of σ-isolated network in Mercourakis space can be found in [9], for linear topological spaces with a metric d such that the identity is σ-slicely continuous in Theorem 2.2 of [30], and for $\mathcal{T}_p - \mathbf{LUR}$ norms in [38]. For* **LUR** *and $\mathcal{T}_p - \mathbf{LUR}$-norms a geometric approach can be found in [32] and [1].*

We need to remember the following:

Definition 3. *A map $\Phi : A \to (Y, \rho)$, from a subset $A \subset E$ to a metric space (Y, ρ), is σ-slicely continuous if there is a sequence of subsets (A_n) in A such that, for every $x \in A$ and every $\epsilon > 0$, there is some $\mathcal{T}_p$-open half space H in E and $q \in \mathbb{N}$ such that $x \in A_q \cap H$ and $\Phi(A_q \cap H)$ has ρ-diameter less than ϵ*

In Section 3.3 of [28] it is explained the construction of a metric ρ on E, which induce a topology finer that $\mathcal{T}_p$ and makes the identity $Id : E :\to (E, \rho)$ σ-slicely continuous, whenever we have a pointwise-**LUR** norm on the normed space $E \subset \ell^\infty(\Gamma)$. Our aim in the next result is to prove the converse result even when hypothesis are restricted to a radial set only. Let us remember that a set A in a vector space E is said to be radial if for every $x \in E, x \neq 0$ there is a number $\lambda > 0$ such that $\lambda x \in A$. A first consequence of the former theorem is the following:

Theorem 5 (Slicely-continuous characterization theorem)**.** *A subspace E of the normed space $l^\infty(\Gamma)$ admits an equivalent $\mathcal{T}_p$-locally uniformly rotund norm if, and only if, there is a radial set $A \subset E$, a metric ρ on A generating a topology finer than $\mathcal{T}_p$ and such that the identity map $Id : (A, \mathcal{T}_p) \to (A, \rho)$ is σ-slicely continuous.*

Proof. One direction is nothing else that our Theorem 3.21 in [28] which gives a metric ρ defined on the whole E . In case we have the metric ρ on a radial set A such that the identity map is σ-slicely isolated, we have a function base of Φ, as described in Proposition 2.24 in [28], which is a σ-slicely isolated family of sets $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n$ in A that is a network for the pointwise topology on A. Let us observe that constructions based on Theorem 2, as the one of Theorem 4, are local ones and it follows that there is an equivalent $\mathcal{T}_p$-lower semicontinuous norm $\|\cdot\|_0$ on E which is $\mathcal{T}_p - \mathbf{LUR}$ at every point of A. Thus a $\mathcal{T}_p - \mathbf{LUR}$ norm at every point $x \in E \setminus \{0\}$. Indeed, let us take a scalar $r(x)$ such that $r(x)x \in A$ and assume that

$$\lim_m (2\|x_m\|_0^2 + 2\|x\|_0^2 - \|x + x_m\|_0^2) = 0.$$

Then

$$\lim_m (2\|r(x)x_m\|_0^2 + 2\|r(x)x\|_0^2 - \|r(x)(x + x_m)\|_0^2) = 0,$$

thus $\mathcal{T}_p - \lim_n r(x)x_n = r(x)x$ since $r(x)x \in A$ and finally $\mathcal{T}_p - \lim_n x_n = x$, so the proof is over. □

The former theorem corresponds with the equivalence $(i) \Leftrightarrow (ii)$ of Theorem 3.1 in [28] in the locally uniformly rotund case, and $1 \Leftrightarrow 2$ in Theorem 3 in the Introduction. Kenderov and Moors's studied in deep fragmentability by metrics in arbitrary Banach spaces [16], the following result

Corollary 1. *A subspace E of the normed space $l^\infty(\Gamma)$ which admits an equivalent $\mathcal{T}_p$-locally uniformly rotund norm is fragmentable by a metric d generating a topology finer than $\mathcal{T}_p$*

Proof. Since the identity map for E to (E, ρ) is σ-slicely continuous, for every $\epsilon > 0$ we can write $E = \cup_{n=1}^{\infty} E_{n,\epsilon}$ to have, for every $n \in \mathbb{N}$ and every $x \in E_{n,\epsilon}$, some $\mathcal{T}_p$-open half space H with $x \in H \cap E_{n,\epsilon}$ and $\rho - \text{diam}(H \cap E_{n,\epsilon}) < \epsilon$, see Proposition 2.39 in[28]. The conclusion now follows applying Proposition 3.2 of Kenderov and Moors, [17]. □

For Borel subsets we have the following:

Corollary 2. *The σ-algebras of Borel sets for the pointwise topology $\mathcal{T}_p$ and the metric ρ on the radial subset A of Theorem 5 are the same.*

Proof. It follows from (iii) Theorem 2.2 in [30]. □

Next characterization shows an strong linking with metrization theory, in particular with the Bing-Nagata metrization theorem saying that a topological space is metrizable if, and only if, its topology has a σ-discrete basis. In order to do it we need more properties of the network we have on every normed space $E \subset \ell^\infty(\Gamma)$ with a $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p$-locally uniformly rotund equivalent norm. The following result corresponds with Theorem 2.5 in [31], where we studied the **LUR** case:

Proposition 1. *Let E be a normed subspace of $\ell^\infty(\Gamma)$ which admits a $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p - \mathbf{LUR}$ norm. Then the pointwise topology $\mathcal{T}_p$ on E admits a network*

$$\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$$

where each one of the families $\mathcal{N}_n$ is $\mathcal{T}_p$-slicely isolated and consists of sets which are the difference of $\mathcal{T}_p$-closed and convex subsets of E. Moreover, there is $\delta_n > 0$ such that $\mathcal{N}_n + B(0, \delta_n)$ is norm discrete as well as $\mathcal{T}_p$-slicely isolated for every $n \in \mathbb{N}$ and the family

$$\bigcup \{\mathcal{N}_n + B(0, \delta_n) : n = 1, 2, \dots\}$$

continues being a network of the topology $\mathcal{T}_p$.

Proof. After Theorem 4 there is a network in E for the pointwise topology that can be written as $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$ where every one of the families $\mathcal{M}_r := \{M_{r,i} : i \in I_r\}$ is $\mathcal{T}_p$-slicely isolated. Let us now follow our proof of Theorem 2.5 in [31] but adapting it to the new situation here. We follow the same notation and write $\varphi_{r,i}$ to denote the F-distance (see Proposition 2.1 in

[31]) to $\overline{\text{co}\ \{M_{r,j} : j \neq i\}}^{\sigma(E^{**},E^*)}$, where F is as above the linear span in E^* of the set of Dirac evaluations $\pi_\gamma(x) := x(\gamma)$ for $\gamma \in \Gamma$. We take

$$N^n_{r,i} := \{x \in \overline{\text{co}\ (M_{r,i})}^{\mathcal{T}_p} : \varphi_{r,i}(x) > 3/4n\},$$

and each one of the families $\mathcal{N}^n_r := \{N^n_{r,i} : i \in I_r\}$ is $\mathcal{T}_p$-slicely isolated by Theorem 2.3 in [31]. Indeed, the lower semicontinuity and convexity of the functions $\varphi_{r,i}$ tell us that $\varphi_{r,j}(y) = 0$ for every $y \in \overline{\text{co}\ (M_{r,i})}^{\mathcal{T}_p}$ and $j \neq i, j \in I_r$. Moreover, it is easily checked that $\varphi_{r,i}(z) > 3/4n - \mu$ if $z \in N^n_{r,i} + B(z,\mu)$ but $\varphi_{r,i}(z) < \mu$ whenever $z \in N^n_{r,j} + B(z,\mu)$. Let us choose δ_n such that $0 < 2\delta_n < 3/4n - \delta_n$, then the norm open sets $\{N^n_{r,i} + B(0,\delta_n) : i \in I_r\}$ are disjoint and they form a norm discrete family. The fact that it is $\mathcal{T}_p$-slicely isolated family follows from the former computations and statement (3) in Theorem 2.3 of [31]. Moreover, each one of the sets $N^n_{r,i}$ is the difference of convex and $\mathcal{T}_p$-closed subsets of E. The union of all these families:

$$\bigcup\{\mathcal{N}^n_r : r, n = 1, 2, ...\}$$

is the network we are looking for. Indeed, given $x \in E$ and W_0 a $\mathcal{T}_p$-neighbourhood of the origin, we fix a convex and closed $\mathcal{T}_p$-neighbourhood of 0 such that $W_1 + B(0,\mu) \subset W_0$ for some $0 < \mu$. The network property says that there is $r \in \mathbb{N}$ and $i \in I_r$ such that $x \in M_{r,i} \subset x + W_1$, thus by convexity we have $x \in \overline{\text{co}(M_{r,i})}^{\mathcal{T}_p} \subset x + W_1$. Moreover, there is n_0 such that for $n \geq n_0$ we have $x \in N^n_{r,i}$ from where it follows that, for n big enough we will have $\delta_n < \mu$ and thus

$$x \in N^n_{r,i} + B(0,\delta_n) \subset x + W_1 + B(0,\mu) \subset x + W_0,$$

since (δ_n) goes to 0 when n goes to infinity. □

We also need the following technical lemma:

Lemma 2. *Let $\{\mathcal{M}_i : i = 1, 2, \dots n\}$ a finite set of uniformly bounded and $\mathcal{T}_p$-slicely isolated families in the normed space $E \subset \ell^\infty(\Gamma)$. The family of non void finite intersections sets, i.e.*

$$\{\emptyset \neq \cap_{i=1}^n M_i : M_i \in \mathcal{M}_i, i = 1, 2, \dots n\},$$

is a $\mathcal{T}_p$-slicely isolated family.

Proof. By induction it is enough to prove the result for two families only. Let us fix

$$\mathcal{D} := \{D_i : i \in I\} \text{ and } \mathcal{B} := \{B_j : j \in J\}$$

two $\mathcal{T}_p$-slicely isolated families. Let us choose $x_0 \in D_{i_0} \cap B_{j_0}$ and select by hypothesis $\mathcal{T}_p$-open half spaces $H = \{y \in E : h(y) > h(x_0) - \delta\}$ and $L = \{y \in E : g(y) > g(x_0) - \delta\}$ so that

$$H \cap D_i = \emptyset \text{ and } L \cap B_j = \emptyset \text{ for every } i \neq i_0, j \neq j_0. \tag{13}$$

Let us reduce the sets $D_{i_0}, B_{\text{J}0}$ accordingly to be able to apply Lemma 3.1 in [25] as follows:

$$D' := \{y \in D_{i_0} : h(y) \leq h(x_0) + \frac{\delta}{2}\}$$

and

$$B' = \{y \in B_{j_0} : g(y) \leq g(x_0) + \frac{\delta}{2}\}.$$

If we denote by

$$A_0 = \cup\{D_i : i \in I, i \neq i_0\} \cup D' \bigcup \cup\{B_j : j \in J, j \neq j_0\} \cup B'$$

then

$$A_1 := \{y \in A_0 : h(y) > h(x_0) - \delta\} = H \cap (D') \bigcup H \cap (\cup\{B_j : j \in J\})$$

and

$$A_2 := \{y \in A_1 : g(y) > g(x_0) - \delta\} = H \cap L \cap D' \bigcup H \cap L \cap B'$$

having in mind (13). If we take θ and δ small enough to have

$$\{y \in A_1 : \delta \leq g(x_0) - g(y) \leq \theta\} = \emptyset$$

an application of Lemma 3.1 in [25] finishes the proof. Indeed we will have a $\mathcal{T}_p$-open half space S such that $x_0 \in S \cap A_0 \subset A_2$ and therefore $S \cap D_i \cap B_j = \emptyset$ whenever either $i \neq i_0$ or $j \neq j_0$.

□

Next result corresponds with Theorem 1.3 in [31], for the locally uniformly rotund case, and equivalence $1 \Leftrightarrow 4$ of Theorem 3:

Theorem 6 (Basis characterization theorem). *Let E be a normed subspace of $\ell^\infty(\Gamma)$. The normed space E has an equivalent $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p -$* **LUR** *norm if, and only if, there is a metric d on E generating a topology finer than $\mathcal{T}_p$ and coarser than the norm topology, with a basis for the metric topology $\mathcal{S}$ that admits a decomposition as*

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$$

where every $\mathcal{S}_n$ is $\mathcal{T}_p$-slicely isolated and norm discrete family of norm open sets.

Proof. If we observe that sets in the family $\{N^n_{r,i} + B(0, \delta_n) : N^n_{r,i} \in \mathcal{N}^n_r\}$, which has been constructed in Proposition 4, are norm open sets of E we arrive to the conclusion that we can change the metric ρ constructed in Theorem 3.21 of [28] by another one d, coarser than the norm metric of E and finer than $\mathcal{T}_p$, with the required properties. Indeed, let us define $d_{n,r}(x, y) = 0$ if both x and y belong to the same set in the family

$$\{N^n_{r,i} + B(0, \delta_n) : N^n_{r,i} \in \mathcal{N}^n_r\},$$

and $d_{n,r}(x, y) = 1$ otherwise for every $n, r \in \mathbb{N}$. Now we set

$$d(x, y) := \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{2^{n+r}} d_{n,r}(x, y)$$

for every $x, y \in E$, and d is the metric we are looking for by Lemma 2. Indeed a basis of the topology generated by d is obtained making finite intersections of sets taken from

$$\bigcup_{n,r=1}^{\infty} \{N_{r,i}^n + B(0,\delta_n) : N_{r,i}^n \in \mathcal{N}_r^n\}.$$

The reverse implication follows from Theorem 4 since a basis for d is a network for any coarser topology and so for $\mathcal{T}_p$. □

Next result stress on a different property related with developable topological spaces, also called Moore spaces in metrization theory, see [12]. It corresponds with equivalence $1 \Leftrightarrow 5$ in Theorem 3 in the Introduction and it completes its proof. For a family of subsets $\mathcal{V}$ of a fixed set X and given point $x \in X$ we write

$$\mathrm{st}(x, \mathcal{V}) = \cup\{V : x \in V, V \in \mathcal{V}\}$$

Theorem 7 (Developable-network theorem)**.** *Let E be a normed subspace of $l^\infty(\Gamma)$. Then E admits an equivalent $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p - \mathbf{LUR}$ norm if, and only if, there are families $\mathcal{H}_n$ of $\mathcal{T}_p$-open half spaces and non void subsets $A_p \subset E$ such that for every $\mathcal{T}_p$-open half space L in E and every $x \in L$ there are integers $n, p \in \mathbb{N}$ such that $x \in A_p$ and*

$$\mathrm{st}(x, \mathcal{H}_n) \cap A_p \subset L.$$

Proof. Theorem 4 tell us that a $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p - \mathbf{LUR}$ norm produces in E a σ-slicely isolated network $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$ for $\mathcal{T}_p$. If we take, for every $n \in \mathbb{N}$, $A_n := \bigcup \mathcal{N}_n$ and $\mathcal{H}_n$ denotes the family of $\mathcal{T}_p$-open half spaces meeting at most one element of $\mathcal{N}_n$, we have that the family of sets

$$\{\mathrm{st}(x, \mathcal{H}_n) \cap A_n : x \in A_n\}$$

is a refinement of the family of sets in $\mathcal{N}_n$, so the conclusion even follows for every $\mathcal{T}_p$-open set L which contains the point x. Conversely, let us assume there are families $\mathcal{H}_n$ of $\mathcal{T}_p$-open half spaces and non void subsets $A_p \subset E$ such that for every $\mathcal{T}_p$-open half space L in E and every $x \in L$ there are integers $n, p \in \mathbb{N}$ which verifies

$$\mathrm{st}(x, \mathcal{H}_n) \cap A_p \subset L.$$

Without lose of generality we may and do assume that our sets A_n are bounded, if not we intersect them with a countable family of balls which cover E. Let us apply the Multiple-slice Localization Theorem 2 to the sequence of sets (A_n) and the sequence of $\mathcal{T}_p$-open half spaces $(\mathcal{H}_n)$ to get an equivalent $\mathcal{T}_p$-lower semicontinuous norm $\|\cdot\|_0$. We claim that this new norm is a $\mathcal{T}_p - \mathbf{LUR}$ norm on E. Indeed, let us assume that

$$\lim_n (2\|x_n\|_0^2 + 2\|x\|_0^2 - \|x + x_n\|_0^2) = 0.$$

Let us fix V a $\mathcal{T}_p$-open neighbourhood of $x \in E$, select $L_1, \dots L_r$ a finite number of $\mathcal{T}_p$-open half spaces with $x \in \cap_{j=1}^r L_j \subset \overline{\cap_{j=1}^r L_j}^{\mathcal{T}_p} \subset V$, and choose $\delta > 0$ small enough to have $B(x, \delta) \subset \cap_{j=1}^r L_j$. By hypothesis there are pair of integers $(m_1 p_1), \dots, (m_r, p_r)$ such that

$$x \in \mathrm{st}(x, \mathcal{H}_{m_j}) \cap A_{p_j} \subset L_j - B(0, \delta), j = 1, \dots, r.$$

If $\sigma = ((m_1, p_1), (m_2, p_2), \dots, (m_r, p_r))$ Theorem 2 give a sequences of $\mathcal{T}_p$-open half spaces

$$\left\{H_n^j \in \mathcal{H}_{m_j} : n = 1, 2, ...\right\},$$

for $j = 1, 2, \dots, r$; such that:

1. There is $n_0 \in \mathbb{N}$ with $x \in \cap_{j=1}^r H_n^j$, and $x_n \in H_n^j$ if $x_n \in A_{p_j}$ for $j = 1, \dots, r$ whenever $n \geq n_0$, and
2. There is some n_δ such that

$$x, x_n \in \overline{\left(\mathrm{co}(\cap_{j=1}^r A_{p_j} \cap H_n^j) + B(0, \delta)\right)}^{\mathcal{T}_p}$$

   for all $n \geq n_\delta$.

By convexity we have $\mathrm{co}(\mathrm{st}(x, \mathcal{H}_{m_j}) \cap A_{p_j}) \subset L_j - B(0, \delta), j = 1, \dots, r$, then we have

$$x_n \in \overline{\left(\mathrm{co}(\cap_{j=1}^r A_{p_j} \cap H_n^j) + B(0, \delta)\right)}^{\mathcal{T}_p} \subset \overline{\cap_{j=1}^r L_j}^{\mathcal{T}_p} \subset V$$

whenever $n \geq n_\delta$ and the proof is over

□

Remark 2. *If there is a set $A \subset E$, families $\mathcal{H}_n$ of $\mathcal{T}_p$-open half spaces and non void subsets $A_p \subset A$ such that for every $\mathcal{T}_p$-open half space L in E and every $x \in L \cap A$ it is assumed that there are integers $n, p \in \mathbb{N}$ such that $x \in A_p$ and*

$$\mathrm{st}(x, \mathcal{H}_n) \cap A_p \subset L,$$

the same proof shows that the new norm $\|\cdot\|_0$ is $\mathcal{T}_p -$ **LUR** *at every point $x \in A$.*

An application for **LUR** renormings is the following:

Corollary 3. *A normed space E admits an equivalent* **LUR** *norm if there are families $\mathcal{H}_n$ of $\sigma(E, E^*)$-open half spaces such that*

$$\bigcap_{n=1}^{\infty} \overline{\mathrm{st}(x, \mathcal{H}_n) \cap S_E}^{w^*} = \{x\}$$

for every $x \in S_E$.

Proof. We claim that

$$\{\mathrm{st}(x, \mathcal{H}_n) \cap S_E : n = 1, 2, \dots\}$$

is a subbasis of neighbourhood of $x \in S_E$ for the weak topology induced on S_E. Indeed, let us fix $g \in S_{E^*}$ and $\mu > 0$ such that $g(x) > \mu$. In case we have

$$y_p \in \cap_{n=1}^{p} \mathrm{st}(x, \mathcal{H}_n) \cap \{y \in E : g(y) \leq \mu\} \cap S_E \neq \emptyset$$

for every $p \in \mathbb{N}$, the sequence (y_p) has a w^*-cluster point

$$y^{**} \in \bigcap_{n=1}^{\infty} \overline{\mathrm{st}(x, \mathcal{H}_n) \cap S_E}^{w^*} \cap B_{E^{**}}$$

but $g(y^{**}) \leq \mu$, thus $y^{**} \neq x$ which is a contradiction proving our claim. Theorem 7 provides an equivalent norm $\|\cdot\|_0$ with such that $\|\cdot\|_0$ is going to be weakly-**LUR** at every point of the unit sphere S_E by Remark 2, so a weakly-**LUR** norm on E. To finish the proof we only need to apply our main theorem in [27], or Corollary 3.23 in [28] to conclude that E has an equivalent **LUR** norm.

□

5 Some applications

With the former characterizations we are able to deal with Question 6.16, p.122 of [28]. Then our purpose now is to present results on nonlinear maps transferring the pointwise **LUR** renorming property in the flavour of [28]. To begin with we introduce the following result, useful to deal with metric spaces only. It corresponds with the main Theorem 1.15 in [28] for **LUR** renormings:

Theorem 8. *Let $E \subset l^{\infty}(\Gamma)$, (Y, ρ) a metric space and*

$$\Phi : E \to (Y, \rho)$$

a σ-slicely continuous map. If there is a sequence of sets (D_n) in Y such that for every $\mathcal{T}_p$-open half space H and $x \in H$ there is some $\delta > 0$, $p \in \mathbb{N}$ with

$$\Phi x \in D_p \text{ and } \Phi^{-1}(D_p \cap B_\rho(\Phi x, \delta)) \subset H,$$

then X admits an equivalent $\mathcal{T}_p$-LUR norm.

Proof. Φ has a σ-slicely isolated function base that we denote by $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$ by Proposition 2.24 in [28]. If we take finite intersections of the form $\Phi^{-1}(D_{p_1} \cap \cdots \cap D_{p_r}) \cap \mathcal{N}_{n_1} \cap \cdots \cap \mathcal{N}_{n_r}$ we obtain $\mathcal{T}_p$-slicely isolated families by Lemma 2, and

$$\bigcup_{n_1,\dots,n_r,p_1,\dots,p_r,r=1}^{\infty} \Phi^{-1}(D_{p_1} \cap \cdots \cap D_{p_r}) \cap \mathcal{N}_{n_1} \cap \cdots \cap \mathcal{N}_{n_r}$$

is a network of the pointwise topology on E. Indeed, given any $\mathcal{T}_p$-open half space H with $x \in H$ there is some $\delta > 0$, $p \in \mathbb{N}$ with

$$\Phi x \in D_p \text{ and } \Phi^{-1}(D_p \cap B_\rho(\Phi x, \delta)) \subset H,$$

Moreover, since $\mathcal{N}$ is a function basis of Φ there is $n \in \mathbb{N}$ and $N \in \mathcal{N}_n$ such that $x \in N \subset \Phi^{-1}(B_\rho(\Phi x, \delta))$, thus $x \in N \cap \Phi^{-1}(D_p) \subset \Phi^{-1}(D_p \cap B_\rho(\Phi x, \delta)) \subset H$, and the proof is over. □

A way to produce the situation of the former theorem is to deal with co-σ-continuous maps as we have done in [28]. In Section 2.2 of [28] precise definitions and main connections are described, for instance with the theory of general analytic metric spaces. For instance, we have the following consequence:

Corollary 4. *Let $E \subset l^\infty(\Gamma)$, (Y, ρ) a metric space and*

$$\Phi : E \to (Y, \rho)$$

a σ-slicely continuous map. If there is a metric d on E such that $\Phi : (E, d) \to (Y, \rho)$ is co-σ-continuous and there exists a sequence of sets (A_n) in E such that for every $\mathcal{T}_p$-open half space H and $x \in H$ there is some $\delta > 0$, $p \in \mathbb{N}$ with

$$x \in A_p \text{ and } A_p \cap B_d(x, \delta) \subset H,$$

*then there is an equivalent $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p -$ **LUR** norm on E.*

Proof. Proposition 2.30 in [28] says that co-σ-continuity of a map is the same as to have separable fibers and every selector σ-continuous. Thus we have $\Phi^{-1}(y) = \overline{\{y(n) : n \in \mathbb{N}\}}^d$ for every $y \in Y$ and the map $\Xi_n(y) := y(n)$ for every $y \in Y$ is σ-continuous to (E, d). Corollary 2.41 in [28] tell us that the map $\Xi_n \circ \Phi$ is σ-slicely continuous for every n, and finally we have

$$x \in \overline{\{\Xi_n \circ \Phi x : n \in \mathbb{N}\}}^d$$

for every $x \in E$, thus Proposition 2.23 in [28] says that the identity map on E is σ-slicely continuous when the metric d is consider on E, so the above Theorem 8 finishes the proof since the identity map from E to (E, d) verifies its hypothesis. □

If a Banach space E has a Gateaux differentiable norm then E is a weak Asplund space, [33]. Weak Asplund spaces and their related classes of spaces deserved special attention to P. Kenderov and his School in Optimization ,[2, 3, 14, 15, 19, 18, 34, 29]. For a comprehensible amount of information on the matter see [7]. Kenderov and Moors observed for a dual space E^* with Gateaux differentiable norm that the Banach space E is σ-fragmentable by the norm, [16]. Nevertheless the following problem seems to be open:

Question 1. *Does a Banach space E admit an equivalent* **LUR** *norm if its dual space E^* has a Gateaux differentiable norm ?*

Remark 3. *If the norm of Banach space E is Fréchet differentiable and the dual norm is Gateaux smooth at the norm attaining functionals it follows from Theorem 8 that E admits an equivalent $\sigma(E,E^*)-\mathbf{LUR}$, and so an equivalent $\mathbf{LUR}$-norm. Details for the proof follows the same arguments as Proposition 4.4 in [28]*

Corollary 5. *Let E be a normed space, (Y,ρ) a metric space and*

$$\Phi : E \to (Y,\rho)$$

a σ-slicely continuous map. If there is a metric d on E such that $\Phi : (E,d) \to (Y,\rho)$ is co-σ-continuous and there exists a sequence of sets (A_n) in E such that for every open half space H and $x \in H$ there is some $\delta > 0$, $p \in \mathbb{N}$ with

$$x \in A_p \text{ and } A_p \cap B_d(x,\delta) \subset H,$$

then there is an equivalent $\mathbf{LUR}$ norm on E.

Proof. The former corollary provides an equivalent $\sigma(E,E^*)-\mathbf{LUR}$ norm on E, thus the conclusion follows from our main theorem in [27], or Corollary 3.23 in [28]. □

Let us remark that Theorems 1.12, 3.35 and 3.46 in [28] are particular cases of former corollary since the conditions imposed there implies the co-σ-continuity of the involved maps

Corollary 6. *For normed spaces $E \subset l^\infty(\Gamma)$, $Y \subset l^\infty(\Delta)$ with a bijection $\Phi : E \to Y$ which is σ-slicely continuous in both directions we have that E admits an equivalent $\mathcal{T}_p-\mathbf{LUR}$ norm if, and only if, Y does it.*

Proof. The implication $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ in Theorem 4.29 of [28] is always true, see Remark 4.30, p. 96, [28]. Thus conditions in Theorem 8 are satisfied and the proof is over. □

For bijective maps we have the following result:

Theorem 9. *Let $E \subset l^\infty(\Gamma)$, $Y \subset l^\infty(\Delta)$ be normed spaces, $\Phi : E \to Y$ be a bijection with sequences of sets (F_n) in E, (D_n) in Y such that:*

1. *For every $\mathcal{T}_p$-open half space $G \subset Y$ and every $x \in E$ with $\Phi x \in G$ there is $p \in \mathbb{N}$ and a $\mathcal{T}_p$-open half space $H \subset E$ with*

$$x \in F_p \cap H \text{ and } \Phi(F_p \cap H) \subset G$$

2. *For every $\mathcal{T}_p$-open half space $H \subset E$, $y \in Y$ with $\Phi^{-1}y \in H$ there is $q \in \mathbb{N}$, a $\mathcal{T}_p$-open half space $G \subset Y$ with*

$$y \in D_q \cap G \text{ and } \Phi^{-1}(D_q \cap G) \subset H$$

Then E admits an equivalent $\mathcal{T}_p - \mathbf{LUR}$ norm if, and only if, Y does it.

Proof. Let us assume that Y admits an equivalent $\mathcal{T}_p$-lower semicontinuous and $\mathcal{T}_p - \mathbf{LUR}$ norm. Theorem 5 provides us with a metric ρ on Y, generating a topology finer than $\mathcal{T}_p$ such that the identity map on Y is σ-slicely continuous. Condition 1 implies that Φ is going to be σ-slicely continuous from E to (Y, ρ) by Corollary 2.42, [28]. Condition 2 implies that hypothesis of Theorem 8 are satisfied since the ρ-topology is finer than $\mathcal{T}_p$ on Y and the proof is over. □

Let us finish with the following open problem:

Question 2. *If $\Phi : E \to Y$ is a Lipschitz homomorphism between Banach spaces E and Y, does it follows that Φ is σ-slicely continuous, and so that E admits an equivalent* **LUR** *norm if, and only if, Y does it?*

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