On weakly locally uniformly rotund Banach spaces

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WLUR Banach spaces

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Abstract

We show that every normed space E with a weakly locally uniformly rotund norm has an equivalent locally uniformly rotund norm. After obtaining a σ–discrete network of the unit sphere S_E for the weak topology we deduce that the space E must have a countable cover by sets of small local diameter which in turn implies the renorming conclusion. This solves a question posed by Deville, Godefroy, Haydon and Zizler. For a weakly uniformly rotund norm we prove that the unit sphere is always metrizable for the weak topology despite it may not have Kadec property. Moreover, Banach spaces having a countable cover by sets of small local diameter coincide with the descriptive Banach spaces studied by Hansell, so we present here some new characterizations of them.

Key words: weakly locally uniformly rotund, locally uniformly rotund, renorming, descriptive Banach spaces.

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1 Introduction.

Throughout this paper E will denote a normed space and $\|\cdot\|$ its norm. B_E will stand for the closed unit ball and S_E the unit sphere.

The norm $\|\cdot\|$ on a normed space E is said to be uniformly rotund (**UR** for short) if $\lim_n \|x_n - y_n\| = 0$ whenever $x_n, y_n \in S_E$, $n \in \mathbb{N}$, are such that $\lim_n \|x_n + y_n\| = 2$, and it is said to be locally uniformly rotund (**LUR** for short) if $\lim_n \|x_n - x\| = 0$ whenever $x_n, x \in S_E$, $n \in \mathbb{N}$, are such that $\lim_n \|x_n + x\| = 2$.

The corresponding notions for the weak topology are obtained replacing $\|\cdot\|$–convergence by weak convergence:

A norm $\|\cdot\|$ on E is said to be weakly uniformly rotund (**WUR** for short) if weak–$\lim_n (x_n - y_n) = 0$ whenever $x_n, y_n \in S_E$, $n \in \mathbb{N}$, are such that $\lim_n \|x_n + y_n\| = 2$, and it is said to be weakly locally uniformly rotund (**WLUR** for short) if weak–$\lim_n (x_n - x) = 0$ whenever $x_n, x \in S_E$, $n \in \mathbb{N}$, are such that $\lim_n \|x_n + x\| = 2$.

Clearly the following diagram holds for a given norm $\|\cdot\|$ on E:

$$\begin{array}{ccc} \textbf{UR} & \Longrightarrow & \textbf{WUR} \\ \Downarrow & & \Downarrow \\ \textbf{LUR} & \Longrightarrow & \textbf{WLUR} \end{array}$$

From the point of view of renorming theory we are interested in characterizing the normed spaces E that have an equivalent norm with some of the above properties in terms of geometrical or topological conditions of the space E itself. As an example let us mention James, Enflo and Pisier's theorem asserting that a Banach space E is superreflexive if and only if it has an equivalent **UR** norm. An account of renorming theory appears in the authoritative text of Deville, Godefroy and Zizler [4], we refer to it for any undefined notion mentioned here.

About **WUR** renormings there has been an important progress lately. Hájek has shown that a Banach space E must be an Asplund space whenever it has a **WUR** norm, [13]. Based on that result and using the projectional resolution of the identity on the dual space [6], Fabian, Hájek and Zizler [7] have proved that a Banach space E has a **WUR** equivalent norm if and only if the bidual unit ball $B_{E^{**}}$ is uniform Eberlein compact with the weak*–topology. Consequently E has an equivalent **LUR** norm too since the dual space E^* is a subspace of a weakly compactly generated Banach space [11], [8]. A direct proof of the weak–K–analyticity of the dual space E^* when E has a **WUR** norm can be found in [25].

For the notion of **WLUR** normed spaces only partial results have so far been obtained. In the book of Deville, Godefroy and Zizler it is shown that a Banach space with a **WLUR** and Fréchet differentiable norm must be **LUR** renormable [4, Chapter VII, Proposition 2.6.] and some consequences for the transfer technique are deduced. In problem VII.1.(i), [4, p. 333], they asked whether an Asplund space that admits a **WLUR** norm must have an equivalent **LUR** norm. If T is a tree, then the existence of a **WLUR** norm on $C(T)$ implies the existence of an equivalent **LUR** norm on $C(T)$ as Haydon showed [16]. He also asks if this is true for $C(K)$ when K is an arbitrary scattered compact space [16, Problem 11.3] (see also [17]). It is mentioned in [4, Chapter VII, p 333] that it is not known if Haydon's result holds for a general Banach space.

We give a positive answer to that question in the general case:

MAIN THEOREM. *Let E be a normed space with a* **WLUR** *norm. Then E has an equivalent* **LUR** *norm.*

It is well known that in the above diagram the converse implications, of course with the exception of the one stated in our main theorem, are not true even for equivalent norms. A previous result in the same spirit is due to the third named author [33] since he proved that a Banach space with Kadec property and a rotund norm has an equivalent **LUR** norm. Let us point out that the Kadec property on a given norm is not comparable with **WLUR**. In fact neither Kadec property implies **WLUR** nor even **WUR** implies Kadec property. Let us recall that a norm is said to have Kadec property if the weak and the norm topologies coincide on the unit sphere.

The techniques to prove our main theorem are based on [26] where a characterization for the **LUR** renormability of a normed space is obtained that fits in the framework of σ–fragmentable Banach spaces, the theory introduced and developed by Jayne, Namioka and Rogers and related from its beginning with renorming properties of non separable Banach spaces [19]. In [10], [22] using topological games it was actually proved that every **WLUR** Banach space is σ–fragmentable. The following notion introduced and studied in [19] is in the core of our arguments:

Definition 1 Let $(Y, \mathcal{T})$ be a Hausdorff topological space and ρ a metric on Y. The topological space $(Y, \mathcal{T})$ is said to have a *countable cover by sets of small local ρ–diameter* (ρ-**SLD** for short) if for each $\varepsilon > 0$, it is possible to write $Y = \bigcup_{n\geq 1} Y_{n,\varepsilon}$,in such a way that for every n and $x \in Y_{n,\varepsilon}$ there exists a $\mathcal{T}$–neighbourhood V of x such that the ρ-*diam*$(V \cap Y_{n,\varepsilon}) < \varepsilon$.

Let E be a normed space, w its weak topology and $\rho(x, y) :=\| x - y \|$, x, $y \in E$, the norm metric. Given A a subset of E, $f \in E^*$ and $\lambda \in \mathbb{R}$, we denote by $S(A, f, \lambda) = \{u \in A : f(u) > \lambda\}$ the open slice of A. *If (A, w) is ρ–*__SLD__* then we shall say that A has the* **JNR** *property.* When it is possible to replace the relative neighbourhoods in A by slices we will say that A has the property **sJNR**. Explicitly, A has the property **sJNR** whenever for each $\varepsilon > 0$ it is possible to write $A = \bigcup_{n\geq 1} A_{n,\varepsilon}$ in such a way that for every $n \in \mathbb{N}$ and $x \in A_{n,\varepsilon}$, there exists a slice $S(A_{n,\varepsilon}, f, \lambda)$ containing x and diam $S(A_{n,\varepsilon}, f, \lambda) < \varepsilon$.

We can state now the Main Theorem in [26, §4]:

Theorem 1 *Let E be a normed space. The following conditions are equivalent:*

a) the unit sphere S_E of E has **sJNR***;*

b) E has **sJNR***;*

c) E has an equivalent **LUR** *norm;*

d) E *has* **JNR** *and an equivalent* **WMLUR** *norm.*

The construction of the above **LUR** norm is based on some probabilistic arguments mainly margingales. Let us remark that very recently M. Raja has obtained a new proof of this theorem which is more geometrical and dispenses with martingales which makes it quite simpler [29]. Let us recall that a norm in a space E is said to be *(weakly) midpoint locally uniformly rotund* (**(W)MLUR** for short) if given sequences (y_k), (z_k) and x in E we have *(*w*–*$\lim_k(y_k - z_k) = 0$ *)* $\lim_k \| y_k - z_k \| = 0$ whenever $\| y_k \|$, $\| z_k \| \leq \| x \|$ and $\lim_k \| y_k + z_k - 2x \| = 0$. Let us point out that **MLUR** does not implies **LUR** renormability [16].

Since **WLUR** implies **WMLUR**, according to condition d) of Theorem 1 our main theorem will be proved as soon as we show that any normed space with a **WLUR** norm has the **JNR** property.

Despite of the proof of our main theorem relies mainly on linear topological arguments, to get a better understanding of the phenomenon involved it seems to us that some topological concepts play an essential role in it. The second part is devoted to study them and to indicate which is their relation with the main theorem, with concepts previously studied in descriptive topology and with M. Raja's approach to renorming [29], [30]. Actually we characterize **JNR** property in terms of a discrete way of approximation. Hopefully this study could shed some light for the open problems stated at the end. As an application see the example in Section 3.

Lately the link of renorming theory with non linear theory is attracting some attention. In this direction Namioka and Pol obtained in [27] topological characterizations of the σ–fragmentability of a subset A of a Banach space in terms of the weak–topology only (without referring to the norm). An earlier characterization is due to Hansell in [14]. Consequently σ–fragmentability is a stable property under weak homeomorphisms. Recently M. Raja [29] has shown that the presence of the **JNR** property in a normed space $(E, \| \cdot \|)$ is equivalent to the existence of a symmetric homogeneous weakly lower semicontinuous function $F : E \to [0, +\infty[$ with $\|\cdot\| \le F(\cdot) \le 3\|\cdot\|$ and such that the norm and the weak topology coincide on the "sphere" $S = \{x \in E : \ F(x) = 1\}$. Therefore the **JNR** property is, roughly speaking, "Kadec property without convexity". L. Oncina showed in [28] that the **JNR** property is a stable property under weak homeomorphisms too. We shall describe here several characterizations of the **JNR** property that become more relevant after the proof we present for our main theorem. Indeed they are related to properties studied by Hansell [14] for descriptive topological spaces that fit in the framework of "generalized metric spaces"[12].

The first topological notion relevant to our discussion is a condition of discreteness of a family of sets with respect to its union. It goes back to people studying some covering properties related with paracompactness [2].

Definition 2 A family $\mathcal{H}$ of subsets of a topological space $(Y, \mathcal{T})$ is said to be *isolated (resp. discrete)* whenever for any $x \in \bigcup\{H : \ H \in \mathcal{H}\}$ (resp. $x \in Y$) there exists $V \in \mathcal{T}$ such that $x \in V$ and the set $\{H : \ H \in \mathcal{H}, \ H \cap V \neq \emptyset\}$ contains exactly one element. (resp. at most one element.)

A family $\mathcal{F}$ of subsets of a topological space $(Y, \mathcal{T})$ is said to be *σ–isolated (resp. σ–discrete)* whenever it can be decomposed $\mathcal{F} = \bigcup_{n \ge 1} \mathcal{F}_n$ in such a way that every $\mathcal{F}_n$ is isolated (resp. discrete).

A family $\mathcal{A}$ of subsets of a topological space $(Y, \mathcal{T})$ is said to be *σ–isolatedly decomposable (resp.σ–discretely decomposable)* if for every $A \in \mathcal{A}$ we have a sequence $\{A_n : \ n \ge 1\}$ such that $A = \bigcup_{n \ge 1} A_n$ and for every $n \in \mathbb{N}$ the family

$$\mathcal{A}_n = \{A_n : \ A \in \mathcal{A}\}$$

is isolated (resp. discrete).

A general class of "descriptive spaces" based on this notion goes back to Frolík [9] and Hansell [14], (see [15] too), where the term "relatively discrete" is used for isolated families [15, Definition 6.1.].

Definition 3 A *network* for a topological space Y is a collection $\mathcal{F}$ of subsets of Y such that whenever $x \in U$ with U open, there exists $F \in \mathcal{F}$ with $x \in F \subset U$.

(Let us point out that x is not assumed to be an interior point of F.) This concept of network, introduced by Arhangel'skii in [1], has been one of the most useful tools for generalized metric spaces [12]. It turns out now that it is a very helpful notion to deal with the topological problems arising with renorming.

The following definition appears in [14] where several of its properties are studied and its connection with σ–fragmentability and renorming well established:

Definition 4 A Banach space E is said to be *descriptive* whenever it has a network for its norm–topology which is σ–isolated in the weak topology.

The following result explains its relevance in our discussion:

Theorem 2 *A Banach space E is descriptive if and only if it has the* **JNR** *property.*

The topological covering property known as "weak θ–refinability" can be characterized by the condition that every open cover of the space has a σ–isolated refinement ([2, Theorem 3.7.]). As Hansell showed in [14] any descriptive Banach space has this property hereditarily for its weak topology, and any σ–fragmentable Banach space which is hereditarily weakly θ–refinable for the weak topology is descriptive. Only recently Dow, Junila and Pelant [5] found examples of Banach spaces $C(K)$ without this covering property. Nevertheless their examples are not σ–fragmentable and it is an open problem to decide whether σ–fragmentability and **JNR** property are the same for a Banach space.

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2 Weakly locally uniformly rotund norms.

In this section our main theorem is proved.

Proposition 1 *Let $(E, \|\cdot\|)$ be a normed space with a* **WLUR** *norm and x a point of its unit sphere S_E. The family*

$$\{S(S_E, f, 1-\varepsilon):\ x \in S(S_E, f, 1-\varepsilon),\ f \in S_{E^*},\ 0 < \varepsilon < 1\}$$

is a base of neighbourhoods at x for the weak topology.

Proof. Indeed if $y_\varepsilon \in S(S_E, f_\varepsilon, 1-\varepsilon)$ then $\|(x+y_\varepsilon)/2\| \geq (1/2) f_\varepsilon(x+y_\varepsilon) > 1-\varepsilon$, hence $\lim_{\varepsilon \to 0} \|(x+y_\varepsilon)/2\| = 1$, and finally weak–$\lim_{\varepsilon \to 0} y_\varepsilon = x$ which proves the proposition. ■

We are going to state a property for the unit sphere of a **WLUR** Banach space with the weak topology close to the well known Montgomery's lemma in metric spaces (see e.g. [24, §30.X])

Key Lemma *Let $\{S(B_E, f_\gamma, \lambda_\gamma):\ \gamma < \Gamma\}$ be a family of slices of the unit ball of a normed space $(E, \|\cdot\|)$ covering the unit sphere S_E. Let $M_\gamma := S(B_E, f_\gamma, \lambda_\gamma) \setminus (\bigcup \{S(B_E : f_\beta, \lambda_\beta):\ \beta < \gamma\})$ for every $\gamma < \Gamma$. If the norm $\|\cdot\|$ of E is* **WLUR** *then the family of disjoint sets $\{M_\gamma \cap S_E:\ \gamma < \Gamma\}$ is σ–discretely decomposable in the unit sphere (S_E, w).*

Proof. Set $D_\gamma := S(B_E : f_\gamma, \lambda_\gamma)$ and for any $n \in \mathbb{N}$ and $\gamma < \Gamma$ let us consider the set

$$D_\gamma^n := \left\{x \in D_\gamma :\ f_\gamma(x) \geq \lambda_\gamma + \frac{1}{n}\right\}.$$

We have $D_\gamma = \bigcup_{n=1}^{+\infty} D_\gamma^n$. Now let

$$M_\gamma^n := D_\gamma^n \setminus \left(\bigcup \{D_\beta :\ \beta < \gamma\}\right),$$

we have again $M_\gamma = \bigcup_{n=1}^{+\infty} M_\gamma^n$ for every $\gamma < \Gamma$. Moreover if $x \in M_\gamma^n$ and $y \in M_\beta^n$ for $\gamma \neq \beta$ then we have either

$$|f_\gamma(x-y)| \geq \frac{1}{n} \quad \text{or} \quad |f_\beta(x-y)| \geq \frac{1}{n}.$$

Indeed, assume for instance $\beta < \gamma$. Since $x \notin D_\beta$ and $y \in M_\beta^n$ it follows $f_\beta(x) \leq \lambda_\beta$ and $f_\beta(y) \geq \lambda_\beta + (1/n)$. So $f_\beta(y) - f_\beta(x) \geq (1/n)$.

To obtain an isolated decomposition we will use now the **WLUR** condition. Indeed, for every $x \in S_E$ let $\gamma(x)$ be the ordinal for which $x \in M_{\gamma(x)}$. Since $\|\cdot\|$ is **WLUR** for each $x \in S_E$ and each $\eta > 0$ there is a $\delta(x, \eta) > 0$ such that

$$\left|f_{\gamma(x)}(x-y)\right| < \eta \text{ whenever } y \in S_E \text{ and } \left\|\frac{x+y}{2}\right\| > 1 - \delta(x, \eta).$$

Let us define the sets

$$S_p(\eta) := \left\{x \in S_E : \ \delta(x, \eta) > \frac{1}{p}\right\} \text{ where } p = 1, 2, \dots, \ \eta > 0.$$

Thus $S_E = \bigcup_{p=1}^{+\infty} S_p(\eta)$ for every $\eta > 0$, and for any $p \in \mathbb{N}$ we have

$$\left|f_{\gamma(x)}(x-y)\right| < \eta \text{ and } \left|f_{\gamma(y)}(x-y)\right| < \eta$$

$$\text{whenever } \left\|\frac{x+y}{2}\right\| > 1 - \frac{1}{p}, \quad x, y \in S_p(\eta). \tag{1}$$

We claim that for each n, $p \in \mathbb{N}$ the family

$$\left\{M_\gamma^n \cap S_p(1/n) : \ \gamma < \Gamma\right\} \tag{2}$$

is discrete for the weak topology. Indeed, according to the choice of the sets $S_p(1/n)$ it follows

$$\gamma(x) = \gamma(y) \text{ whenever } \left\|\frac{x+y}{2}\right\| > 1 - \frac{1}{p}, \quad x, y \in \bigcup\left\{M_\gamma^n \cap S_p(1/n) : \ \gamma < \Gamma\right\}$$

since otherwise we would have $\left|f_{\gamma(x)}(x-y)\right| \geq (1/n)$ or $\left|f_{\gamma(y)}(x-y)\right| \geq (1/n)$ which contradicts (1) above. Let us choose for any $x \in S_E$ a $f_x \in S_{E^*}$ such that $\langle x, f_x\rangle = 1$. If $S\left(S_E, f_x, 1-(1/p)\right)$ intersects two sets of the family (2) then we must have

$$y \in S\left(S_E, f_x, 1-(1/p)\right) \cap M_\alpha^n \cap S_p(1/n)$$

for some $\alpha < \Gamma$ and

$$z \in S\left(S_E, f_x, 1-(1/p)\right) \cap M_\beta^n \cap S_p(1/n)$$

for some $\beta < \Gamma$. Then $\|(y+z)/2\| > 1-(1/p)$ and so $\beta = \gamma(z) = \gamma(y) = \alpha$. Therefore the family (2) is discrete in the weak topology.

Now to finish the proof it is enough to observe that

$$M_\gamma = \bigcup_{n=1}^{+\infty} \bigcup_{p=1}^{+\infty} \left(M_\gamma^n \cap S_p(1/n) \right), \qquad \forall \gamma < \Gamma.$$

■

Corollary 1 *Let $(E, \|\cdot\|)$ be a normed space with a* **WLUR** *norm then (S_E, w) has a σ–discrete network.*

Proof. Fix ε, $0 < \varepsilon < 1$. Let $\mathcal{D}_\varepsilon$ be a family of slices of the unit ball covering the unit sphere, $\mathcal{D}_\varepsilon = \{S(B_E, f_\gamma, 1-\varepsilon) : \ \gamma < \Gamma_\varepsilon\}$, with $f_\gamma \in S_{E^*}$ and the same width $1-\varepsilon$. If $M_\gamma^\varepsilon := S(B_E, f_\gamma, 1-\varepsilon) \setminus (\bigcup \{S(B_E : f_\beta, 1-\varepsilon) : \ \beta < \gamma\})$, $\gamma < \Gamma_\varepsilon$, then according to the main lemma the family $\mathcal{M}_\varepsilon := \left\{ M_\gamma^\varepsilon \cap S_E : \ \gamma < \Gamma_\varepsilon \right\}$ is σ–discretely decomposable on (S_E, w). Then $\mathcal{M} := \bigcup_{n=1}^{+\infty} \mathcal{M}_{1/n}$ must be σ–discretely decomposable since it is a countable union of families with this property.

Since for every $\varepsilon > 0$ the family $\mathcal{M}_\varepsilon$ is a covering of S_E, Proposition 1 shows that $\mathcal{M}$ is a network in (S_E, w). ■

For the proof of our main theorem we will need two more lemmas that are straightforward adaptations of Lemmas 11 and 12 of [26].

Lemma 1 (see Lemma 12 in [26]) *Let Y be a non–void set with a metric ρ and a topology $\mathcal{T}$ defined on it such that $(Y, \mathcal{T})$ has a σ–discrete network. Assume that for every $x \in Y$ there is a ρ–separable subset Z_x of Y such that $x \in \overline{\bigcup \{Z_{x_n} : \ n \in \mathbb{N}\}}^\rho$ whenever $x \in \overline{\{x_n : \ n \in \mathbb{N}\}}^{\mathcal{T}}$. Then $(Y, \mathcal{T})$ is ρ–***SLD***.*

Proof. We can proceed in the same way as in Lemma 12 of [26] where now $\mathcal{C} = \bigcup_{n=1}^{+\infty} \mathcal{C}_n$ is a σ–discrete network for the topology $\mathcal{T}$. Indeed, with the same notation as there, given $t \in Y$ there exist $n_k \in \mathbb{N}$ and $\gamma_k \in \Gamma_{n_k}$ such that $\left\{ V_{\gamma_k}^{n_k} \right\}_{k=1}^{+\infty}$ are the sets of the network containing t. The previously chosen points $\left\{ v_{\gamma_k}^{n_k} \right\}_{k=1}^{+\infty}$ clearly verifies that

$$t \in \overline{\{v_{\gamma_k}^{n_k} : \ k \in \mathbb{N}\}}^{\mathcal{T}}.$$

Now the statement follows in the same manner as in [26]. ■

Lemma 2 (see Lemma 11 in [26]) *Let E be a normed space, $(F,\mathcal{T})$ a topological space with a σ–discrete network and A a subset of E. Let $\varphi : A \to F$ be a map such that for every $x \in A$ there is a separable subspace Z_x of E in such a way that $x \in \overline{span\{Z_{x_n} :\ n \in \mathbb{N}\}}^{\|\cdot\|}$ whenever $\{x_n\}_{n=1}^{+\infty}$ is a bounded sequence with*

$$\varphi(x) \in \overline{\{\varphi(x_n) :\ n \in \mathbb{N}\}}^{\mathcal{T}}.$$

*Then the topological space $(A, \varphi^{-1}(\mathcal{T}))$ is $\|\cdot\|$–**SLD** (where $\varphi^{-1}(\mathcal{T})$ is the topology $\{\varphi^{-1}(V) :\ V \in \mathcal{T}\}$ that is the coarser topology for which φ is continuous).*

Proof. It follows from Lemma 1 above by the same method of linearization as Lemma 11 follows from Lemma 12 in [26]. ■

Remark. In the next section we study topological characterizations of ρ–**SLD** property from where self–contained proofs of Corollary 2 below follow.

Corollary 2 *If E is a normed space and A a subset of it such that (A, w) has a σ–discrete network then A has the* **JNR** *property.*

Proof. It follows from Lemma 2 applied to the identity map from A into (A, w). ■

Remark 1 In particular if A is a subset of a normed space such that (A, w) is metrizable then A must have the **JNR** property.

Proof of the Main Theorem. According to Theorem 1 it suffices to show that S_E has the **sJNR** property. Moreover from Proposition 1 it follows that there is a base of weak neighbourhoods made up by slices, then we only need to show that S_E has the **JNR** property. Now it suffices to apply Corollaries 1 and 2 . ■

3 Descriptive Banach spaces and the JNR property.

Definition 5 Let $(Y,\mathcal{T})$ be a topological space and X a subset of Y, a covering $\mathcal{C}$ of X is said to be *approximating to X in* $(Y,\mathcal{T})$ whenever for every $U \in \mathcal{C}$ we can select a separable subset W_U, $W_U \subset Y$ such that

$$x \in \overline{\bigcup\{W_U :\ x \in U,\ U \in \mathcal{C}\}}^{\mathcal{T}}, \qquad \forall x \in X.$$

When $Y = X$ we say that the topological space $(Y, \mathcal{T})$ has an *approximating covering.*

Every network $\mathcal{N}$ of a topology $\mathcal{T}$ on a set Y provides an approximating covering. Indeed choose P_U in U for every $U \in \mathcal{N}$ and we have

$$x \in \overline{\{P_U : \ U \in \mathcal{N}, \ x \in U\}}^{\mathcal{T}}.$$

We introduce here the concept of approximating covering as a tool for gluing "separable pieces" which could be understood as the "topological analog" for the projectional resolutions of the identity in a Banach space.

The following proposition and its corollary contains Lemma 1 in Section 2 with all details for a complete proof:

Proposition 2 *Let Y be a set with a metric ρ and a topology $\mathcal{T}$ defined on it. Let X be a non–void subset of Y. Then the following conditions are equivalent:*

*a) $(X, \mathcal{T})$ is ρ–**SLD**;*

b) Every discrete family of sets in (X, ρ) is σ–isolatedly decomposable in $\mathcal{T}$;

c) (X, ρ) has a network which is σ–isolated for the topology $\mathcal{T}$;

d) There is a covering approximating to X in (Y, ρ) which is σ–isolated for the topology $\mathcal{T}$;

e) There exists a sequence of subsets $D_n \subset X$ and a sequence of $\mathcal{T}$ to ρ continuous functions $g_n : D_n \to Y$ such that

$$t \in \overline{\{g_n(t) : \ t \in D_n\}}^{\rho}, \qquad \forall t \in X.$$

Proof a)$\Longrightarrow$b) Let $\mathcal{A}$ be a discrete family of sets in (X, ρ). Set $X^{(m)}$ the set of points x in X for which the cardinal of the set $\{A \in \mathcal{A} : \ B_\rho(x, 1/m) \cap A \neq \emptyset\}$ is not greater than one. From the discreteness of the family $\mathcal{A}$ we get $X = \bigcup_{m\geq 1} X^{(m)}$. Moreover according to a) it follows that for any $m \geq 1$ we can write

$$X = \bigcup_{n\geq 1} X_{m,n},$$

where for any $x \in X_{m,n}$ there exists $V \in \mathcal{T}$ such that $x \in V$ and ρ–diam$(V \cap X_{m,n}) < 1/m$. Then if for any $A \in \mathcal{A}$ we write $A_{m,n} := A \cap X_{m,n} \cap X^{(m)}$ we get

$$A = \bigcup_{m,n\geq 1} A_{m,n}$$

and from the choice of the sets $A_{m,n}$'s it is easy to see that for any m, $n \geq 1$, the family $\{A_{m,n} : \ A \in \mathcal{A}\}$ is isolated for the topology $\mathcal{T}$.

b)$\Longrightarrow$c) Since every metric space has a σ–discrete basis [21], we can fix a basis $\mathcal{B} = \bigcup_{n\geq 1} \mathcal{B}_n$ for the ρ–topology such that every $\mathcal{B}_n$ is a discrete family in (X, ρ). For every $B \in \mathcal{B}_n$ we have

$$B = \bigcup_{p\geq 1} B_p$$

where $\{B_p : B \in \mathcal{B}_n\}$ is $\mathcal{T}$–isolated. Then the family $\bigcup_{n\geq 1}\bigcup_{p\geq 1} \{B_p : B \in \mathcal{B}_n\}$ is a network for the ρ–topology which is $\mathcal{T}$–σ–isolated.

c)$\Longrightarrow$d) Let $\mathcal{C} = \bigcup_{n\geq 1} \mathcal{C}_n$ be a network for the ρ–topology on X where every $\mathcal{C}_n$ is isolated for $\mathcal{T}$. For every $U \in \mathcal{C}_n$ chose an element $z_{n,U} \in U$ and set $Z_{n,U} := \{z_{n,U}\}$. Then $\mathcal{C}$ is clearly a σ–isolated covering of $(X, \mathcal{T})$ that is approximating to X in (Y, ρ) .

d)$\Longrightarrow$e) Let $\mathcal{C}$ be a covering approximating to X in (Y, ρ) such that $\mathcal{C} = \bigcup_{n\geq 1} \mathcal{C}_n$ and every $\mathcal{C}_n$ is a $\mathcal{T}$–isolated family. Let $\{W_U : U \in \mathcal{C}\}$ be the family of ρ–separable subspaces fulfilling the requirement of Definition 5. Let us fix for every $U \in \mathcal{C}$ a sequence $\{x(U, k) : k \in \mathbb{N}\}$ such that

$$W_U \subset \overline{\{x(U, k) : k \in \mathbb{N}\}}^{\rho}.$$

Then

$$x \in \overline{\{x(U, k) : k \in \mathbb{N}, \ x \in U, \ U \in \mathcal{C}\}}^{\rho}, \quad \forall x \in X \tag{3}$$

Set $D_n := \bigcup\{C : C \in \mathcal{C}_n\}$, since $\mathcal{C}_n$ is $\mathcal{T}$–isolated it makes sense to define the functions $g_{n,k} : D_n \to Y$ by the equality $g_{n,k}(x) = x(U,k)$ if $x \in U$, $U \in \mathcal{C}_n$. Every $g_{n,k}$ is $\mathcal{T}$–locally constant and so $\mathcal{T}$–to–ρ continuous. Moreover from (3) it follows

$$x \in \overline{\bigcup\{g_{n,k}(x) : k \geq 1, x \in D_n\}}^{\rho}.$$

e)$\Longrightarrow$a). Given $\varepsilon > 0$, we define the sets

$$X_{n,\varepsilon} := \{t \in D_n : \rho\left(g_n(t), t\right) < \varepsilon/6\}.$$

The condition e) gives that $X = \bigcup_{n\geq 1} X_{n,\varepsilon}$. Moreover if we fix $t \in X_{n,\varepsilon}$ from the continuity of g_n from $(D_n, \mathcal{T})$ into (Y, ρ) we get a $\mathcal{T}$–neighbourhood W of t such that for any $t' \in D_n \cap W$ we must have $\rho\left(g_n(t), g_n(t')\right) < \varepsilon/6$. Consequently for any $t' \in X_{n,\varepsilon} \cap W$ we have

$$\rho(t, t') \leq \rho(t, g_n(t)) + \rho(g_n(t), g_n(t')) + \rho(g_n(t'), t') < \varepsilon/6 + \varepsilon/6 + \varepsilon/6 = \varepsilon/2$$

so ρ–diameter$(X_{n,\varepsilon} \cap W) \leq \varepsilon$. ■

As we have seen before, natural situations to apply the former proposition is when $\mathcal{T}$ is coarser than the ρ–topology. Moreover if we assume the linking condition stated in our Lemma 1 above we have the following

Corollary 3 *Let Y be a set with a metric ρ and a coarser topology $\mathcal{T}$ defined on it. If for every $x \in Y$ it has been associated a ρ–separable subset Z_x of Y such that $x \in \overline{\bigcup\{Z_{x_n} : n \in \mathbb{N}\}}^{\rho}$ whenever $x \in \overline{\{x_n : n \in \mathbb{N}\}}^{\mathcal{T}}$ for x, $x_n \in Y$, then the following conditions are equivalent for a given subset X of Y:*

*a) $(X, \mathcal{T})$ is ρ–**SLD**;*

b) $(X, \mathcal{T})$ has a σ–isolated network;

c) There is a σ–isolated approximating covering of X in $(Y, \mathcal{T})$;

d) There exists a sequence of subsets, $D_n \subset X$ and a sequence of $\mathcal{T}$–locally constant functions $g_n : D_n \to Y$, such that

$$t \in \overline{\{g_n(t) : t \in D_n\}}^{\mathcal{T}} \quad \forall t \in X.$$

Proof. a)$\Longrightarrow$b) It follows from Proposition 2 since a network for the ρ–topology will be a network for the coarser topology $\mathcal{T}$ too.

b)$\Longrightarrow$c)$\Longrightarrow$d). The proof is the same as in Proposition 2 replacing the ρ–topology by the topology $\mathcal{T}$.

d)$\Longrightarrow$a) It is a consequence of our hypothesis linking the $\mathcal{T}$ and ρ–approximation. Indeed let us fix the positive integer n and the $\mathcal{T}$–locally constant function $g_n : D_n \to Y$, for $t \in D_n$ choose points $t(n,k)$ such that

$$Z_{g_n(t)} \subset \overline{\{t(n,k) : \ k \in \mathbb{N}\}}^{\rho}.$$

We can define $\mathcal{T}$–locally constant functions

$$g_{n,k} : D_n \to Y$$

by $g_{n,k}(t) = t(n,k)$, $t \in D_n$. Now according to our hypothesis

$$t \in \overline{\{g_n(t) : \ t \in D_n\}}^{\mathcal{T}} \Longrightarrow t \in \overline{\bigcup \left\{ Z_{g_n(t)} : \ t \in D_n \right\}}^{\rho}$$

and consequently $t \in \overline{\bigcup \{g_{n,k}(t) : \ t \in D_n, \ k \in \mathbb{N}\}}^{\rho}$. To finish the proof it is enough to apply Proposition 2. ■

Remark 2 1. The equivalence a)$\Longleftrightarrow$b) in Proposition 2 is due to L. Oncina [28]. We are very grateful to him for his permission to include the proof here.

2. Topological spaces having a σ–isolated network have been studied by Hansell [14], [15]. They are characterized as continuous images of a metric space through maps sending discrete families to σ–isolatedly decomposable families. If in addition the

metric space is complete the spaces are called *descriptive* in [14] and isolated–analytic in [15]. In any case they form a general class for extending classical analytic spaces to the non–separable case and they go all the way back to a previous work of Frolík [9]. Proposition 2 shows that the ρ–**SLD** property due to Jayne, Namioka and Rogers, is equivalent to the σ–isolated network condition studied by Hansell.

3. A class of generalized metric spaces is a class of spaces defined by a property possessed by all metric spaces which are *close* to metrizability in some sense [12]. Proposition 2 above provides some characterizations of one of these classes, namely the topological spaces with a countable cover by sets of small local diameter introduced by Jayne, Namioka and Rogers in [19]. The σ–spaces are defined by replacing the *base* by *network* in the Bing–Nagata–Smirnov metrization theorem, i.e. a topological space is a σ–space if it has a σ–discrete network. Here we are dealing with a further refinement replacing *discrete* by *isolated*. Closely related is the class of semi–metrizable spaces which fits between Moore spaces and semi–stratifiable spaces. Our conditions d) of Proposition 2 and c) of Corollary 3 will help us later since every semi–stratifiable space is going to have a σ–discrete approximating covering despite they are not σ–spaces in general [12, Example 9.10.].

4. The implications d)$\Longrightarrow$e)$\Longrightarrow$ a) in Proposition 2 are based upon Srivatsa's proof of his selection theorem [32] and [20]. The idea of approximating by points that can be taken outside of X comes from Srivatsa approach for the well known Jayne–Rogers selection theorem [32].

If we apply Proposition 2 and its corollary to normed spaces with the norm–metric and the weak topology we shall obtain the proof of the Theorem 2 stated in the introduction. Indeed we can be more precise here and show the following:

Theorem 3 *Let E be a normed space and A a non–void subset of it. The following* *assertions are equivalent:*

a) A has the **JNR** *property (i.e. (A, w) is $\|\cdot\|_E$–***SLD***);*

b) A has a network for the $\|\cdot\|_E$–topology which is σ–isolated for the weak topology;

c) A has a σ–isolated network for the weak topology;

d) A has a σ–isolated approximating covering in (E, w);

e) There is a sequence of subsets $D_n \subset A$ and a sequence of continuous functions $g_n : (D_n, w) \to (E, \|\cdot\|)$ such that

$$t \in \overline{span\ \{g_n(t) :\ t \in D_n\}}^{\|\cdot\|} \quad \forall t \in A. \tag{4}$$

Proof. It is clear from the proof of Proposition 2 and Corollary 3 that the only implication to prove is e)$\Longrightarrow$a). We will proceed in a standard way.

Let us fix an $\varepsilon > 0$. For any $x \in A$, $m \in \mathbb{N}$, $p = (p_i)_1^m \in \mathbb{N}^m$ and $q = (q_i)_1^m \in \mathbb{Q}^m$ we set

$$D^p := \bigcap D_{p_i}, \quad f_{p,q}(x) := \sum q_i g_{p_i}(x).$$

If we put

$$S_{m,p,q} := \{x \in D^p :\ \|x - f_{p,q}(x)\| < \varepsilon\}$$

from (4) we get $A = \bigcup S_{m,p,q}$. Since $f_{p,q} : D^p \to E$ are weak–norm continuous we can find a weak neighbourhood W of x such that $\|f_{p,q}(x) - f_{p,q}(y)\| < \varepsilon$ whenever $y \in D^p \cap W$. Then for $y \in W \cap S_{m,p,q}$ we have

$$\|x - y\| < \|x - f_{p,q}(x)\| + \|f_{p,q}(x) - f_{p,q}(y)\| + \|y - f_{p,q}(y)\| < 3\varepsilon.$$

So $\|\cdot\|$–diam $(W \cap S_{m,p,q}) \le 6\varepsilon$. ■

Remark 3 The equivalence b)$\Longleftrightarrow$c) is already contained in Hansell's preprint of 1989 [14]. When A is a Souslin set of E these conditions are equivalent to say that A is descriptive in the terminology of Hansell [14, Theorem 1.3.]. In particular when A is the whole space E and E is a Banach space they are equivalent to having the **JNR** property or to be isolated–analytic, for the weak topology [15]. Results in the same spirit for σ–fragmentability appear in [14], [18], [22] and [27]. Since every **WCG** Banach space is **LUR** renormable, every weakly compact subset of a Banach space verifies any of the equivalent conditions of the corollary above, (see remark 2.7 in [32]). It is an interesting question to provide of a direct proof of this fact without using, in a way or another, the projectional resolution of the identity approach.

Remark 4 A Banach space has the property **JNR** if and only if its unit sphere has it. A proof of this fact can be obtained by a slight modification of the proof of Theorem 2.8 in [19, p 174].

We are going to show that semi–stratifiable spaces are of importance in those questions. We begin by giving the precise definition:

Definition 6 ([12], Theorem 5.8.) A topological space $(Y, \mathcal{T})$ is said to be *semi–stratifiable* if there is a function $g : \mathbb{N} \times Y \to \mathcal{T}$ such that

i) $\{x\} = \bigcap_{n=1}^{+\infty} g(n, x)$, for every $x \in Y$, and

ii) $y \in \bigcap_{n=1}^{+\infty} g(n, x_n) \Longrightarrow (x_n)_{n=1}^{+\infty}$ converges to y.

It is well known that a topological space Y is semi–metrizable if and only if it is semi–stratifiable and first countable [12, Theorem 9.8.], and that every semi–stratifiable space is subparacompact [12, Theorem 5.11.]. Despite that even the semi–metrizable spaces are not in general σ–spaces the following results show that they always have a σ–discrete approximating covering which will be of interest for applications.

Proposition 3 *Let $(Y, \mathcal{T})$ be a semi–stratifiable topological space. Then Y has a σ–discrete approximating covering.*

Proof. Let $g : \mathbb{N} \times Y \to \mathcal{T}$ be the mapping of Definition 6. For every positive integer n we set the open cover

$$\mathcal{S}_n := \{g(n,x) : \ x \in Y\}.$$

Since semi–stratifiable spaces are subparacompact we must have a σ–discrete refinement $\mathcal{F}_n$ of $\mathcal{S}_n$. For every $F \in \mathcal{F}_n$ we select $x_F \in Y$ such that $F \subset g(n, x_F)$ and we set $W_F := \{x_F\}$. It is clear now by the condition (ii) of Definition 6 that $\bigcup_{n=1}^{+\infty} \mathcal{F}_n$ is an approximating covering which is σ–discrete too. ■

Example 1 We have seen in Section 2 that the unit sphere of a **WLUR** norm has a σ–discrete network for the weak topology from where the **LUR** renormability follows. The semi–stratifiable spaces lead us to a more general setting related with $\|\cdot\|$ on E such that the dual $\|\cdot\|^*$ is Gâteaux differentiable in the norm–attaining linear functionals of S_{E^*}. Given a Banach space E equipped with a norm $\|\cdot\|$ we denote by

$$\mathrm{NA}_1(\|\cdot\|) := \{f \in S_{E^*} : \ f(x) = \|f\|, \text{ for some } x \in S_E\}.$$

Debs, Godefroy and Saint Raymond show in [3, Lemma 10] that a norm $\|\cdot\|$ is **LUR** if, and only if, there exists a map $\sigma : \mathrm{NA}_1(\|\cdot\|) \to S_E$ w^*–to–norm continuous, such that $< f, \sigma(f) >= 1$, $\forall f \in \mathrm{NA}_1(\|\cdot\|)$. Given a rotund norm $\|\cdot\|$ on E we have for every $f \in \mathrm{NA}_1(\|\cdot\|)$ a unique vector $\sigma(f) \in S_E$ with $< f, \sigma(f) >= 1$. A simple adaptation of the arguments given in [3] shows that σ is $\|\cdot\|^*$–to–weakly continuous if, and only if, the dual $\|\cdot\|^*$ is Gâteaux differentiable at every point of $\mathrm{NA}_1(\|\cdot\|)$. When σ is only w^*–to–weak continuous, (this is the case of a **WLUR** norm) we can show that (S_E, w) is a semi–stratifiable space. Indeed let us denote by f_x any linear functional on S_E with $< x, f_x >= 1$ and $x \in S_E$. We define the map

$$\begin{aligned} g : \mathbb{N} \times S_E &\to \{W \cap S_E : \ W \text{ weak open set}\} \\ g(n,x) &:= S\left(S_E, f_x, 1 - (1/n)\right). \end{aligned}$$

Then by rotundity we have

$$\{x\} = \bigcap_{n=1}^{\infty} g(n,x) \text{ for every } x \in S_E \tag{5}$$

and by w^* to weak continuity of σ we have

$$y \in \bigcap_{n=1}^{\infty} g\left(n, x_n\right),\ y \in S_E \Longrightarrow \left(x_n\right) \text{ converges weakly to } y. \tag{6}$$

Indeed, (f_{x_n}) has a w^*–cluster point g in B_{E^*} and $g(y) = 1$ from where it follows that $g \in \mathrm{NA}_1(\|\cdot\|)$ and $\sigma(g) = y$. The w^*–to–weak continuity of σ implies that $(\sigma(f_{x_n}) = x_n)$ weakly converges to y and the proof is done. Consequently S_E has the **JNR** property and since $\|\cdot\|^*$ is also Gâteaux–smooth at $\mathrm{NA}_1(\|\cdot\|)$, the given norm $\|\cdot\|$ is **WMLUR** and the Banach space E will be **LUR** renormable too. Summarizing if a rotund norm $\|\cdot\|$ on a Banach space E is such that the map $\sigma : \mathrm{NA}_1(\|\cdot\|) \to S_E$ given by $< f, \sigma(f) >= 1$, $\forall f \in \mathrm{NA}_1(\|\cdot\|)$ is weak*–to–weak continuous then E is **LUR** renormable too. When we renorm the space E we change the set $\mathrm{NA}_1(\|\cdot\|)$ and the map σ becomes weak*–to–$\|\cdot\|$ continuous.

Let us mention that Kenderov and Moors have recently shown that a Banach space is σ–fragmentable if and only if it is fragmented by a metric finer than the weak topology [23], deducing that if a dual space has a Gâteaux differentiable norm then the predual space is σ–fragmentable.

4 Concluding remarks

Let us finish with a precise description of the kind of generalized metric space structure we have on the unit sphere of a **WLUR** Banach space.

Definition 7 For a subset A of Y and a collection $\mathcal{U}$ of subsets of Y we denote

$$\mathrm{st}(A,\mathcal{U}) := \bigcup \{U \in \mathcal{U} :\ U \cap A \neq \emptyset\}.$$

For $x \in Y$ we write $\mathrm{st}(x,\mathcal{U})$ instead of $\mathrm{st}(\{x\},\mathcal{U})$. A sequence $(\mathcal{G}_n)$ of open covers of a topological spaces $(Y, \mathcal{T})$ is called a *development for* Y if for each $x \in Y$, the sets

$$\{\mathrm{st}(x, \mathcal{G}_n) :\ n \in \mathbb{N}\}$$

is a base of neighbourhoods at x. A *Moore space* is a regular space with a development.

Proposition 4 *Let E be a normed space with a* **WLUR** *norm. Then the unit sphere S_E with the weak topology is a Moore space.*

Proof. For every $x \in S_E$ we choose $f_x \in S_{E^*}$ with $< x, f_x >= 1$. For every $n \in \mathbb{N}$ we define $\mathcal{G}_n := \{S\left(S_E, f_x, 1-(1/n)\right) : \ x \in S_E\}$, that is the family of all slices of S_E given by the linear functionals f_x, $x \in S_E$, with a fixed width $1-(1/n)$. It is clear now that for every $x \in S_E$, $\{\mathrm{st}\left(x, \mathcal{G}_n\right) : n \in \mathbb{N}\}$ is a base of neighbourhoods at x. Indeed if V is weak–open set with $x \in V$ and $\mathrm{st}\left(x, \mathcal{G}_n\right) \cap S_E \setminus V \neq \emptyset$, $n \in \mathbb{N}$, we must have for every positive integer n a point $y_n \in \mathrm{st}\left(x, \mathcal{G}_n\right)$ with $y_n \notin V$. Since $y_n \in \mathrm{st}\left(x, \mathcal{G}_n\right)$ there exists f_{z_n} with $f_{z_n}(x) > 1-(1/n)$, and $f_{z_n}\left(y_n\right) > 1-(1/n)$, so we have $\left\|\left(x+y_n\right)/2\right\| > 1-(1/n)$ and the **WLUR** of the norm implies that (y_n) converges weakly to x which is a contradiction with the choice $y_n \notin V$, $\forall n \in \mathbb{N}$. Consequently $(\mathcal{G}_n)_{n=1}^{\infty}$ is a development of (S_E, w) which is a Moore space. ■

For a **WUR** norm we can prove more

Proposition 5 *Let E be a normed space with a* **WUR** *norm. Then the unit sphere S_E with the weak topology is metrizable.*

Proof. Let $(\mathcal{G}_n)$ be the development defined in Proposition 4. It is enough to apply Moore metrization theorem [12, Theorem 1.4.] and to show in that case that given a weak–open set V with $x \in V$, $x \in S_E$, there exists another weak–open set W, $x \in W$, and a positive integer n such that

$$\mathrm{st}\left(W, \mathcal{G}_n\right) \subset V.$$

Reasoning as in the former proposition, if this is not the case we must have for every positive integer n

$$\left(\bigcup\{G : \ G \cap S\left(S_E, f_x, 1-(1/n)\right) \neq \emptyset, \ G \in \mathcal{G}_n\}\right) \cap \left(S_E \setminus V\right) \neq \emptyset.$$

Therefore for every positive integer n we find $G_n \in \mathcal{G}_n$ with $G_n \cap S(S_E, f_x, 1-(1/n)) \neq \emptyset$ and $y_n \in G_n$, $y_n \notin V$. If $G_n = S(S_E, f_{z_n}, 1-(1/n))$ and $w_n \in G_n \cap S(S_E, f_x, 1-(1/n))$ we have $f_{z_n}(w_n) > 1-(1/n)$, $f_{z_n}(y_n) > 1-(1/n)$, $f_x(w_n) > 1-(1/n)$, and so $\|(y_n + w_n)/2\| > 1-(1/n)$, $\|(w_n + x)/2\| > 1-(1/n)$ and $y_n \notin V$, $\forall n \in \mathbb{N}$. Since the norm is **WUR** we have that weak–$\lim_n (y_n - z_n) = 0$ and weak–$\lim_n w_n = x$, so weak–$\lim_n y_n = x$ which is a contradiction with the choice $y_n \notin V$, $\forall n \in \mathbb{N}$.

Remark 5 There are **WUR** norms that have not Kadec property so the metric in the unit sphere whose associated topology is the weak topology is not the $\|\cdot\|$–metric.

Open Problems

Problem 1 Is there an example of a σ–fragmentable Banach space which is not descriptive (i.e. has not **JNR**)? (see [5], [14])

Problem 2 Is the renormability by a Kadec norm, or by a **LUR** norm an invariant property under weak or under Lipschitz isomorphism?

Problem 3 If a Banach spaces is descriptive and it has a rotund norm $\|\cdot\|$, does it follows that it has an equivalent **LUR** norm? Even in the particular case when (S_E, w) is metrizable we do not know the answer. A positive answer would be the natural extension of the third named author result asserting that Kadec property and rotundity imply **LUR** renormability.

Problem 4 Does it follow easily, i.e. without using Amir–Lindenstrauss approach, that any weakly compact set of a Banach space has the **JNR** property.

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