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On the equivalence of weak and Schauder bases

By

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McArthur asks in [14] about the relation between the closed graph theorem and the continuity theorem for the linear coefficients of a basis in a locally convex space. The purpose of this paper is to give a result which shows the linking between the weak basis theorem and the closed graph theorem. We shall apply it to the closed graph theorems of Pták, Saxon, De Wilde and Valdivia obtaining many of the known cases of validity for the weak basis theorem as well as new applications. The same result holds for bases of complete subspaces.

I. Introduction and notations. The linear spaces we shall use here are defined over the field K of real or complex numbers. The word "space" means separated locally convex space (l.c.s.). For a space $E[\mathfrak{T}]$ we put $\hat{E}[\hat{\mathfrak{T}}]$ for the completion of $E[\mathfrak{T}]$ and $\tilde{E}[\tilde{\mathfrak{T}}]$ for the local completion. $\tilde{E}$ is the intersection of all locally complete subspaces of $\hat{E}$ containing E and $\tilde{\mathfrak{T}}$ is the restriction of $\hat{\mathfrak{T}}$ in $\tilde{E}$. Standard references for notation and concepts are, [11], and [12].

A sequence $\{e_n : n = 1, 2, \ldots\}$ in a l.c.s. E is said to be a basis of E if every $x \in E$ has a unique representation $x = \sum_{n=1}^{\infty} c_n(x) e_n$, $c_n(x) \in K$ $n = 1, 2 \ldots$ Let E' be the topological dual of E. When $\{c_n : n = 1, 2, \ldots\} \subset E'$ the basis is called a Schauder basis. A weak basis is a basis for the weak topology, that means a basis of $E[\sigma(E, E')]$. For every positive integer m, we denote by T_m the linear mapping $T_m : E \to E$ defined by $T_m(x) = \sum_{n=1}^{m} c_n(x) e_n$. A basis is said to be equicontinuous if the sequence $\{T_m : m = 1, 2, \ldots\}$ is equicontinuous. Every equicontinuous basis is a Schauder basis and the converse is true for barrelled spaces. An equicontinuous basis of E is an equicontinuous basis of its completion because the operators T_m can be uniquely extended to $\hat{E}$ forming an equicontinuous sequence which converges to the identity on the dense subspace E and hence on $\hat{E}$.

Banach, [2], proved that a basis of a (B)-space is always a Schauder basis. This result was generalized to (F)-spaces by Newns, [15]. Banach proved, further, that a weak basis of a (B)-space is always a basis. Bessaga and Pełczynsky, [4], extended this result to (F)-spaces. We known then that every weak basis in a Fréchet space is a Schauder basis. This result is known as the weak basis theorem. Bennet and Cooper, [3], proved it for strict (LF)-spaces and Floret, [10], for sequentially retractive (LF)-spaces. M. De Wilde [5], obtained a rather general result for bornological, sequentially complete and webbed spaces. Efimova, [9], has recently proved the weak basis theorem for regular inductive limits of a sequence of normed barrelled spaces. M. Valdivia has showed to us the result for metrizable barrelled spaces.

The purpose of this paper is to give a result which shows the linking between the weak basis theorem and the closed graph theorem. We shall apply it to the closed graph theorems of Pták, Saxon, De Wilde, and Valdivia obtaining all the aforesaid results as a particular case and we shall give new cases of application. The idea behind the results given here is that the projective equicontinuous topology of McArthur, [14], can be endowed with a good structure for the closed graph theorem. That is usually done embedding the space with McArthur's topology in a sequence space with the range enlarged and endowing it with a finer good structure for the closed graph theorem.

II. The weak basis theorem. For a l.c.s. F we denote by $l^\infty(F)$ the space of bounded sequences in F endowed with the topology of uniform convergence.

Theorem 1. *Let E be a subspace of the l.c.s. $F[\mathfrak{T}]$ such that every linear mapping from E into $l^\infty(F)$ with closed graph in $E \times l^\infty(F)$ is continuous. If E has a weak basis $\{e_n: n = 1, 2, \ldots\}$, it is a Schauder and equicontinuous basis of $E[\mathfrak{T}]$ and consequently of its completion $\hat{E}[\hat{\mathfrak{T}}]$.*

Proof. Let $\mathfrak{T}^*$ be the locally convex topology on E with a fundamental system of neighbourhoods of the origin given by the subsets $V^* = \bigcap_{m=1}^{\infty} T_m^{-1}(V)$, where V is any neighbourhood of the origin in E, [14]. V^* is contained in the weak closure of V and so $\mathfrak{T}^*$ is finer than $\mathfrak{T}$ and a Hausdorff topology. The sequence $\{T_m: E[\mathfrak{T}^*] \to E[\mathfrak{T}]: m = 1, 2, \ldots\}$ is clearly equicontinuous. We shall prove that $\mathfrak{T}^* = \mathfrak{T}$ and this equality gives us the result. Indeed, in such a case $\{T_m: m = 1, 2, \ldots\}$ is an equicontinuous sequence of mappings on $E[\mathfrak{T}]$ that converges to the identity on the dense subspace generated by the vectors of the basis, hence the convergence is on all the space $E[\mathfrak{T}]$. Let us remark that a net $\{x_\alpha, \alpha \in D, \geqq\}$ converges to the origin in $E[\mathfrak{T}^*]$ if and only if the net $\{T_m(x_\alpha): \alpha \in D, \geqq\}$ converges to the origin in $E[\mathfrak{T}]$ uniformly in $m = 1, 2 \ldots$ We define the mapping $T: E[\mathfrak{T}] \to l^\infty(F)$ by $Tx = \{T_m(x): m = 1, 2, \ldots\}$. T induces on E the topology $\mathfrak{T}^*$, if we prove the continuity of T the conclusion follows immediately. According to the hypothesis of the theorem, it will be enough to show that T has its graph closed in $E \times l^\infty(F)$. Let $\{x_\alpha: \alpha \in D, \geqq\}$ be a net in E which converges to x and such that the net $\{T(x_\alpha): \alpha \in D, \geqq\}$ converges to y in $l^\infty(F)$. T_m is an operator of finite rank and so $T_m(x_\alpha) = \sum_{n=1}^{m} c_n(x_\alpha) e_n$, $\lim \{T_m(x_\alpha): \alpha \in D, \geqq\} = \sum_{n=1}^{m} \alpha_n e_n$ $m = 1, 2 \ldots$ and the limit is uniform in $m = 1, 2, \ldots$. Let $y_m = \sum_{n=1}^{m} \alpha_n e_n$, showing that the sequence $\{y_m: m = 1, 2, \ldots\}$ converges to x in the weak topology, the unicity of the representation gives us that $T_m x = y_m$ $m = 1, 2 \ldots$, which means that $Tx = y$ and the graph of T will be closed. Let $f \in E'$, $\lim \{f(T_m(x_\alpha)): \alpha \in D, \geqq\} = f(y_m)$ uniformly in $m = 1, 2 \ldots$. According to the Moore-Smith theorem about permutability of limits we have that:

$$\lim_{m\to\infty} f(y_m) = \lim_{m\to\infty} \lim_{\alpha\in D} f(T_m(x_\alpha)) = \lim_{\alpha\in D} \lim_{m\to\infty} f(T_m(x_\alpha)) = \lim_{\alpha\in D} f(x_\alpha) = f(x).$$

Therefore, $\sum_{n=1}^{\infty} \alpha_n e_n = x$ in the weak topology and the proof is finished. □

If a space E fulfills the condition of Theorem 1 we shall say that E holds the weak basis theorem. For the applications coming it is interesting to remark that given a finer topology on $l^\infty(F)$ that works for the closed graph theorem the conclusion shall be obtained.

III. Some applications.

Theorem 2. *The weak basis theorem holds on every barelled and metrizable space.*

Proof. We take F the completion of the barelled and metrizable space E. $l^\infty(F)$ is a Fréchet space and Pták's closed graph theorem, [12], together with Theorem 1 gives us the conclusion. □

Theorem 2 has been proved by Valdivia in a different way.

A space E is Baire-like, [16], if given an increasing sequence of closed, absolutely convex subsets of E covering E there is one of them which is a neighbourhood of the origin in E. Amemiya and Kōmura theorem, [1], says that a metrizable and barrelled space is Baire-like. Saxon's closed graph theorem, [16], works between Baire-like spaces and (LB)-spaces. Applying it together with Theorem 1 we can obtain the following result:

Theorem 3. *The weak basis theorem holds on every locally convex hull E of Baire-like spaces with E having a sequence of bounded subsets covering E. In particular on every inductive limit of an increasing sequence of Baire-like spaces with bounded linking maps.*

Proof. Let F be the completion of E. There is an increasing sequence of absolutely convex bounded subsets $\{D_n: n = 1, 2\dots\}$ covering E. A theorem of De Wilde and Houet, [8], and Valdivia, [17], says that $F = \cup\{\bar{D}_n := 1, 2, \dots\}$, and that $\{n\bar{D}_n: n = 1, 2, \dots\}$ is a fundamental sequence of bounded subsets of F which are Banach discs, where we are denoting by $\bar{D}_n$ the closure of D_n in F. Let us denote by B_n the subset $n\bar{D}_n$, $n = 1, 2\dots$ and by F_{B_n} the linear hull of B_n endowed with the norm of the gauge of B_n, [11]. Therefore $l^\infty(F) = \cup\{l^\infty(F_{B_n}): n = 1, 2, \dots\}$ and we have a topology of (LB)-space on $l^\infty(F)$ finer than the original one. Saxon's closed graph theorem applys for mappings from E into $l^\infty(F)$ and we obtain the result having in mind Theorem 1. □

Now we have the following improvement of Efimova's weak basis theorem, [9], removing the regularity condition:

Corollary 3.1. *The weak basis theorem holds on every inductive limit of a sequence of normed barrelled spaces. In particular on arbitrary (LB)-spaces.*

Proof. It follows immediately from Theorem 3 because of the Amemiya-Kōmura theorem, [1]. □

As far as we know the result for (LB)-spaces was only known with the additional condition of local completeness as a consequence of De Wilde's weak basis theorem, [5].

A space E is suprabarrelled, [18], if given an increasing sequence of subspaces of E covering E there is one of them which is barrelled and dense in E. Valdivia's closed graph theorem, [18], works between suprabarrelled and (LF)-spaces.

Theorem 4. *The weak basis theorem holds on every strict inductive limit of a sequence of suprabarrelled and metrizable spaces.*

Proof. Let $\{E_n\colon n = 1, 2, \ldots\}$ be a sequence of suprabarrelled and metrizable spaces and E the inductive limit. Let F be the completion of E and $\bar{E}_n$ the closure of E_n in F which is the completion of E_n and thus a Fréchet space. We have that $F = \cup \{\bar{E}_n\colon n = 1, 2, \ldots\}$ by a theorem of De Wilde and Houet, [8]. From the barrellednes of F it follows that F is the strict inductive limit of the sequence of Fréchet spaces $\{\bar{E}_n\colon n = 1, 2, \ldots\}$, [17]. Therefore we have a topology of (LF)-space on $l^\infty(F)$ because $l^\infty(F) = \cup\{l^\infty(\bar{E}_n)\colon n = 1, 2, \ldots\}$ bearing in mind the localization theorem of bounded sets in strict (LF)-spaces. Valdivia's closed graph theorem can be applied together with Theorem 1 to reach the conclusion. □

IV. On De Wilde's weak basis theorem. Following De Wilde, [5], [7], a sequence $A_1 \supset A_2 \supset \cdots A_k \supset \cdots$ of non void subsets of a space E is said to be a completing sequence if there is a sequence (λ_k) of positive numbers such that, if $0 \leqq \mu_k \leqq \lambda_k$ and $x_k \in A_k$ $k = 1, 2 \ldots$, then the series $\sum_{k=1}^{\infty} \mu_k x_k$ is convergent in E. A web $\mathscr{W}$ in a space E is a family of subsets of E, $\mathscr{W} = \{C_{m_1, m_2, \ldots, m_k}\}$, where $k, m_1, m_2, \ldots m_k$ are positive integers and such that the following relations are satisfied:

$E = \cup \{C_n\colon n = 1, 2, \ldots\}$, $C_{m_1, m_2, \ldots m_k} = \cup \{C_{m_1, m_2, \ldots, m_k, m}\colon m = 1, 2 \ldots\}$ $k = 1, 2, \ldots$.

A web $\mathscr{W}$ is completing or $\mathscr{C}$-web if for each sequence (m_n) of positive integers the sequence $\{C_{m_1, m_2, \ldots, m_n}\colon n = 1, 2, \ldots\}$ is completing. A space with a completing web is called a webbed space or $\mathscr{C}$-web space.

De Wilde's closed graph theorem is verified for mappings between Baire spaces and webbed spaces. A space E is totally barrelled, [21], if given a sequence of subspaces of E covering E there is one of them which is barrelled and its closure has finite codimension in E. Valdivia shows in [20] an extension of De Wilde's theorem for linear mappings defined from totally barrelled spaces into spaces with a completing web of absolutely convex subsets. On looking carefully at Valdivia's proof it is clear that the result remains true for spaces with a web $\mathscr{W} = \{C_{m_1, m_2, \ldots m_k}\}$ such that for each sequence (m_n) of positive integers, if $A_{m_1, m_2, \ldots m_k}$ is the absolutely convex cover of $C_{m_1, m_2, \ldots m_k}$, the sequence $\{A_{m_1, m_2, \ldots m_k}\colon k = 1, 2, \ldots\}$ is completing. For instance, for webbed spaces which are locally complete.

In his thesis, De Wilde, [5], proved the following result:

Theorem 5. *Let E be a bornological, sequentially complete and webbed space. If E has a weak basis* $\{e_n: n = 1, 2, \ldots\}$, *then it is a Schauder basis on E.*

This result is a consequence of Theorem 1 together with De Wilde's closed graph theorem. Indeed, for a sequentially complete and webbed space E, $l^\infty(E)$ has a $\mathscr{C}$-web. Moreover if E is bornological it must be ultrabornological and we get a proof working with $F = E$ in Theorem 1.

The important fact inside the former result is that for a webbed and locally complete space E, $l^\infty(E)$ has a $\mathscr{C}$-web too, [7]. Moreover it is locally complete and Valdivia's closed graph theorem, [20], can be applied to obtain the following improvement:

Theorem 6. *The weak basis theorem holds on every locally convex hull of totally barrelled spaces which has a local completion with a* $\mathscr{C}$*-web.*

Open problem. We do not know if the weak basis theorem holds for ultrabornological and strictly webbed spaces, [5], [7]. Even in the case of an arbitrary (LF)-space it seems to be difficult to find the adequate space F in order to apply Theorem 1. Of course we have positive answers for (LF)-spaces locally complete (De Wilde, [5]), for metrizable (LF)-spaces (Theorem 2) and for (LB)-spaces (Corollary 3.1 and Theorem 6).

V. On Schauder decomposition. Let $E[\mathfrak{T}]$ be a l.c.s. A sequence of non trivial subspaces $\{M_i: i = 1, 2, \ldots\}$ of E is a basis of subspaces for E if and only if to each $x \in E$ corresponds a unique sequence $\{P_i(x): i = 1, 2, \ldots\}$ where $P_i(x) \in M_i$ for every i and the series $\sum_{i=1}^{\infty} P_i(x)$ converges to x in the topology $\mathfrak{T}$. It is clear that P_i are linear projections from E into M_i and verifies $P_i P_j = 0$ if $i \neq j$. A basis of subspaces with the property that each P_i is continuous is called a Schauder basis. If the subspaces M_i are closed in E the Schauder basis is called a Schauder decomposition of E, [14]. On looking carefully at the proof of Theorem 1 it is clear that it remains true for a weak basis of complete subspaces instead of a weak basis obtaining that it must be a Schauder decomposition of E. All the applications given are consequently verified in this more general situation. In such a way extensions of the known result of McArthur, [13], for Fréchet spaces are obtained.

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