Deville's master lemma and Stone's discreteness in renorming theory

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1 Introduction

Let $(X, \|\cdot\|)$ be a normed space. The norm $\|\cdot\|$ in X is said to be locally uniformly rotund (**LUR** for short) if

$$[\lim_n (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0] \Rightarrow \lim_n \|x - x_n\| = 0$$

for any sequence (x_n) and x in X. The construction of this kind of norms in separable Banach spaces lead Kadec to the proof of the existence of homeomorphism between all separable Banach spaces, [1]. For a non separable Banach space is not always possible to have such an equivalent norm, for instance the space l^∞ does not have it. When such a norm exists its construction is usually based on a good system of coordinates that we must have on the normed space X from the very beginning, for instance in a biorthogonal system

$$\{(x_i, f_i) \in X \times X^* : i \in I\}$$

with some additional properties such as being a strong Markusevich basis, [16]. Sometimes there is not such a system and the norm is constructed modelling enough convex functions on the given space X to add all of them up with the powerful lemma of Deville, see lemma VII 1.1 in [2].

We are going to present here a lemma connecting Deville's master lemma with our approach for locally uniformly rotund renormings as developped in [12]. We have been extensively using Stone's theorem on the paracompactness of a metric space to play with discreteness when looking for equivalent locally uniformly rotund norms on a given Banach space X. We shall see in this paper that our condition of being slicely isolated corresponds with the so called rigidity condition inside Deville's lemma. Our free-coordinate approach to **LUR** renormings is explained here with the construction of convex functions describing slicely relatively discrete families of sets in a normed space X.

2 Lower semicontinuous convex functions and LUR renormings.

We are going to study the interplay between convex functions and **LUR** renormings. The method to construct **LUR** norms based on Deville's master lemma is developped in chapter VII of [2], where it is called the decomposition method. Deville's lemma has been extensively used by R. Haydon in his seminal papers [4], [5], as well as in [6]. Deville's lemma needs the existence of suitable convex functions $(\phi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ on the Banach space X, it provides an equivalent norm $|||\cdot|||$ on X such that the **LUR** condition on a given point x gives up a maximazing sequence of convex functions ϕ_{i_n} at x that interplays with a **LUR** condition on ψ_{i_n}, see lemma 1.1 in chapter VII of [2]. Let us precisely recall it here:

Lemma 1 (Deville, Godefroy and Zizler decomposition method). *Let $(X, \|\cdot\|)$ be a normed space, let I be a set and let $(\varphi)_{i\in I}$ and $(\psi)_{i\in I}$ be families of non-negative convex functions on X which are uniformly bounded on bounded subsets of X. For every $x \in X$, $m \in \mathbb{N}$ and $i \in I$ define*

$$\varphi(x) = sup\left\{\varphi_i(x) : i \in I\right\}, \tag{1}$$

$$\theta_{i,m}(x) = \varphi_i(x)^2 + 2^{-m}\psi_i(x)^2, \tag{2}$$

$$\theta_m(x) = sup\left\{\theta_{i,m}(x) : i \in I\right\}, \tag{3}$$

$$\theta(x) = \|x\|^2 + \sum_{m=1}^{\infty} 2^{-m}(\theta_m(x) + \theta_m(-x)). \tag{4}$$

Then the Minkowski functional of $B = \{x \in X : \theta(x) \le 1\}$ is an equivalent norm $\|\cdot\|_B$ on X such that if $x_n, x \in X$ satisfy the LUR condition:

$$\lim_n [2\|x_n\|_B^2 + 2\|x\|_B^2 - \|x_n + x\|_B^2] = 0,$$

then there is a sequence (i_n) in I such that:

1. $lim_n\varphi_{i_n}(x) = lim_n\varphi_{i_n}(x_n) = lim_n\varphi_{i_n}((x + x_n)/2) = sup\left\{\varphi_i(x) : i \in I\right\}$
2. $\lim_n[\frac{1}{2}(\psi_{i_n}^2(x_n) + \psi_{i_n}^2(x)) - \psi_{i_n}^2(\frac{1}{2}(x_n + x))] = 0.$

Deville's lemma is based on the construction of an equivalent **LUR** norm on a weakly compactly generated Banach space by the second named author in [15], where the convex functions are measuring distances to suitable finite dimensional subspaces as well as evaluations on some coordinate functionals in tha dual space X^*; see [16], theorem 7.3. The method we have developped in [12] is mainly based on Stone's theorem about paracompactness of metric spaces. The σ-discrete base for the norm topology of a normed space X can be refined to obtain a σ-slicely isolated network if, and only if, the normed space X admits an equivalent **LUR** norm, [12]. Recent contributions show an interplay between both methods, [5, 8, 9]. It is our intention here to show the connection between both approaches. The linking property will be the notion of slicely relatively discretness, or slicely isolatedness, that glues the discretness of Stone's theorem with the linear topological structure of the dual pair associated to X. Let us recall precise definitions and results:

Definition 1. *Let X be a normed space and F be a norming subspace in the dual X^*. A family $\mathcal{B} := \{B_i : i \in I\}$ of subsets in X is called $\sigma(X,F)$-slicely isolated (or $\sigma(X,F)$-slicely relatively discrete) if it is a disjoint family of sets such that for every*

$$x \in \cup\{B_i : i \in I\}$$

there are a $\sigma(X,F)$-open half space H and $i_0 \in I$ such that

$$H \cap \cup\{B_i : i \in I, i \neq i_0\} = \emptyset \text{ and } x \in B_{i_0} \cap H.$$

Our approach for **LUR** renormings is based on the topological concept of network. Let us recall that a family of subsets $\mathcal{N}$ in a topological space $(T, \mathcal{T})$ is a network for the topology $\mathcal{T}$ when for every open set $W \in \mathcal{T}$ and every $x \in W$ there is some $N \in \mathcal{N}$ such that $x \in N \subset W$. A main result with our approach is the following:

Theorem 1 ([12], chapter III). *Let X be a normed space and F a norming subspace in the dual X^*. X admits a $\sigma(X,F)$-lower semicontinuous and equivalent locally uniformly rotund norm if, and only if, the norm topology has a network $\mathcal{N}$ that can be written as $\mathcal{N} = \cup_{n=1}^{\infty}\mathcal{N}_n$ where every one of the families $\mathcal{N}_n$ is $\sigma(X,F)$- isolated.*

A first result we shall prove here is that we can replace the network with a basis of the norm topology in the former theorem, see theorem **??**.

We shall begin with the construction of convex and lower semicontinuous functions related to the norm-distance function to a fixed convex set. Such a norm distance is a convex function, however to control the lower semicontinuity too we need a small modification given in the next result:

Proposition 1. *Let X be a normed space and F a norming subspace in the dual space X^*. If C is a bounded and convex subset of X and we define*

$$\varphi(x) := inf\left\{sup\{|<x-c^{**}, f>| : f \in B_{X^*} \cap F\} : c^{**} \in \overline{C}^{\sigma(X^{**},X^*)}\right\}$$

Then φ is a convex $\sigma(X,F)$-lower semicontinuous map from X to $\mathbb{R}^+$.

Proof.- The fact that C is convex implies that $\overline{C}^{\sigma(X^{**},X^*)}$ is convex too and φ is a convex function. Indeed, let us take x and y in X and fix $0 \leq \sigma \leq 1$ and $\epsilon > 0$. If we choose c_x^{**} and c_y^{**} such that

$$sup\,\{| < x - c_x^{**}, f > | : f \in B_{X^*} \cap F\} \leq \varphi(x) + \epsilon,$$

and

$$sup\,\big\{| < y - c_y^{**}, f > | : f \in B_{X^*} \cap F\big\} \leq \varphi(y) + \epsilon,$$

then

$$\begin{aligned} sup\,\big\{< \sigma x + (1-\sigma)y - (\sigma c_x^{**} + (1-\sigma)c_y^{**}), f >: f \in B_{X^*} \cap F\big\} \leq \\ sup\,\{< \sigma x - \sigma c_x^{**}, f >: f \in B_{X^*} \cap F\} + \\ sup\,\big\{< (1-\sigma)y - (1-\sigma)c_y^{**}), f >: f \in B_{X^*} \cap F\big\} \leq \\ \sigma(\varphi(x) + \epsilon) + (1-\sigma)(\varphi(y) + \epsilon) \leq \sigma\varphi(x) + (1-\sigma)\varphi(y) + \epsilon \end{aligned} \tag{5}$$

and so we have that

$$\varphi(\sigma x + (1-\sigma)y) \leq \sigma\varphi(x) + (1-\sigma)\varphi(y) + \epsilon$$

for every $\epsilon > 0$ from where the convexity follows. Let us see the lower semicontinuity now, so let us fix $r \geq 0$ and take a net $\{x_\alpha : \alpha \in A\}$ in X with $\varphi(x_\alpha) \leq r$ for every $\alpha \in A$ and let $x \in X$ be the $\sigma(X, F)$-limit of the net $\{x_\alpha : \alpha \in A\}$. We will see that $\varphi(x) \leq r$ too. Let us fix an $\epsilon > 0$ and choose $c_\alpha^{**} \in \overline{C}^{\sigma(X^{**},X^*)}$ such that

$$sup\,\{| < x_\alpha - c_\alpha^{**}, f > | : f \in B_{X^*} \cap F\} \leq r + \epsilon$$

for every $\alpha \in A$. Since C is bounded we can find a cluster (x^{**}, c^{**}) point of the net $\{(x_\alpha, c_\alpha^{**}) : \alpha \in A\}$ in $X^{**} \times X^{**}$ for the topology $\sigma(X^{**}, X^*)$. Then we have that x^{**} does coincide with x when both linear functionals are restricted to F and thus

$$< x^{**} - c^{**}, f >=< x - c^{**}, f >\leq (r + \epsilon) \text{ for all } f \in B_{X^*} \cap F$$

and so $\varphi(x) \leq r + \epsilon$. Since the reasoning is valid for every $\epsilon > 0$ we have $\varphi(x) \leq r$ as required . □

Definition 2. *For a map φ defined as in the former proposition we will say that it is the F-distance to the set $\overline{C}^{\sigma(X^{**},X^*)}$.*

We now arrive to the following interplay result:

Theorem 2. *Let $(X, \|\cdot\|)$ be a normed space and F be a norming subspace in X^*. Let $\mathcal{B} := \{B_i : i \in I\}$ be a uniformly bounded family of subsets of X. The following are equivelent:*

1. *The family $\mathcal{B}$ is $\sigma(X,F)$-slicely isolated*

2. *There is a family $\mathcal{L} := \{\varphi_i : X \to \mathbb{R}^+, i \in I\}$ of convex $\sigma(X,F)$-lower semicontinuous functions such that*

$$\{x \in X : \varphi_i(x) > 0\} \bigcap (\bigcup \{B_j : j \in I\}) = B_i$$

   for every $i \in I$.

3. *There is a family $\mathcal{L} := \{\psi_i : X \to \mathbb{R}^+, i \in I\}$ of convex $\sigma(X,F)$-lower semicontinuous functions and numbers $0 \leq \alpha \leq \beta$ such that*

$$\psi_i(B_i) > \beta \geq \alpha \geq \psi_i(B_j)$$

   for every $i, j \in I$.

*P*roof.-Let us assume that the family $\mathcal{B}$ is $\sigma(X,F)$-slicely isolated. Applying proposition 1 we may consider φ_i to be the F- distance to the convex bounded set:

$$\overline{co\{B_j : j \neq i, j \in I\}}^{\sigma(X^{**},X^*)}$$

for every $i \in I$. Our hypothesis on the slicely isolated character of the family $\mathcal{B}$ tells us that when the point x belongs to the set B_{i_0} of the family $\mathcal{B}$, then there is a $\sigma(X,F)$-open half space H in X with $x \in H$ and $H \cap B_i = \emptyset$ for all $i \in I$ with $i \neq i_0$. Let us write $H = \{y \in X : f(y) > \mu\}$ where $f \in B_{X^*} \cap F$, and then we have $\varphi_{i_0}(y) \geq f(y) - \mu > 0$ for every $y \in H$, and so $\varphi_{i_0}(x) > 0$, and $\varphi_i(x) = 0$ for all $i \in I$ with $i \neq i_0$. The condition 2 clearly implies 3 with $\alpha = \beta = 0$. Finally, if we assume 3, given a family $\mathcal{L} := \{\psi_i : X \to \mathbb{R}^+, i \in I\}$ of convex and $\sigma(X,F)$-lower semicontinuous functions such that the conditions in 3 are satisfied we will have that $\psi_i(y) \leq \alpha$ for every $y \in co\,\{B_j : j \neq i, j \in I\}$ by convexity of the function ψ_i, so for every $y \in \overline{co\,\{B_j : j \neq i, j \in I\}}^{\sigma(X,F)}$ too by the lower semicontinuity of ψ_i. Consequently we have $x \notin \overline{co\,\{B_j : j \neq i, j \in I\}}^{\sigma(X,F)}$ for every $x \in B_i$ and every $i \in I$. An straightforward application of Hahn Banach's theorem finishes the proof of the $\sigma(X,F)$-slicely isolated property for the family $\mathcal{B}$. $\square$

A normed space X with a locally uniformly rotund norm decomposes the σ-discrete basis of the norm topology in a σ-slicely isolated (or relatively discrete) network. We are going to prove now it is always posible to recover a basis of the norm topology with both properties: it is going to be σ-discrete and slicely isolated at the same time.

Proposition 2. *Let X be a normed space with a norming subspace $F \subset X^*$ and $\|\cdot\|_F$ the equivalent norm associated with it; i.e.*

$$\|\cdot\|_F := sup\{| < \cdot, f > | : f \in B_{X^*} \cap F\}.$$

Given a uniformly bounded and $\sigma(X, F)$-slicely isolated family $\mathcal{A} := \{A_i : i \in I\}$ of subsets in X there exist decompositions $A_i = \cup_{n=1}^{\infty} A_i^n$ with

$$A_i^1 \subset A_i^2 \subset ...A_i^n \subset A_i^{n+1} \subset ... \subset A_i$$

for every $i \in I$ and such that the families

$$\{A_i^n + B_{\|\cdot\|_F}(0, 1/4n) : i \in I\}$$

are $\sigma(X, F)$-slicely isolated and norm discrete for every $n \in \mathbb{N}$.

Proof.- Let us denote with φ_i de F-distance to $\overline{co(A_j : j \neq i)}^{\sigma(X^{**}.X^*)}$. The former proposition gives us the scalpel to split up the sets of the family using this convex functions. Indeed, let us define $A_i^n := \{x \in A_i : \varphi_i(x) > 1/n\}$ and we have that $A_i = \cup_{n=1}^{\infty} A_i^n$. Moreover, if $x \in A_i^n + B_{\|\cdot\|_F}(0, 1/4n)$ then we have

$$\varphi_i(x) > 3/4n$$

Indeed, let us write $x = y + z$, $y \in A_i^n$, $\|z\|_F < 1/4n$, since $\varphi_i(y) > 1/n$ we can select a number ρ with $\varphi_i(y) > \rho > 1/n$ and we will have for every fixed $c^{**} \in \overline{co(A_j : j \neq i)}^{\sigma(X^{**}.X^*)}$ that $\|y - c^{**}\|_F > \rho$. So we can find some $f \in B_{X^*} \cap F$ with $f(y - c^{**}) > \rho$. Now we see that $f((y + z) - c^{**}) > \rho - 1/4n$ and so $\|x - c^{**}\|_F > \rho - 1/4n$ for every $c^{**} \in \overline{co(A_j : j \neq i)}^{\sigma(X^{**}.X^*)}$. Consequently we see that $\varphi_i(x) \geq \rho - 1/4n > 3/4n$.
On the other hand for $y \in A_j$ with $j \neq i$, we know that $\varphi_i(y) = 0$, then for $x \in A_j^n + B_{\|\cdot\|_F}(0, 1/4n)$ if we write $x = y + z$, with $y \in A_j^n$ and $\|z\|_F < 1/4n$ we have, for fixed $c^{**} \in \overline{co(A_j : j \neq i)}^{\sigma(X^{**}.X^*)}$, that:

$$\|x - c^{**}\|_F < \|y - c^{**}\|_F + 1/4n$$

from where it follows that

$$\varphi_i(x) = inf\{\|x - c^{**}\|_F : c^{**} \in \overline{co(A_j : j \neq i)}^{\sigma(X^{**}.X^*)}\} \leq 1/4n$$

since $\varphi_i(y) = 0$. All together means that the family

$$\{A_i^n + B_{\|\cdot\|_F}(0, 1/4n) : i \in I\}$$

verifies the conditions in 3 of the former proposition with the functions $(\varphi_i)_{i\in I}$ and constants $\alpha = 1/4n, \beta = 3/4n$. Thus it is $\sigma(X, F)$-slicely isolated as we wanted to prove. Moreover, if we fix $\delta > 0$ and such that

$$1/4n + \delta < 3/4n - \delta$$

the former family is discrete for the norm topology. Indeed for any $z \in X$ we have that

$$B_{\|\cdot\|_F}(z, \delta) \cap \cup\{A_i^n + B_{\|\cdot\|_F}(0, 1/4n) : i \in I\}$$

has non empty intersection with at most one member of the family because every time the intersection is non empty we can see that $\varphi_i(z) > 3/4n - \delta$ if

$$B_{\|\cdot\|_F}(z, \delta) \cap \{A_i^n + B_{\|\cdot\|_F}(0, 1/4n)\} \neq \emptyset$$

but $\varphi_i(z) < 1/4n + \delta$ when

$$B_{\|\cdot\|_F}(z, \delta) \cap \{A_j^n + B_{\|\cdot\|_F}(0, 1/4n)\} \neq \emptyset$$

for any $j \neq i$ and $j \in I$. This fact can be seen as above writing now $z = x+y$ with $x \in B_{\|\cdot\|_F}(z, \delta) \cap \{A_i^n + B_{\|\cdot\|_F}(0, 1/4n)\}$ and $\|y\|_F < \delta$ in the first case and $x \in B_{\|\cdot\|_F}(z, \delta) \cap \{A_j^n + B_{\|\cdot\|_F}(0, 1/4n)\}$ with $\|y\|_F < \delta$ for the second one. □

We now arrive to the main result for this section:

Theorem 3. *Let X be a normed space with a norming subspace $F \subset X^*$. Then X admits an equivalent $\sigma(X, F)$-lower semicontinuous and LUR norm if, and only if, the norm topology admits a σ-discrete basis $\mathcal{B} = \cup\mathcal{B}_n$ such that every one of the families $\mathcal{B}_n$ is $\sigma(X, F)$-slicely isolated and norm discrete.*

*P*roof.- If the normed space X admits an equivalent $\sigma(X, F)$-lower semicontinuos and LUR norm it has a network for the norm topology $\mathcal{N}$ such that $\mathcal{N} = \cup_{n=1}^{\infty}\mathcal{N}_n$ where every one of the families $\mathcal{N}_n$ is $\sigma(X, F)$-slicely isolated. It is not a restriction to assume that every one of the families $\mathcal{N}_n$ is uniformly bounded since intersections with a fixed ball of a $\sigma(X, F)$-slicely isolated family continues being $\sigma(X, F)$-slicely isolated. If we apply the former proposition to every $\mathcal{N}_n$ we obtain the families we are looking for. Indeed, let us write $\mathcal{N}_p = \{N_i^p : i \in I_p\}$ and $N_i^p = \cup_{n=1}^{\infty} N_i^{p,n}$ for the decomposition made up with the former proposition. It now follows that

$$\bigcup_{n,m=1}^{\infty} \{N_i^{p,n} + B_{\|\cdot\|_F}(0, 1/4n) : i \in I_n\}$$

is the basis of the norm topology we are looking for. Indeed, for a given $x \in X$ and $\epsilon > 0$ we find some $p \in \mathbb{N}$ and $i \in I_p$ with $x \in N_i^p \subset B(x, \epsilon/2)$. There is $m_0 \in \mathbb{N}$ such that $x \in N_i^{p,m}$ whenever $m \geq m_0$. It now follows that for integers big enough m we have $N_i^{p,m} + B_{\|\cdot\|_F}(0, 1/4m) \subset B(x, \epsilon)$ since $x \in N_i^p \subset B(x, \epsilon/2)$. □

3 The connection lemma

Now we are in position to present our main result here. For a slicely isolated family of sets it is always possible to construct an equivalent norm, such that, the LUR condition on the new norm for a sequence and a point x implies that the sequence is eventually in the same set of the family to which the limit point x belongs.

Lemma 2 (Connection lemma). *Let $(X, \|\cdot\|)$ be a normed space and F be a norming subspace in X^*. Let $\mathcal{B} := \{B_i : i \in I\}$ be an uniformly bounded and slicely isolated family of subsets of X for the $\sigma(X, F)$-topology. Then there is an equivalent and $\sigma(X, F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{B}}$ on X such that for every sequence $\{x_n : n \in \mathbb{N}\}$, and x in X with $x \in B_{i_0}$ for $i_o \in I$, the condition*

$$\lim_n \left(2\,\|x_n\|_{\mathcal{B}}^2 + 2\,\|x\|_{\mathcal{B}}^2 - \|x_n + x\|_{\mathcal{B}}^2\right) = 0$$

implies that:

1. *There is n_0 such that*

   $$x_n, (x_n + x)/2 \notin \overline{co\,\{B_i : i \neq i_0, i \in I\}}^{\sigma(X,F)}$$

   for all $n \geq n_0$

2. *For every positive δ there is $n_\delta \in \mathbb{N}$ such that*

   $$x_n \in \overline{(co(B_{i_0}) + \delta B_X)}^{\sigma(X,F)}$$

   whenever $n \geq n_\delta$.

Proof: Let us fix the index $i \in I$ and define the nonnegative, convex and $\sigma(X,F)$-lower semicontinuous function φ_i as the F- distance to

$$\overline{co\{B_j : j \neq i, j \in I\}}^{\sigma(X^{**},X^*)}.$$

Let us choose a point $a_i \in B_i$ and set $D_i = co(B_i)$ for every $i \in I$, and $D_i^\delta := D_i + B(0,\delta)$, where we denote by $B(0,\delta)$ the open ball of radious δ for the equivalent norm given by

$$\| \cdot \|_F := \sup \{ | < \cdot, f > | : f \in F \cap B_{X^*} \},$$

i.e., $B(0,\delta) := \{x \in X : \|x\|_F < \delta\}$, for every $\delta > 0$ and $i \in I$. We denote by $p_{i,\delta}$ the Minkowski functional of the convex body $\overline{D_i^\delta}^{\sigma(X,F)} - a_i$. Then we can define the $\sigma(X,F)$-lower semicontinuous norm p_i by the formula

$$p_i^2(x) = \sum_{q=1}^{\infty} \frac{1}{q^2 2^q} p_{i,1/q}(x)^2$$

for every $x \in X$. Indeed, since $B(0,\delta) + a_i \subset \overline{D_i^\delta}^{\sigma(X,F)}$ we have for every $x \in X$, and $\delta > 0$, that $p_{i,\delta}(\delta x / \|x\|_F) \leq 1$, thus $\delta p_{i,\delta}(x) \leq \|x\|_F$ and the above series converges. Finally we define the nonnegative, convex and $\sigma(X,F)$-lower semicontinuous function $\psi_i(x) := p_i(x - a_i)$ for every $x \in X$. We are now in position to apply Deville's master lemma, see lemma 1 above, to get an equivalent and $\sigma(X,F)$-lower semicontinuous norm $\| \cdot \|_{\mathcal{B}}$ on X such that the condition

$$\lim_n \left(2 \|x_n\|_{\mathcal{B}}^2 + 2 \|x\|_{\mathcal{B}}^2 - \|x_n + x\|_{\mathcal{B}}^2 \right) = 0$$

for a sequence $\{x_n : n \in \mathbb{N}\}$ and x in X implies the existence of a sequence of indexes (i_n) in I such that:

1. $lim_n \varphi_{i_n}(x) = lim_n \varphi_{i_n}(x_n) = lim_n \varphi_{i_n}((x + x_n)/2) = sup \{\varphi_i(x) : i \in I\}$
2. $\lim_n [\frac{1}{2}(\psi_{i_n}^2(x_n) + \psi_{i_n}^2(x)) - \psi_{i_n}^2(\frac{1}{2}(x_n + x))] = 0$

Our hypothesis on the slicely isolated character of the family $\mathcal{B}$ tell us after theorem 2 that when the point x belongs to the set B_{i_0} of the family $\mathcal{B}$, we have $\varphi_{i_0}(x) > 0$, but $\varphi_i(x) = 0$ for all $i \in I$ with $i \neq i_0$. From the condition 1 above it now follows that there exists a postive integer n_0 such that $i_n = i_0$, $\varphi_{i_o}(x_n) > 0$ and $\varphi_{i_0}(\frac{1}{2}(x + x_n)) > 0$ for all $n \geq n_0$, from where the conclusion 1 of the lemma follows. Moreover, the condition 2 above now implies that

$$\lim_n [2^{-1}(\psi_{i_0}^2(x_n) + \psi_{i_0}^2(x)) - \psi_{i_0}^2(2^{-1}(x_n + x))] = 0,$$

and so by the convex arguments, and for every positive integer q, we have that

$$\lim_n [2^{-1}((p_{i_0,1/q}(x_n - a_{i_0}))^2 + (p_{i_0,1/q}(x - a_{i_0}))^2) - (p_{i_0,1/q}(2^{-1}(x_n + x) - a_{i_0}))^2] = 0,$$

and consequently

$$\lim_n p_{i_0,1/q}(x_n - a_{i_0}) = p_{i_0,1/q}(x - a_{i_0}).$$

If we fix a positive number δ and we set the integer q such that $1/q < \delta$, since $x - a_{i_0} \in D_{i_0}^{1/q} - a_{i_0}$ we have that $p_{i_0,1/q}(x - a_{i_0}) < 1$ because $D_{i_0}^{1/q} - a_{i_0}$ is norm open and therefore, there is a positive integer n_δ such that for $n \geq n_\delta$ we have that $p_{i_0,1/q}(x_n - a_{i_0}) < 1$ and thus $x_n - a_{i_0} \in \overline{D_{i_0}^{\delta}}^{\sigma(X,F)} - a_{i_0}$, and indeed $x_n \in \overline{(co(B_{i_0}) + B(0,\delta))}^{\sigma(X,F)}$, so the proof is over. □

A direct consequence of the connection lemma is a straightforward proof of the renorming implication in theorem 1

Corollary 1. *In a normed space X with a norming subspace F in X^* we have an equivalent $\sigma(X,F)$-lower semicontinuous and locally uniformly rotund norm whenever there are slicely isolated families for the $\sigma(X,F)$ topology*

$$\{\mathcal{B}_n : n = 1, 2, ...\}$$

such that for every x in X and every $\epsilon > 0$ there is some positive integer n with the property that $x \in B \in \mathcal{B}_n$ and that $\|\cdot\| - diam(B) < \epsilon$.

*P*roof.- It is not a restriction to assume that every one of the families $\mathcal{B}_n$ is uniformly bounded since we can make intersections with countably many balls centered in the origin and covering X without losing the character of slicely isolatedness and the network condition of the whole family. So we can consider the norms $\|\cdot\|_{\mathcal{B}_n}$ constructed using the connection lemma for each one of the families $\mathcal{B}_n$ and to define the new norm by the formula:

$$\|x\|_{\mathcal{B}}^2 := \sum_{n=1}^{\infty} c_n \|x\|_{\mathcal{B}_n}^2$$

for every $x \in X$, where the sequence (c_n) is chosen accordingly for the convergence of the series. This is possible because all the norms $\|\cdot\|_{\mathcal{B}_n}$ are equivalent to the original one and there are numbers d_n such that

$$\|\cdot\|_{\mathcal{B}_n} \leq d_n \|\cdot\|,$$

so it is enough to take $c_n := \frac{1}{d_n^2 2^n}$. If we cosider a sequence $\{x_n : n \in \mathbb{N}\}$ and x in X such that

$$\lim_n \left(2\|x_n\|_{\mathcal{B}}^2 + 2\|x\|_{\mathcal{B}}^2 - \|x_n + x\|_{\mathcal{B}}^2\right) = 0,$$

and we fix an $\epsilon > 0$, we know that there is q and $B_0 \in \mathcal{B}_q$ with $x \in B_0 \subset B(x, \epsilon)$. The condition

$$\lim_n \left(2\|x_n\|_{\mathcal{B}}^2 + 2\|x\|_{\mathcal{B}}^2 - \|x_n + x\|_{\mathcal{B}}^2\right) = 0$$

implies that

$$\lim_n \left(2\|x_n\|_{\mathcal{B}_q}^2 + 2\|x\|_{\mathcal{B}_q}^2 - \|x_n + x\|_{\mathcal{B}_q}^2\right) = 0$$

by convex arguments. The connection lemma now says that for every positive δ there is $n_\delta \in \mathbb{N}$ such that

$$x_n \in \overline{(co(B_0) + \delta B_X)}^{\sigma(X,F)}$$

whenever $n \geq n_\delta$. Thus $\|x_n - x\| \leq \epsilon + \delta$ for $n \geq n_\delta$ and $\lim_n x_n = x$ in $(X, \|\cdot\|)$ as we wanted to prove.

□

Remark.- A normed space X with a norming subspace F admits an equivalent $\sigma(X, F)$-lower semicontinuous and **LUR** norm if, and only, for every $\epsilon > 0$ we have that $X = \cup_{n=1}^{\infty} X_n^\epsilon$ and for every $n \in \mathbb{N}$ and every $x \in X_n^\epsilon$ there is a $\sigma(X, F)$-open half space H with $x \in H$ and $\|\cdot\|$-diam$(H \cap X_n^\epsilon) \leq \epsilon$, in other words if, and only if, the identity map from the $\sigma(X, F)$ to the $\|\cdot\|$-topology is σ-slicely continuous, see [12].

The fact that we are in the conditions of the former corollary when the identity map from the $\sigma(X, F)$ to the $\|\cdot\|$-topology is σ-slicely continuous follows from Stone's theorem about the paracompactnes of metric spaces, [10]. Indeed, if we have a discrete family of sets $\{D_i : i \in I\}$ in X with $\|x - y\| > \delta$ for every $x \in D_i$ and $y \in D_j$ with $i \neq j$, and $\|\cdot\|$-diam$D_i \leq \epsilon$ for every $i \in I$, we can define the refinement

$$D_i^n := D_i \cap X_n^\delta$$

for all $i \in I$, so that the family $\{D_i^n : i \in I\}$ is going to be $\sigma(X, F)$-isolated with $D_i = \cup_{n=1}^{\infty} D_i^n$. Given an open cover of the normed space X with sets of diameter less than or equal to ϵ it has a σ-discrete refinement with families $\{D_i : i \in I\}$ as above. Collecting all families for all $\epsilon = 1/m$ we are in the conditions of the former corollary and so we have an equivalent $\sigma(X, F)$-lower semicontinuous and **LUR** norm on X

With the same proof we have the following:

Corollary 2. *If X and Y ar normed spaces, F and G are norming subspaces in X^* and Y^* respectively, and $T : X \longrightarrow Y$ is a continuous linear map which is $\sigma(X,F)$ to $\sigma(Y,G)$ continuous too, then we have an equivalent $\sigma(X,F)$-lower semicontinuous norm on X $\|\cdot\|_T$ such that*

$$\lim_n \left(2\|x_n\|_T^2 + 2\|x\|_T^2 - \|x_n + x\|_T^2\right) = 0$$

implies that

$$\lim_n Tx_n = Tx$$

whenever T is $\sigma(X,F)$-slicely continuous to the norm; for instance when Y admits an equivalent $\sigma(Y,G)$- lower semicontinuous and LUR norm. In particular if Y has a LUR norm and $T^*(Y^*)$ is norme dense in X^*, then the normed space X admits an equivalent LUR norm.

Remark.- Our approach here do not need any convexification argument as the ones based in Bourgain Namioka supperlemma, [14, 3], or those developped in [12]. Indeed the convex structure here is inside the proof of the connection lemma, it is in the fact that the functions used are already convex, it is the convexity of the functions φ_i and ψ_i, as they have been defined, which gives free of charge the construction of the equivalent norm using now Deville's master lemma. In our approach here the convexification is done on the elements of the σ-slicely isolated network only. In all previous approaches it was done on the sets X_n^ϵ from the decomposition $X = \cup_{n=1}^\infty X_n^\epsilon$ above.

4 Compactness and renormings

For a bounded set B in a normed space X, the Kuratowski index of non-compactness of B is defined by

$$\alpha(B) := \inf\{\epsilon > 0 : B \subseteq Z + B(0,\epsilon) \text{ for some norm compact subset } Z\}$$

The main results in the work [3] provides extensions of corollary 1 when the Kuratowski index of non-compactness is used instead of the diameter. We are going to extend the corollary 1 for spaces of the kind $C(K)$, where K is a separable compact space, as an application of our connection lemma above, but using a more general measure of non-compactness:

Definition 3. *For a bounded subset A of the normed space X with norming subspace $F \subseteq X^*$ we define the index of non $\sigma(X,F)$-compactness by*

$$\chi^F(A) := inf\,\{\epsilon > 0 : A \subseteq Z + B(0,\epsilon) \text{ for some } Z \in \mathcal{W}\}$$

where we denote by $\mathcal{W}$ the family of all $\sigma(X,F)$-relatively compact subsets of X.

The theorem reads as follows:

Theorem 4. *Let K be a separable compact space such that there are slicely isolated families for the pointwise topology*

$$\{\mathcal{B}_n : n = 1, 2, ...\}$$

such that for every x in $C(K)$ and every $\epsilon > 0$ there is some positive integer n, with the property that $x \in B \in \mathcal{B}_n$ and $\chi^{T_p}(B) < \epsilon$. Then the Banach space $C(K)$ admits an equivalent and T_p-lower semicontinuous locally uniformly rotund norm.

*P*roof.- Without any lost of generality we can, and we do assume, that every one of the families $\mathcal{B}_n$ is uniformly bounded since the intersection of a slicely isolated family of sets with a fixed ball is slicely isolated too. Let us construct, with the use of the connection lemma, equivalent pointwise lower semicontinuous norms $\|\cdot\|_{\mathcal{B}_n}$ for every positive integer n that verify the conclusion of the lemma for the family $\mathcal{B}_n$. If we choose a countable set $T =: \{t_n : n = 1, 2, ...\}$ in the separable compact K, we can now define a new norm with the formula:

$$|||x|||^2 := \sum_n^{\infty} c_n(\|x\|^2_{\mathcal{B}_n} + x(t_n)^2)$$

for every $x \in C(K)$, where the sequence (c_n) is chosen accordingly for the convergence of the series. Now the condition

$$\lim_n \left(2\,|||x_n|||^2 + 2\,|||x|\,||^2 - |||x_n + x|||^2\right) = 0$$

for a given sequence (x_n) and x in $C(K)$ implies that the sequence itself is a pointwise relatively compact subset of $C(K)$. Indeed, for every $\epsilon > 0$, choosing the family $\mathcal{B}_n$ such that $x \in B \in \mathcal{B}_n$ and the index of pointwise compactness $\chi^{T_p}(B) < \epsilon$, the connection lemma tells us that there is a positive integer n_0 such that $\chi^{T_p}(\{x_n : n = n_0, n_0 + 1, ...\}) < \epsilon$ and so for all the sequence $\chi^{T_p}(\{x_n : n = 1, 2, ...\}) < \epsilon$. So the sequence (x_n) forms a relatively pointwise compact subset of $C(K)$. Moreover, and again by the convex arguments we have that $\lim_n x_n(t) = x(t)$ for every $t \in T$. Thus for any pointwise limit point y of a subsequence $\{x_{n_k} : k = 1, 2, ...\}$ and every $s \in K$ we choose a net $\{u_\alpha : \alpha \in (A, \succ)\}$ in the dense subset T with $s = \lim_\alpha u_\alpha$ in the compact K, and the interchange limit property of relatively pointwise compact subsets of $C(K)$ studied by Grothendieck tells us that the following happens:

$$\lim_\alpha \lim_k x_{n_k}(u_\alpha) = \lim_\alpha x(u_\alpha) = x(s) = \lim_k \lim_\alpha x_{n_k}(u_\alpha) = \lim_k x_{n_k}(s) = y(s).$$

Consequently any cluster point y of the sequence (x_n) coincides with the given x and so the pointwise limit of the sequence (x_n) is the continuous function x, thus the space $C(K)$ is pointwise locally uniformly rotund with the new norm $|||\cdot|||$, and it has an equivalent pointwise lower semicontinuous and locally uniformly rotund norm by our results in [10]. $\square$

Let us finish with an open problem:

Question.-

Given an scattered compact space K, is there any characterization of the LUR renormability of $C(K)$ by means of any σ-discreteness property for the family of all clopen subsets of K? Indeed, we know that if $\mathcal{A}$ is the familly of all clopen subsets of the scattered compact spaces K and $C(K)$ admits an equivalent pointwise lower semicontinuous and LUR norm, then the family of clopen sets is a countable union $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ of families such that every one of them provides a set of characteristic functions $\{\mathbf{1}_A : A \in \mathcal{A}_n\}$ which is pointwise slicely discrete, but it is unknown what else is needed to have a reverse implication true.

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